Finite element exterior calculus and the geometrical basis of numerical stability

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Outline

References

Finite element exterior calculus, homological techniques, and applications, Acta Numerica 2006, p. 1–155

> *"Any young (or not so young) mathematician who spends the time to master this paper will have tools that will be useful for his or her entire career."* — Math Reviews

Mixed finite element methods for linear elasticity with weakly imposed symmetry, Math. Comp. 2007

NEW! *Geometric decompositions and local bases for spaces finite element differential forms,* to appear in CMAME

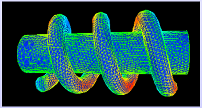
everything is at http://umn.edu/~arnold

Motivations

Why do we need more theory for finite elements?

Why do we need FEEC?

- The finite element method is incredibly successful.
- FEM often amenable to mathematical analysis, allowing validation and comparison of methods.



But plenty of challenges remain, for algorithms and analysis!

- Approximability, consistency, and *stability* $\Longrightarrow$ convergence
- Stability, like well-posedness, can be extremely subtle
- ⚠ Well-posedness + approximability + consistency $\not\Longrightarrow$ stability
- Exterior calculus, de Rham cohomology, Hodge theory, . . . , are geometric tools to get at well-posedness.
- FEEC adapts these tools to the discrete level to get at stability.

Steady heat conduction problem: finite elements in H^1

$$-\operatorname{div} C \operatorname{grad} u = f \qquad \text{strong}$$

$$\int_\Omega C \operatorname{grad} u \cdot \operatorname{grad} v \, dx = \int_\Omega f\, v\, dx \quad \forall v \qquad \text{weak}$$

$$\int_\Omega \left(\frac{1}{2} C \operatorname{grad} u \cdot \operatorname{grad} u - f\, u \right) dx \xrightarrow{u} \text{minimum} \qquad \text{variational}$$

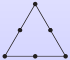
$$\int_\Omega |\operatorname{grad} u|^2 \, dx < \infty \iff u \in H^1(\Omega)$$

$H^1 : u \in L^2(\Omega)$, $\operatorname{grad} u \in L^2(\Omega; \mathbb{R}^n)$

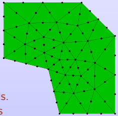
The right FE space:

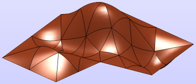
Lagrange elements $\{\, v \in H^1(\Omega) \;:\; v|_T \in \mathcal{P}_r(T) \quad \forall T \in \mathcal{T}_h \,\}$

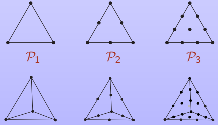
Lagrange finite elements



Shape fns: $\mathcal{P}_2$
DOFs: vertex vals. & edge averages







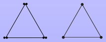
First order (mixed) formulation

$$A\sigma = \operatorname{grad} u, \quad -div\sigma = f \qquad \text{strong}$$

$$\int_\Omega A\sigma \cdot \tau \, dx = -\int_\Omega \operatorname{div} \tau\, u\, dx \quad \forall \tau,$$
$$-\int_\Omega \operatorname{div} \sigma\, v\, dx = \int_\Omega f\, v\, dx \quad \forall v \qquad \text{weak}$$

$$\int_\Omega \left(\frac{1}{2} A\sigma \cdot \sigma + \operatorname{div} \sigma\, u + f\, u\right) dx \xrightarrow{\sigma,\, u} \text{stationary pt.} \qquad \text{variational}$$

$$\sigma \in H(\operatorname{div}, \Omega), \quad u \in L^2(\Omega)$$

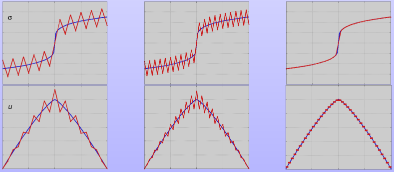
Lagrange elements?  Unstable!

