Mixed Finite Elements for Elasticity

Douglas N. Arnold
Institute for Mathematics and its Applications

joint work with Ragnar Winther, University of Oslo

Scientific Computing
Xi'an Jiaotong University
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displacement $u : \Omega \to \mathbb{R}^2$

stress $\sigma : \Omega \to \mathbb{S} := \mathbb{R}^{2\times 2}_{\text{sym}}$

$$A\sigma = \epsilon\, u := [\nabla u + (\nabla u)^T]/2$$

$$\operatorname{div} \sigma = f$$

$\sigma \in H(\operatorname{div}, \Omega, \mathbb{S}), u \in L^2(\Omega, \mathbb{R}^2)$ satisfy

$$\int_\Omega A\sigma : \tau\, dx + \int_\Omega \operatorname{div} \tau \cdot u\, dx = 0 \qquad \forall \tau \in H(\operatorname{div}, \Omega, \mathbb{S})$$

$$\int_\Omega \operatorname{div} \sigma \cdot v\, dx = \int_\Omega f \cdot v \qquad \forall v \in L^2(\Omega, \mathbb{R}^2)$$

$(\sigma, u) \in H(\operatorname{div}, \Omega, \mathbb{S}) \times L^2(\Omega, \mathbb{R}^2)$ saddle point of

$$\mathcal{L}(\tau, v) = \int_\Omega \left(\frac{1}{2} A\tau : \tau + \operatorname{div} \tau \cdot v - f \cdot v\right) dx$$

Mixed methods seek a saddle point over finite dimensional subspaces.

The question is:

How can we construct finite element spaces

$$\Sigma_h \subset H(\mathrm{div}, \Omega, \mathbb{S}), \qquad V_h \subset L^2(\Omega, \mathbb{R}^2)$$

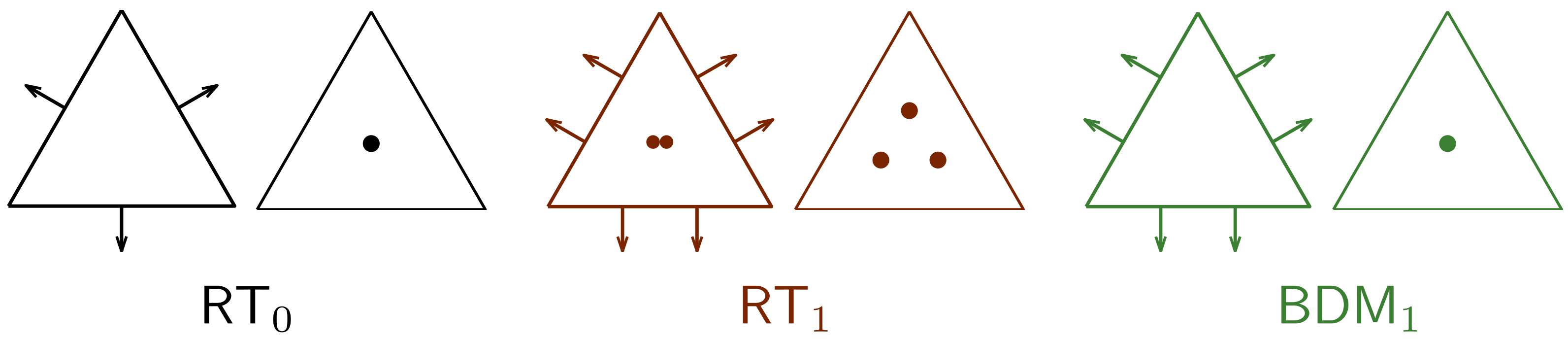
with good stability and convergence properties?

$(\sigma, u) \in H(\mathrm{div}, \Omega, \mathbb{R}^2) \times L^2(\Omega, \mathbb{R})$ is a critical point of

$$\int_\Omega (\frac{1}{2} A\tau \cdot \tau + \mathrm{div}\,\tau\, v - f v)\, dx$$

vector *scalar*

A variety of stable choices of space exist, most notably the Raviart–Thomas and Brezzi–Douglas–Marini families.



