Math 2263	**Name (Print):** ______________
Fall 2014	**Student ID:** ______________
Midterm 1	**Section Number: 001** ______
October 2, 2014	**Teaching Assistant:** ______________
Time Limit: 50 minutes	**Signature:** ______________

This exam contains 7 problems. Answer all of them. Point values are in parentheses. You must show your work to get credit for your solutions - correct answers without work will not be awarded points.

Do not give numerical approximations to quantities such as $\sin 5$, π, $\ln(3)$ or $\sqrt{2}$. However, you should simplify $\cos\frac{\pi}{2} = 0$, $e^0 = 1$, and so on.

1	15 pts	
2	12 pts	
3	15 pts	
4	10 pts	
5	15 pts	
6	15 pts	
7	18 pts	
TOTAL	100 pts	

1. (a) (6 points) Find the point at which the given lines intersect:

$$L_1 : x = 1 + t, \quad y = 1 - t, \quad z = 2t \qquad \text{and} \qquad L_2 : x = 4 + 2s, \quad y = 1 + s, \quad z = 1 - s.$$

Solution. We need to find t and s such that $1 + t = 4 + 2s$, $1 - t = 1 + s$ and $2t = 1 - s$. Solving first two equations we get $t = 1, s = -1$ which satisfy the third equation too. So, the point of intersection is $(2, 0, 2)$.

(b) (9 points) Find an equation for the plane which contains both lines.

Solution. The direction vectors of the lines L_1 and L_2 are $\vec{v_1} = \langle 1, -1, 2\rangle$ and $\vec{v_2} = \langle 2, 1, -1\rangle$. Note that $\vec{v_1}$ and $\vec{v_2}$ both lie on the plane. The normal vector to the plane $\vec{v_1} \times \vec{v_2} = \langle -1, 5, 3\rangle$. The plane passes through the point $(2, 0, 2)$. Hence, the equation for the plane is

$$(-1)(x - 2) + 5(y - 0) + 3(z - 2) = 0, \qquad \text{or} \qquad x - 5y - 3z + 4 = 0.$$

2. (12 points) Find an equation for the surface in (x, y, z)-space obtained by rotating the ellipse $x^2 + 4y^2 = 1$ of the (x, y)-plane **about the x-axis**.

Solution. Take any point (x, y, z) on the surface. The distance from the x-axis is $\sqrt{y^2 + z^2}$, which replaces $|y|$ in the given equation. So the equation for the surface of revolution is $x^2 + 4y^2 + 4z^2 = 1$.

Alternate Solution. If we rotate an ellipse $x^2+4y^2 = 1$ with center at $(0, 0)$ in the (x, y)-plane about the x-axis, we will get an ellipsoid in 3-dimensional space with center at $(0, 0, 0)$.

Say, the equation of the ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Intersection with xy-plane is $x^2 + 4y^2 = 1$, which gives $a = 1, b = \frac{1}{2}$.

Intersection with yz-plane is a circle of radius $\frac{1}{2}$, that is $y^2 + z^2 = \frac{1}{4}$, which gives $c = \frac{1}{2}$. So, the equation of the ellipsoid is

$$x^2 + 4y^2 + 4z^2 = 1.$$

3. (a) (5 points) Find the domain of the function $f(x, y) = \sqrt{1 - x^2} - \sqrt{y}$.

Solution. The domain of f: $D = \{(x, y) : 1 - x^2 \geq 0, y \geq 0\} = \{(x, y) : |x| \leq 1, y \geq 0\}$.

(b) (10 points) Evaluate the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + xy - y^2}{x^2 - y^2}$$

or state that it does not exist, giving reasons.

Solution. We take limit $(x, y) \to (0, 0)$ along the line $y = mx$, $m \neq \pm 1$. The corresponding limit becomes

$$\lim_{x\to 0} \frac{x^2 + x \cdot mx - (mx)^2}{x^2 - (mx)^2} = \lim_{x\to 0} \frac{1 + m - m^2}{1 - m^2} = \frac{1 + m - m^2}{1 - m^2}.$$

The limit clearly depends on m. For example, the limit $= 1$ if $m = 0$ but the limit $= \frac{5}{3}$ if $m = \frac{1}{2}$. Hence, the limit does not exist.

4. (10 points) Suppose $z = f(x, y)$ is a function with partial derivatives $f_x(0, 3) = -1$ and $f_y(0, 3) = 2$. If x and y are both functions of t:

$$x = 1 - t \quad \text{and} \quad y = 2t + t^2,$$

find $\frac{dz}{dt}$ at $t = 1$.

Solution. By chain rule,

$$\frac{dz}{dt} = f_x(x(t), y(t))\frac{dx}{dt} + f_y(x(t), y(t))\frac{dy}{dt}.$$

We have $\frac{dx}{dt} = -1$ and $\frac{dy}{dt} = 2 + 2t$. Plugging in $t = 1$, we obtain

$$\left.\frac{dz}{dt}\right|_{t=1} = f_x(0, 3) \cdot (-1) + f_y(0, 3) \cdot 4 = (-1) \cdot (-1) + 2 \cdot 4 = 9.$$

5. (15 points) For the function $f(x,y) = e^{-y}\sin 2x$, find the second partial derivatives

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}, \quad f_{xy} = \frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad f_{yy} = \frac{\partial^2 f}{\partial y^2}.$$

Solution.

$$f_x = 2e^{-y}\cos(2x), \quad f_y = -e^{-y}\sin(2x).$$

$$f_{xx} = -4e^{-y}\sin(2x), \quad f_{xy} = -2e^{-y}\cos(2x), \quad f_{yy} = e^{-y}\sin(2x).$$

6. (15 points) The point $(x, y, z) = (2, -1, 0)$ lies on the surface S:

$$x^2 - 3y^2 + xz - 4z^2 = 1.$$

Find the equation of the tangent plane to the surface S at $(2, -1, 0)$, in the form $ax+by+cz = d$.

Solution. We have $F(x, y, z) = x^2 - 3y^2 + xz - 4z^2$. So,

$$F_x = 2x + z, \quad F_y = -6y, \quad F_z = x - 8z.$$

The equation of the tangent plane to the surface S at $(2, -1, 0)$ is:

$$F_x(2, -1, 0)(x - 2) + F_y(2, -1, 0)(y + 1) + F_z(2, -1, 0)(z - 0) = 0.$$

That is,

$$4(x - 2) + 6(y + 1) + 2z = 0 \qquad \text{or,} \quad 2x + 3y + z = 1.$$

7. The temperature at any point (x, y) is given by $T(x, y) = 10 - x^2 - 2y^2$.

(a) (8 points) Find the rate of change of temperature at point $P = (1, 0)$ in the direction toward the point $(0, 2)$.

Solution.

$$T_x(x, y) = -2x, \quad T_y(x, y) = -4y.$$

Let $Q = (0, 2)$. The unit vector along $\vec{PQ} = \langle -1, 2 \rangle$ is $\vec{u} = \langle \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$. The rate of change of T at point $P = (1, 0)$ in the direction toward the point Q is

$$D_{\vec{u}}T(1, 0) = \nabla T(1, 0) \cdot \vec{u} = \langle -2, 0 \rangle \cdot \vec{u} = \frac{2}{\sqrt{5}}.$$

(b) (10 points) In which direction does the temperature increase fastest at P? Find the maximum rate of increase of the temperature at P.

Solution. The temperature increase fastest at P in the direction of the gradient vector $\nabla T(1, 0) = \langle -2, 0 \rangle$.

The maximum rate of increase of the temperature at P is $|\nabla T(1, 0)| = \sqrt{(-2)^2 + 0^2} = 2$.