Thermal problem in 1D

Babuška–Narasimhan

$$\sigma = u', \quad -\sigma' = f \qquad \text{on } (-1,1)$$

$$\frac{1}{2}\int_{-1}^{1}(\sigma^2 + \sigma' u + f\,u)\,dx \xrightarrow[H^1\times L^2]{\sigma,\,u} \text{stationary point}$$

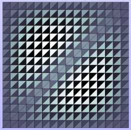


$\mathcal{P}_1$-$\mathcal{P}_1$ (20 elts) $\mathcal{P}_1$-$\mathcal{P}_1$ (40 elts) $\mathcal{P}_1$-$\mathcal{P}_0$ (40 elts)

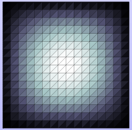
Thermal problem in 2D

$$\sigma = \operatorname{grad} u, \quad -\operatorname{div}\sigma = f$$

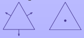
$$\int_\Omega (\frac{1}{2}|\sigma|^2 + \operatorname{div}\sigma\, u + f\, u)\,dx \xrightarrow[H(div)\times L^2]{\sigma,\,u} \text{stationary point}$$



$\mathcal{P}_1$–$\mathcal{P}_0$



$\mathcal{P}_1^-\Lambda^1$–$\mathcal{P}_0$

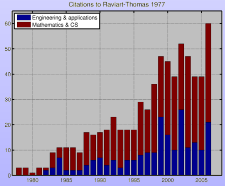


Raviart–Thomas elements ($\mathcal{P}_r^-\Lambda^1$)

Shape functions $\mathcal{P}_1^-\Lambda^1$: span $\left[\binom{1}{0}, \binom{0}{1}, \binom{x}{y}\right]$ DOFs:

A mixed FEM for 2nd order elliptic problems, Proc. conf. Math'l Aspects of the FEM, Rome 1975. Springer Lect. Notes in Math #606, 1977.

Generalizes to all degrees, and all dimensions ($n = 3$: Nédélec '80)



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Numerische Mathematik
Mathematics of Computation
RAIRO – M^{2}AN
Num. Methods for PDEs

Eng. & Apps
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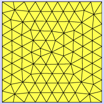
COMPEL

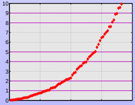
Maxwell eigenvalue problem, unstructured mesh

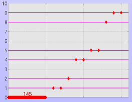
$$\int_\Omega \mu^{-1}\operatorname{curl} E \cdot \operatorname{curl}\tilde{E} = \omega^2 \int_\Omega \epsilon E \cdot \tilde{E} \quad \forall \tilde{E}$$

Right space is $H(\text{curl})$

$$\lambda = m^2 + n^2 = 0, 1, 1, 2, 4, 4, 5, 5, 8, \ldots$$





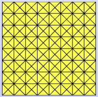


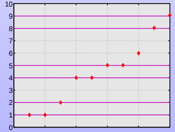
$(\text{Lag.}\mathcal{P}_1)^2$

$\mathcal{P}_1^-\Lambda^1$

Maxwell eigenvalue problem, regular mesh

$\lambda = m^2 + n^2 = 1, 1, 2, 4, 4, 5, 5, 8, \ldots$





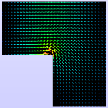
254	574	1022	1598
1.0043	1.0019	1.0011	1.0007
1.0043	1.0019	1.0011	1.0007
2.0171	2.0076	2.0043	2.0027
4.0680	4.0304	4.0171	4.0110
4.0680	4.0304	4.0171	4.0110
5.1063	5.0475	5.0267	5.0171
5.1063	5.0475	5.0267	5.0171
5.9229	5.9658	5.9807	5.9877
8.2713	8.1215	8.0685	8.0438

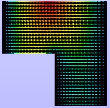
Boffi-Brezzi-Gastaldi '99

Vector Laplacian

$$\operatorname{curl}\operatorname{curl} u - \operatorname{grad}\operatorname{div} u = f \text{ in } \Omega$$
$$u \cdot n = 0, \quad \operatorname{rot} u = 0 \text{ on } \partial\Omega$$

$$\int_\Omega \frac{1}{2}(|\operatorname{curl} u|^2 + |\operatorname{div} u|^2) - f \cdot u \xrightarrow{u} \text{minimum}$$