The spaces used for vector/scalar problems have two key properties that are used to establish stability and convergence.

- $\operatorname{div} \Sigma_h \subset V_h$
- There exists a projection operator Π_h onto Σ_h, bounded in $\mathcal{L}(H^1, L^2)$ uniformly in h, and satisfying the commutativity property $\operatorname{div} \Pi_h \sigma = P_h \operatorname{div} \sigma$ where P_h is the L^2 projection onto V_h

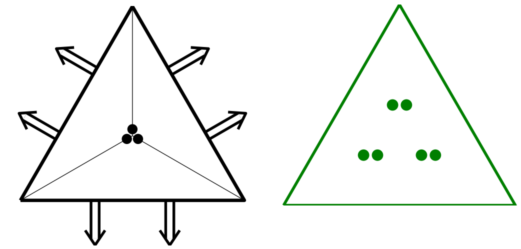
A beautiful convergence theory results: Raviart–Thomas, Falk–Osborn, Douglas–Roberts, . . .

Previous tensor/vector mixed elements

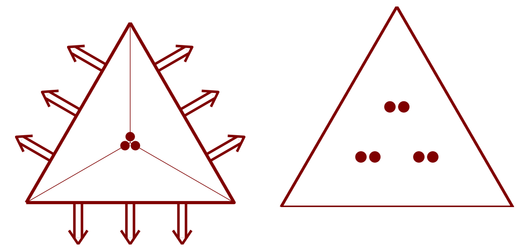
Efforts to obtain similarly nice stress–displacement mixed finite elements had produced *no* stable elements using polynomial shape functions.

Composite elts: Johnson–Mercier '78

cf. Fraeijs de Veubeke '65;Watwood–Hartz '68



Arnold–Douglas–Gupta '84



. . .

Modified variational forms:

Amara–Thomas '79, Arnold–Brezzi–Douglas '84 (PEERS),

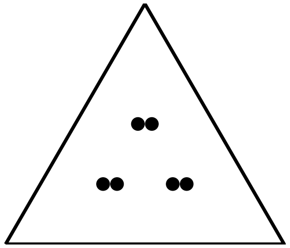
Stenberg '86 . . . , Stein and Rolfes '90;

Mignot–Surrey '81; Arnold–Falk '88; . . .

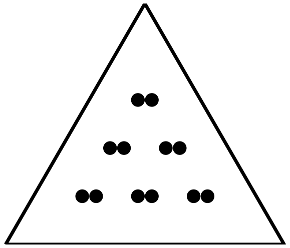
A new family of triangular elements

Pick any polynomial degree $k \geq 1$

For the *displacement* in $L^2(\Omega, \mathbb{R}^2)$ we simply use discontinuous p.w. polynomials of degree $\leq k$.