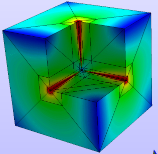
Lagrange finite elements converge nicely
but not to the solution!
(same problem with any conforming FE)

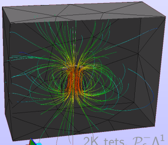
A mixed formulation based on appropriate finite elements works fine

$$\int_\Omega (\frac{1}{2}|\sigma^2| - \operatorname{curl}\sigma \cdot u - \frac{1}{2}|\operatorname{div} u|^2 - f\, u)\, dx \xrightarrow[H(\operatorname{curl})\times H(\operatorname{div})]{\sigma, u} \text{stationary point}$$

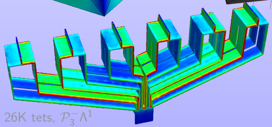
EM calculations based on the generalized RT elements

Schöberl, Zaglmayr 2006, NGSolve





2K tets, $\mathcal{P}_6^- \Lambda^1$



26K tets, $\mathcal{P}_3^- \Lambda^1$

Also: White EMSolve, Demkowicz 3Dhp90, Durufle Montjoie,. . .

Exterior calculus and PDE

The continuous problem

Differential forms

For $\Omega \subset \mathbb{R}^n$, $\Lambda^k(\Omega)$ consists of functions $\Omega \to \mathrm{Alt}^k \mathbb{R}^n$

so if $\omega \in \Lambda^k(\Omega)$, $\quad \omega_x(v_1, \ldots, v_k) \in \mathbb{R}$, $\quad x \in \Omega, v_i \in \mathbb{R}^n$

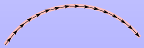
$\Lambda^0(\Omega)$: real-valued functions on Ω

$\Lambda^1(\Omega)$: covector fields, $\omega = \sum_{i=1}^n f_i dx_i$, f_i functions $(dx_i(\mathbf{e}_j) = \delta_{ij})$

$\Lambda^2(\Omega)$: $\omega = \sum_{i<j} f_{ij} dx_i \wedge dx_j$, $(dx_i \wedge dx_j := dx_i \otimes dx_j - dx_j \otimes dx_i)$

$d^k : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$ $\qquad d(f\, dx_{j} \wedge \cdots \wedge dx_k) = \sum_i \frac{\partial f}{\partial x_i} dx_i \wedge dx_j \wedge \cdots \wedge dx_k$

$\int_f \omega \in \mathbb{R}$ $\qquad \omega \in \Lambda^k$, $\dim f = k$



De Rham complex and cohomology

$$0 \to \Lambda^0(\Omega) \xrightarrow{d^0} \Lambda^1(\Omega) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \Lambda^n(\Omega) \to 0$$

$\mathfrak{Z}_k := \ker(d^k)$

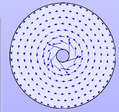
$\mathfrak{B}_k := \mathrm{range}(d^{k-1})$

$$\dim \mathfrak{Z}_k/\mathfrak{B}_k = \begin{cases} \#\text{ of components}, & i = 0, \\ \#\text{ of holes}, & i = 1, \\ \#\text{ of voids}, & i = 2, \\ \ldots \end{cases}$$

vector proxies in $\mathbb{R}^3$: $\sum f_i dx_i \leftrightarrow (f_1, f_2, f_3)$, $\sum f_{ij} dx_i dx_j \leftrightarrow (f_{23}, -f_{13}, f_{12})$

$$0 \to C^\infty(\Omega) \xrightarrow{\text{grad}} C^\infty(\Omega; \mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\Omega; \mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\Omega) \to 0$$

> "Physical vector quantities may be divided into two classes, in one of which the quantity is defined with reference to a line, while in the other the quantity is defined with reference to an area."
>
> – James Clerk Maxwell,
> *Treatise on Electricity & Magnetism, 1891*



Hodge theory

Making use of the inner product:

- Hodge star: $* : \Lambda^k(\Omega) \xrightarrow{\cong} \Lambda^{n-k}(\Omega)$
- formal adjoint: $\delta = \pm * d * : \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$

$$\Lambda^{k-1}(\Omega) \underset{\delta}{\overset{d}{\rightleftarrows}} \Lambda^k(\Omega) \underset{\delta}{\overset{d}{\rightleftarrows}} \Lambda^{k+1}(\Omega)$$

- Hodge Laplacian: $d\delta + \delta d : \Lambda^k \to \Lambda^k$
- harmonic forms: $\mathfrak{H}^k := \{\, \zeta \in \mathfrak{Z}^k \mid \zeta \perp \mathfrak{B}^k \,\} \cong \mathfrak{Z}^k / \mathfrak{B}^k$
- Hodge decomposition: $L^2\Lambda^k(\Omega) = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus (\mathfrak{Z}^k)^\perp$
- Poincaré's inequality: $\|\omega\|_{L^2} \le c\, \|d\omega\|_{L^2}$, $\quad \omega \in (\mathfrak{Z}^k)^\perp$
- Sobolev spaces: $H\Lambda^k(\Omega) = \{\, \omega \in L^2\Lambda^k(\Omega) \mid d\omega \in L^2\Lambda^{k+1}(\Omega) \,\}$

$$0 \to H\Lambda^0(\Omega) \xrightarrow{d} H\Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^n(\Omega) \to 0$$

$$0 \to H^1(\Omega) \xrightarrow{\text{grad}} H(\text{curl}, \Omega) \xrightarrow{\text{curl}} H(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \to 0$$

Some applications

Physical quantities:

0-forms: temperature; electric field potential
1-forms: temperature gradient; electric field
2-forms: heat flux; magnetic flux
stress is a covector-valued 2-form
3-forms: heat density; charge density; mass density

PDEs:

- $-\operatorname{div}\operatorname{grad} u = f$
- $(\operatorname{curl}\operatorname{curl} - \operatorname{grad}\operatorname{div})u = f$
- $\operatorname{curl}\operatorname{curl} u = f$, $\operatorname{div} u = 0$
- $\operatorname{div} u = f$, $\operatorname{curl} u = 0$
- Maxwell's equations
- elasticity
- dynamic problems, eigenvalue problems, lower order-terms
- variable coefficients, nonlinearities. . .

Hodge Laplace problem

Given $f \in \Lambda^k$ $(0 \le k \le n)$, find $u \in \Lambda^k$ with $(d\delta + \delta d)u = f$ (plus BC)

Harmonic functions determine well-posedness:

$$\exists u \iff f \perp \mathfrak{H}^k, \qquad u \text{ is determined only mod } \mathfrak{H}^k$$

This mixed formulation is always well-posed: Given $f \in L^2\Lambda^k(\Omega)$, find

$\sigma \in H\Lambda^{k-1}, \quad u \in H\Lambda^k, \quad p \in \mathfrak{H}^k$:

$$\begin{aligned} &\langle \sigma, \tau \rangle - \langle d\tau, u \rangle = 0 && \forall \tau \in H\Lambda^{k-1} \\ &\langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle = \langle f, v \rangle && \forall v \in H\Lambda^k \\ &\langle u, q \rangle = 0 && \forall q \in \mathfrak{H}^k \end{aligned}$$

Equivalently $\frac{1}{2}\langle \sigma, \sigma \rangle - \frac{1}{2}\langle du, du \rangle - \langle d\sigma, u \rangle - \langle u, p \rangle + \langle f, u \rangle \to$ saddle point

Well-posedness of the Hodge Laplacian

$\sigma \in H\Lambda^{k-1}, \quad u \in H\Lambda^k, \quad p \in \mathfrak{H}^k$:

$$\begin{aligned} &\langle \sigma, \tau \rangle - \langle d\tau, u \rangle = 0 && \forall \tau \in H\Lambda^{k-1} \\ &\langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle = \langle f, v \rangle && \forall v \in H\Lambda^k \\ &\langle u, q \rangle = 0 && \forall q \in \mathfrak{H}^k \end{aligned}$$

Need to control $\|\sigma\|_{H\Lambda} + \|u\|_{H\Lambda} + \|p\|$ by a bounded choice of τ, v, and q.

$\tau = \sigma$ controls $\|\sigma\|$, $v = d\sigma$ controls $\|d\sigma\|$, $v = p$ controls $\|p\|$
$v = u$ controls $\|du\|$, *How to control* $\|u\|$*??*

Hodge decomp.: $u = d\eta + s + z, \quad \eta \in H\Lambda^{k-1}, s \in \mathfrak{H}^k, z \in (\mathfrak{Z}^k)^\perp$

$\tau = \eta$ controls $\|d\eta\|$ and $q = s$ controls $\|s\|$. To bound $\|z\|$ we use Poincaré's inequality:

$$\|z\| \le c\|dz\| = c\|du\| \quad \text{(which is under control)}$$

Discretization

We have a well-posed variational PDE problem. How do we discretize it stably?

Abstract setting

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \Lambda^{k-1} & \xrightarrow{d^{k-1}} & \Lambda^k & \xrightarrow{d^k} & \Lambda^{k+1} & \longrightarrow & \cdots \\ & & \uparrow \cup & & \uparrow \cup & & \uparrow \cup & & \\ \cdots & \longrightarrow & \Lambda_h^{k-1} & \xrightarrow{d^{k-1}} & \Lambda_h^k & \xrightarrow{d^k} & \Lambda_h^{k+1} & \longrightarrow & \cdots \end{array}$$

Complex of Hilbert spaces with d^k bounded and closed range.

For discretization, construct a finite dimensional subcomplex.

Define $\mathfrak{H}_h^k = (\mathfrak{B}_h^k)^\perp \cap \mathfrak{Z}_h^k$.
Discrete Hodge decomp. follows: $\Lambda_h^k = \mathfrak{B}_h^k \oplus \mathfrak{H}_h^k \oplus (\mathfrak{Z}_h^k)^\perp$

Galerkin's method: Λ^{k-1}, Λ^k, $\mathfrak{H}^k \longrightarrow \Lambda_h^{k-1}$, Λ_h^k, $\mathfrak{H}_h^k$

When is it stable?

Bounded cochain projections

Key property: The finite dimensional subcomplex admits a bounded cochain projection.

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \Lambda^{k-1} & \xrightarrow{d^{k-1}} & \Lambda^k & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \pi_h^{k-1}} & & \downarrow{\scriptstyle \pi_h^k} & & \\
\cdots & \longrightarrow & \Lambda_h^{k-1} & \xrightarrow{d^{k-1}} & \Lambda_h^k & \longrightarrow & \cdots
\end{array}$$

- π_h^k bounded
- π_h^k a projection
- $\pi_h^k d^{k-1} = d^{k-1}\pi_h^{k-1}$
- $\lim_{h\to 0} \pi_h^k v = v$, $v \in \Lambda^k$

Theorem

- *The induced map on cohomology is an isomorphism for h small.*
- $\operatorname{gap}(\mathfrak{H}^k, \mathfrak{H}_h^k) \to 0$
- *The discrete Poincaré inequality holds uniformly in h.*
- *Galerkin's method is stable and convergent.*

Proof of discrete Poincaré inequality

Thm. There is a positive constant c, independent of h, such that

$$\|\omega\| \le c\|d\omega\|, \quad \omega \in \mathfrak{Z}_h^{k\perp}.$$

Proof. Given $\omega \in \mathfrak{Z}_h^{k\perp}$, define $\eta \in \mathfrak{Z}^{k\perp} \subset H\Lambda^k(\Omega)$ by $d\eta = d\omega$. By the Poincaré inequality, $\|\eta\| \le c\|d\omega\|$, so it is enough to show that $\|\omega\| \le c\|\eta\|$. Now, $\omega - \pi_h\eta \in \Lambda_h^k$ and $d(\omega - \pi_h\eta) = 0$, so $\omega - \pi_h\eta \in \mathfrak{Z}_h^k$. Therefore

$$\|\omega\|^2 = \langle \omega, \pi_h\eta\rangle + \langle \omega, \omega - \pi_h\eta\rangle = \langle \omega, \pi_h\eta\rangle \le \|\omega\|\|\pi_h\eta\|,$$

whence $\|\omega\| \le \|\pi_h\eta\|$, and the result follows.