$k = 1$

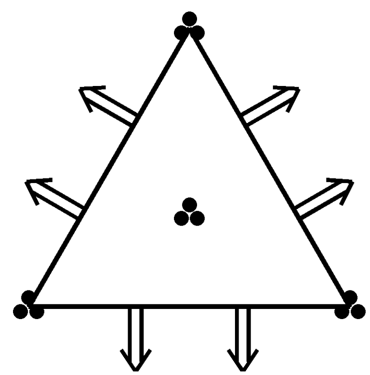


$k = 2$

A new family of triangular elements

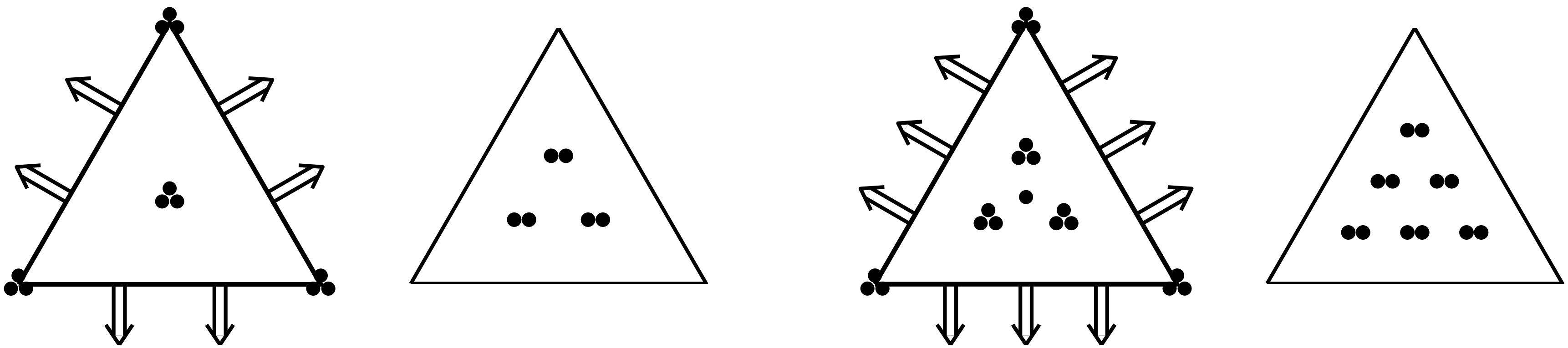
For the *stress* in $H(\mathrm{div}, \Omega, \mathbb{S})$ the shape functions are
$\Sigma_T = \{\, \tau \in \mathcal{P}_{k+2}(T, \mathbb{S}) \mid \mathrm{div}\, \tau \in \mathcal{P}_k(T, \mathbb{R}^2) \}$. $k = 1$ DOF are

- the values of three components at each vertex (9)
- the values of the moments of degree 0 and 1 of the normal components on each edge (12)
- the value of the moment of degree 0 on the triangle (3)



24 stress DOF

Theorem. *1. The DOF are unisolvent.*
2. The assembled finite element space belongs to $H(\operatorname{div}, \Omega, \mathbb{S})$ *and satisfies* $\operatorname{div} \Sigma_h = V_h$.
3. The associated operator $\Pi_h : C(\Omega, \mathbb{S}) \to \Sigma_h$ *satisfies* $\operatorname{div} \Pi_h \tau = P_h \operatorname{div} \tau$ *for all* $\tau \in C(\Omega, \mathbb{S}) \cap H(\operatorname{div}, \Omega, \mathbb{S})$.



Theorem.

$$\|\sigma - \sigma_h\|_{L^2} \leq Ch^{k+2}\|\sigma\|_{H^{k+2}}$$

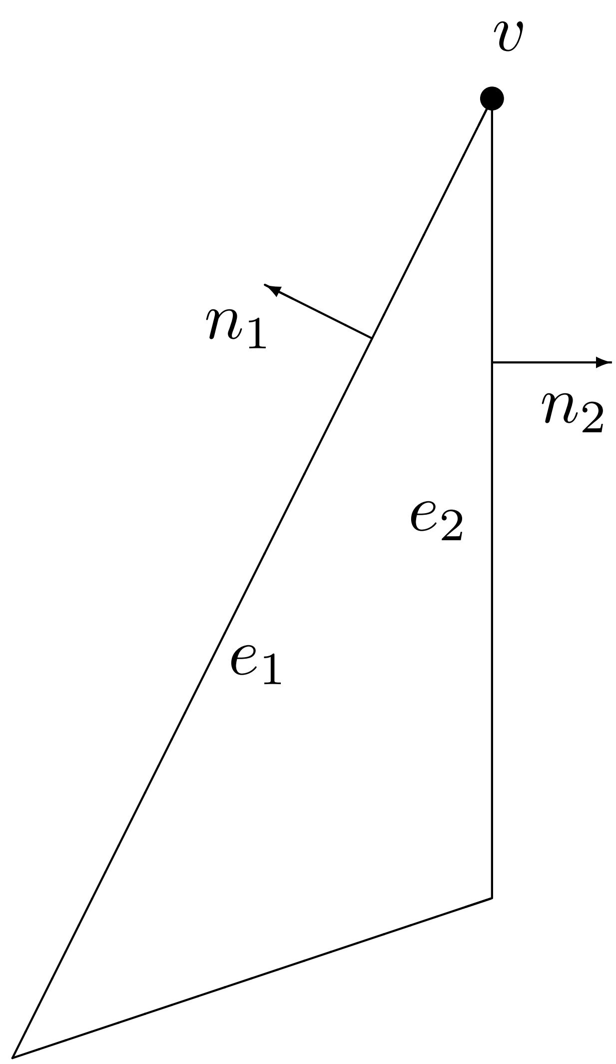
$$\|u - u_h\|_{L^2} \leq Ch^{k+1}\|u\|_{H^{k+2}}$$