Finite element differential forms

How do we construct finite element spaces that fit together in de Rham subcomplexes with bounded cochain projections?

Finite element differential forms

Let $\mathcal{T} = \mathcal{T}_h$ be a triangulation of $\Omega \subset \mathbb{R}^n$. We wish to construct finite element spaces $\Lambda^k(\mathcal{T}) \subset H\Lambda^k(\Omega)$ which form a finite dimensional subcomplex with bounded cochain projections.

We will construct them as usual for finite elements:

On each simplex $T \in \mathcal{T}$ we specify

- a space of polynomials shape functions
- degrees of freedom, each associated to a face of the simplex

It turns out that for each form degree k and polynomial degree r, there are just two "natural" finite element subspaces of $H\Lambda^k(\Omega)$:

$\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$

$$\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h) = \begin{cases} \mathcal{P}_r\Lambda^k(\mathcal{T}_h), & k = 0, \\ \mathcal{P}_{r-1}\Lambda^k(\mathcal{T}_h), & k = n, \\ \text{strictly between}, & 0 < k < n \end{cases}$$

The Koszul complex

Key tool: the Koszul differential $\kappa : \Lambda^k \to \Lambda^{k-1}$:

$$(\kappa\omega)_x(v^1, \dots, v^{k-1}) = \omega_x(X, v^1, \dots, v^{k-1}), \qquad X = x - x_0$$

$$0 \longleftarrow \mathcal{P}_r\Lambda^0 \xleftarrow{\kappa} \mathcal{P}_{r-1}\Lambda^1 \xleftarrow{\kappa} \cdots \xleftarrow{\kappa} \mathcal{P}_{r-n}\Lambda^n \longleftarrow 0$$

C.f., the polynomial de Rham complex

$$0 \longrightarrow \mathcal{P}_r\Lambda^0 \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n \longrightarrow 0$$

For $\Omega \subset \mathbb{R}^3$

$$0 \leftarrow \mathcal{P}_r(\Omega) \xleftarrow{\cdot X} \mathcal{P}_{r-1}(\Omega;\mathbb{R}^3) \xleftarrow{\times X} \mathcal{P}_{r-2}(\Omega;\mathbb{R}^3) \xleftarrow{X} \mathcal{P}_{r-3}(\Omega) \leftarrow 0$$

Key relation: $(d\kappa + \kappa d)\omega = (r+k)\omega \quad \forall \omega \in \mathcal{H}_r\Lambda^k$ (homogeneous polys)

$$\therefore \mathcal{H}_r\Lambda^k = d\mathcal{H}_{r+1}\Lambda^{k-1} \oplus \kappa\mathcal{H}_{r-1}\Lambda^{k+1}$$

Definition of $\mathcal{P}_r^-\Lambda^k$

$$\mathcal{P}_r\Lambda^k = \mathcal{P}_{r-1}\Lambda^k + \mathcal{H}_r\Lambda^k$$
$$= \mathcal{P}_{r-1}\Lambda^k + \kappa\mathcal{H}_{r-1}\Lambda^{k+1} + d\mathcal{H}_{r+1}\Lambda^{k-1}$$

$$\mathcal{P}_r^-\Lambda^k := \mathcal{P}_{r-1}\Lambda^k + \kappa\mathcal{H}_{r-1}\Lambda^{k+1}$$

God made $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$,
all the rest is the work of man.

Degrees of freedom

The other ingredient of a finite element space are the *degrees of freedom*, i.e., a decomposition of the dual spaces $(\mathcal{P}_r\Lambda^k(T))^*$ and $(\mathcal{P}_r^-\Lambda^k(T))^*$, into subspaces associated to subsimplices f of T.

DOF for $\mathcal{P}_r\Lambda^k(T)$: to a subsimplex f of dim. $d \ge k$ we associate

$$\omega \mapsto \int_f \mathrm{Tr}_f\, \omega \wedge \eta, \quad \eta \in \mathcal{P}^-_{r+k-d}\Lambda^{d-k}(f)$$

DOF for $\mathcal{P}_r^-\Lambda^k(T)$:

$$\omega \mapsto \int_f \mathrm{Tr}_f\, \omega \wedge \eta, \quad \eta \in \mathcal{P}_{r+k-d-1}\Lambda^{d-k}(f)$$ Hiptmair

The resulting FE spaces have exactly the continuity required by $H\Lambda^k$:
Theorem. $\mathcal{P}_r\Lambda^k(\mathcal{T}) = \{\, \omega \in H\Lambda^k(\Omega) \ : \ \omega|_T \in \mathcal{P}_r\Lambda^k(T) \quad \forall T \in \mathcal{T} \,\}$.
Similarly for $\mathcal{P}_r^-$.

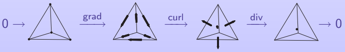
Finite element differential forms and classical mixed FEM

- $\mathcal{P}_r^-\Lambda^0(\mathcal{T}) = \mathcal{P}_r\Lambda^0(\mathcal{T}) \subset H^1$ Lagrange elts 
- $\mathcal{P}_r^-\Lambda^n(\mathcal{T}) = \mathcal{P}_{r-1}\Lambda^n(\mathcal{T}) \subset L^2$ discontinuous elts 
- $n = 2$: $\mathcal{P}_r^-\Lambda^1(\mathcal{T}) \subset H(\mathrm{curl})$ Raviart–Thomas elts 
- $n = 2$: $\mathcal{P}_r\ \Lambda^1(\mathcal{T}) \subset H(\mathrm{curl})$ Brezzi–Douglas–Marini elts 
- $n = 3$: $\mathcal{P}_r^-\Lambda^1(\mathcal{T}) \subset H(\mathrm{curl})$ Nedelec 1st kind edge elts 
- $n = 3$: $\mathcal{P}_r\ \Lambda^1(\mathcal{T}) \subset H(\mathrm{curl})$ Nedelec 2nd kind edge elts 
- $n = 3$: $\mathcal{P}_r^-\Lambda^2(\mathcal{T}) \subset H(\mathrm{div})$ Nedelec 1st kind face elts 
- $n = 3$: $\mathcal{P}_r\ \Lambda^2(\mathcal{T}) \subset H(\mathrm{div})$ Nedelec 2nd kind face elts 

Finite element de Rham subcomplexes

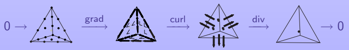
From these spaces we want to build discrete de Rham complexes with bounded projections. It turns out that there are lots of ways to do this (2^{n-1} for each r). Extreme cases are:

$$0 \to \mathcal{P}_r^- \Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{P}_r^- \Lambda^1(\mathcal{T}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^- \Lambda^n(\mathcal{T}) \to 0$$



Whitney 1957, Bossavit 1988

$$0 \to \mathcal{P}_r \Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{P}_{r-1} \Lambda^1(\mathcal{T}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n} \Lambda^n(\mathcal{T}) \to 0$$



Application to Elasticity

What else can you do with FEEC?