Lowest order element is $O(h^3)$ for stress in L^2, $O(h^2)$ for displacement.

Unlike the usual $H(\mathrm{div}, \Omega, \mathbb{R}^2)$ finite elements, our $H(\mathrm{div}, \Omega, \mathbb{S})$ elements involve vertex degrees of freedom.

These are not required for conformity with $H(\mathrm{div}, \Omega, \mathbb{S})$, and may complicate implementation.

However: *any* $H(\mathrm{div}, \Omega, \mathbb{S})$ element employing continuous shape functions must have vertex values among the DOF.



Key structural aspects of the plane elasticity system are encoded in the exact differential complex

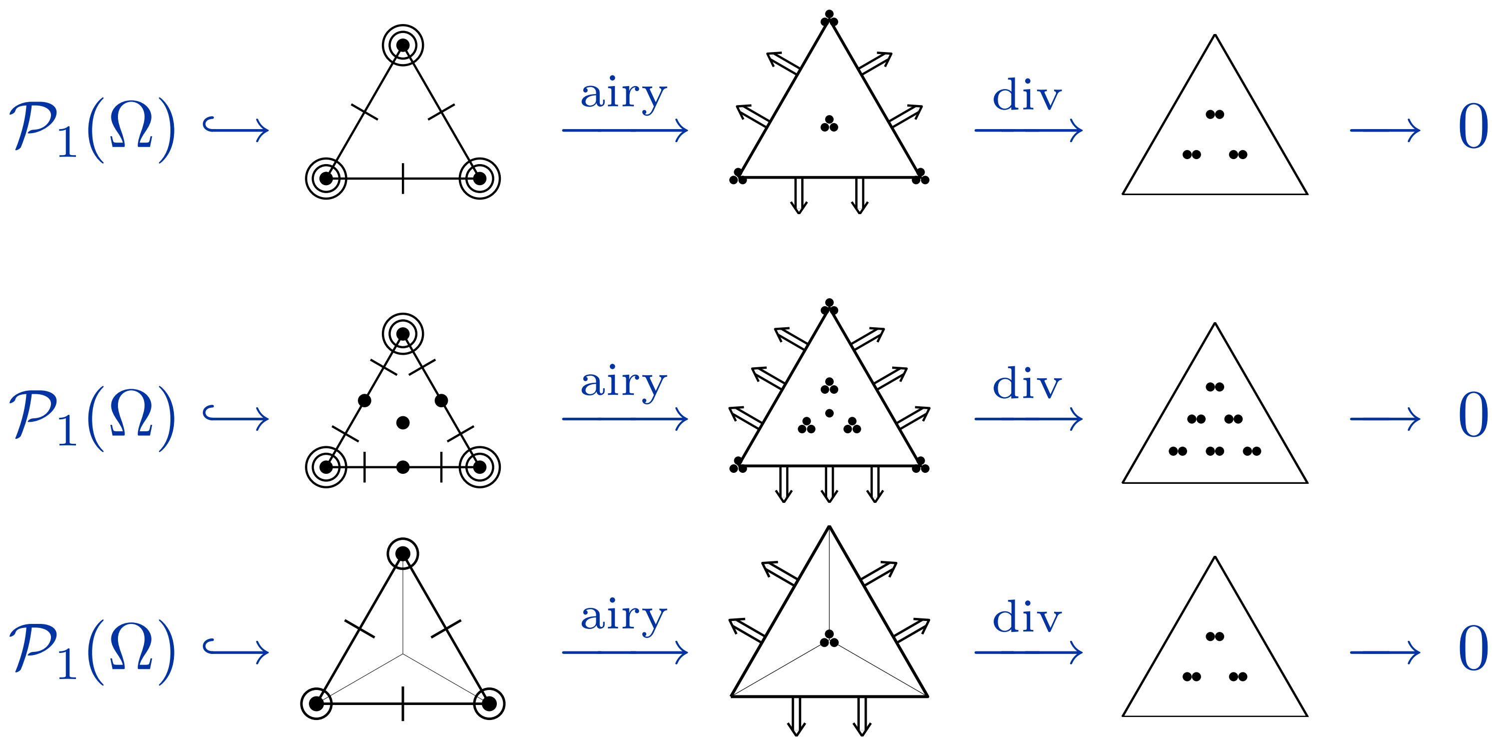
$$\begin{array}{ccccccccc}
\mathcal{P}_1(\Omega) \hookrightarrow & C^\infty(\Omega) & \xrightarrow{\text{airy}} & C^\infty(\Omega,\mathbb{S}) & \xrightarrow{\text{div}} & C^\infty(\Omega,\mathbb{R}^2) & \longrightarrow & 0 \\
& \downarrow I_h & & \downarrow \Pi_h & & \downarrow P_h & & \\
\mathcal{P}_1(\Omega) \hookrightarrow & Q_h & \xrightarrow{\text{airy}} & \Sigma_h & \xrightarrow{\text{div}} & V_h & \longrightarrow & 0
\end{array}$$