Stress–displacement mixed finite elements for elasticity

Find stress $\sigma : \Omega \to \mathbb{R}^{3\times 3}_{\text{sym}}$, displacement $u : \Omega \to \mathbb{R}^3$ such that

$$A\sigma = \epsilon(u), \quad \operatorname{div}\sigma = f$$

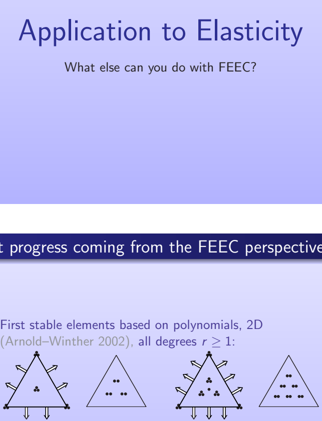
$$\int_\Omega \left(\frac{1}{2} A\sigma : \sigma + \operatorname{div}\sigma \cdot u + f \cdot u \right) dx \xrightarrow[H(\operatorname{div};\mathbb{S})\times L^2(\mathbb{R}^n)]{\sigma, u} \text{stationary point}$$

Search for stable finite elements dates back to the '60s, very limited success.

> . . . to derive elements that exhibit complete continuity of the appropriate components along interfaces. . . was achieved by Raviart and Thomas in the case of the heat conduction problem. . . . Extension to the full stress problem is difficult and as yet such elements have not been successfully noted.
>
> — Zienkiewicz, Taylor, Zhu
>
> *The Finite Element Method: Its Basis & Fundamentals*, 6th ed., vol. 1, 2005

Recent progress coming from the FEEC perspective

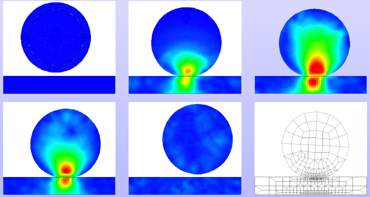
- First stable elements based on polynomials, 2D (Arnold–Winther 2002), all degrees $r \geq 1$:



- 3D stable elements, all degrees $r \geq 1$ (Arnold–Awanou–Winther 2007): for $r = 1$ stress space has 162 degrees of freedom (27 per component on average)

A computation using the new elements

From Eberhard, Hueber, Jiang, Wohlmuth 2006



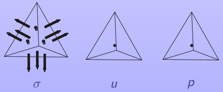
Mixed formulation with weak symmetry

Idea goes back to Fraeijs de Veubeke 1975, Amara–Thomas 1979
In the classical Hellinger–Reissner principle, symmetry of the stress tensor (balance of angular momentum) is assumed to hold exactly.
Instead we impose it weakly with a Lagrange multiplier (the rotation).

$$\int_\Omega \left(\frac{1}{2} A\sigma:\sigma + \operatorname{div}\sigma\cdot u + f\cdot u\right)\,dx \xrightarrow[H(\operatorname{div};\mathbb{S})\times L^2(\mathbb{R}^n)]{\sigma,u} \text{stationary point}$$

$$\int_\Omega \left(\frac{1}{2} A\sigma:\sigma + \operatorname{div}\sigma\cdot u + \sigma:p + f\cdot u\right)\,dx \xrightarrow[H(\operatorname{div};\mathbb{M})\times L^2(\mathbb{R}^n)\times L^2(\mathbb{K})]{\sigma,u,p} \text{S.P.}$$

FEEC has led to very simple stable elements



Features of the new mixed elements

- Based on HR formulation with weak symmetry; very natural
- Lowest degree element is very simple: full $\mathcal{P}_1$ for stress, $\mathcal{P}_0$ for displacement and rotation
- Works for every polynomial degree
- Works the same in 2 and 3 (or more) dimensions
- Robust to material constraints like incompressibility
- Provably stable and convergent

Conclusions

- Exterior calculus clarifies the nature of physical quantities and the structure of the PDEs involving them.
- Capturing the right structure on the discrete level can be essential to get stable methods.
- FEEC provides a very natural framework for the design and understanding of subtle stability issues that arise in the discretization of a wide variety of PDE systems. It brings to bear tools from geometry, topology, and algebra to develop discretizations which are compatible with the geometric, topological, and algebraic structure of the PDE system, and so obtain stability.
- FEEC has been used to unify, clarify, and refine many known finite element methods. It is a mathematically rigorous theory.
- The $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$ spaces are the natural finite element discretizations for differential forms and the de Rham complex.
- Through FEEC we believe we have completed the long search for "just the right" mixed finite elements for elasticity.