The stability conditions are encoded in the exactness and commutativity of a related discrete short exact sequence.

Introducing $Q_h = \text{airy}^{-1}(\Sigma_h)$ and defining an interpolant I_h by commutativity we get a discrete differential complex resolving $\mathcal{P}_1$.

$$\text{airy}\,\phi := \begin{pmatrix} \frac{\partial^2\phi}{\partial y^2} & -\frac{\partial^2\phi}{\partial x\partial y} \\ -\frac{\partial^2\phi}{\partial x\partial y} & \frac{\partial^2\phi}{\partial x^2} \end{pmatrix}$$

For our elements with $k = 1$, $Q_h = \text{airy}^{-1}\,\Sigma_h$ is exactly the Hermite quintic (Argyris) finite element, the simplest H^2 element with polynomial trial functions.



The implication mixed elasticity element $\Longrightarrow$ H^2 element is a major obstruction.

Summary

We have constructed a family of mixed finite elements for elasticity

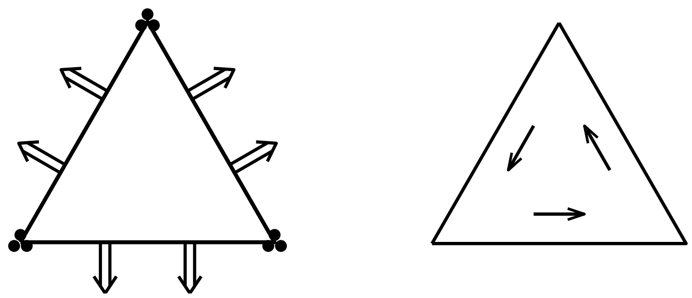
- stable
- conforming in $H(\mathrm{div}, \Omega, \mathbb{S})$
- high order (third order and up for stress)

Potential disadvantages are

- slightly over-smooth
- vertex degrees of freedom
- relativity complicated

A simplified conforming element

$$\tilde{V}_T = \{\, \binom{a-cy}{b+cx} \,|\, a,b,c \in \mathbb{R} \,\},\ \tilde{\Sigma}_T = \{\, \tau \in \Sigma_T \,|\, \operatorname{div} \tau \in \tilde{V}_T \,\}$$



21 stress DOF

$$\|\sigma - \sigma_h\|_{L^2} \le Ch^2 \|\sigma\|_{H^2}$$

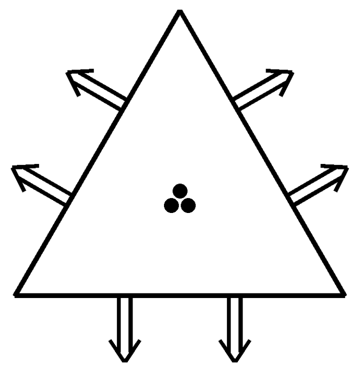
$$\|u - u_h\|_{L^2} \le Ch \|u\|_{H^2}$$

Again, *displacement* we use discontinuous p.w. linears.

Stress: $\Sigma_T = \{\, \tau \in \mathcal{P}_2(T, \mathbb{S}) \,|\, n \cdot \tau n \in \mathcal{P}_1(e, \mathbb{R}) \quad \forall$ edges $e \,\}$

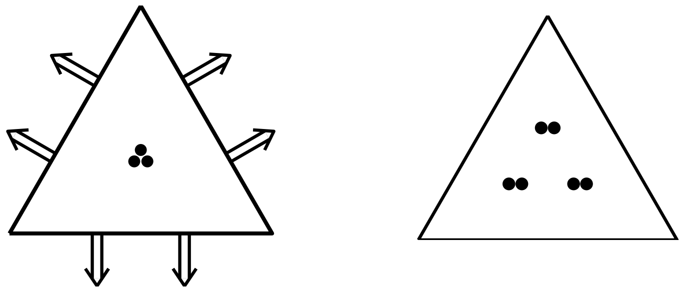
DOF are

- the values of the moments of degree 0 and 1 of the normal components on each edge (12)
- the value of the moment of degree 0 on the triangle (3)



15 stress DOF

same DOF as Watwood–Hartz/Johnson Mercier



Consistency error:

$$E_h(u,\tau) = \int_\Omega (A\sigma : \tau + \operatorname{div}_h \tau \cdot u)\, dx, \quad \tau \in \Sigma_h$$
$$= \sum_e \int_e [t \cdot \tau n] u \cdot t\, ds$$

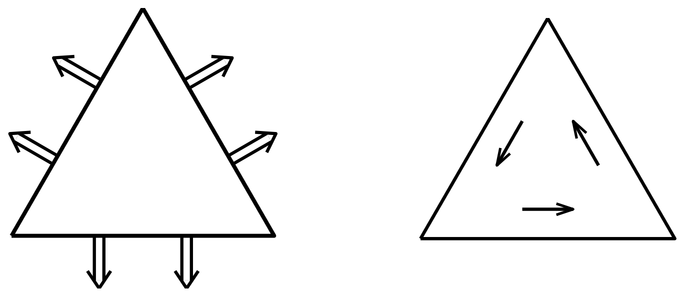
We need to bound this in terms of $\|\tau\|_0$.
Even though $[t \cdot \tau n] \perp \mathcal{P}_1$ we only get $O(h)$:

$$|E_h(u,\tau)| \le Ch\|\tau\|_0\|u\|_2$$

$$\|\sigma - \sigma_h\|_0 \le ch\|u\|_{H^2}, \quad \|u - u_h\|_0 \le ch\|u\|_{H^2}$$

A simplified version

$$\tilde{V}_T = \{ \binom{a-cy}{b+cx} \mid a,b,c \in \mathbb{R} \}, \; \tilde{\Sigma}_T = \{ \tau \in \Sigma_T \mid \operatorname{div} \tau \in \tilde{V}_T \}$$



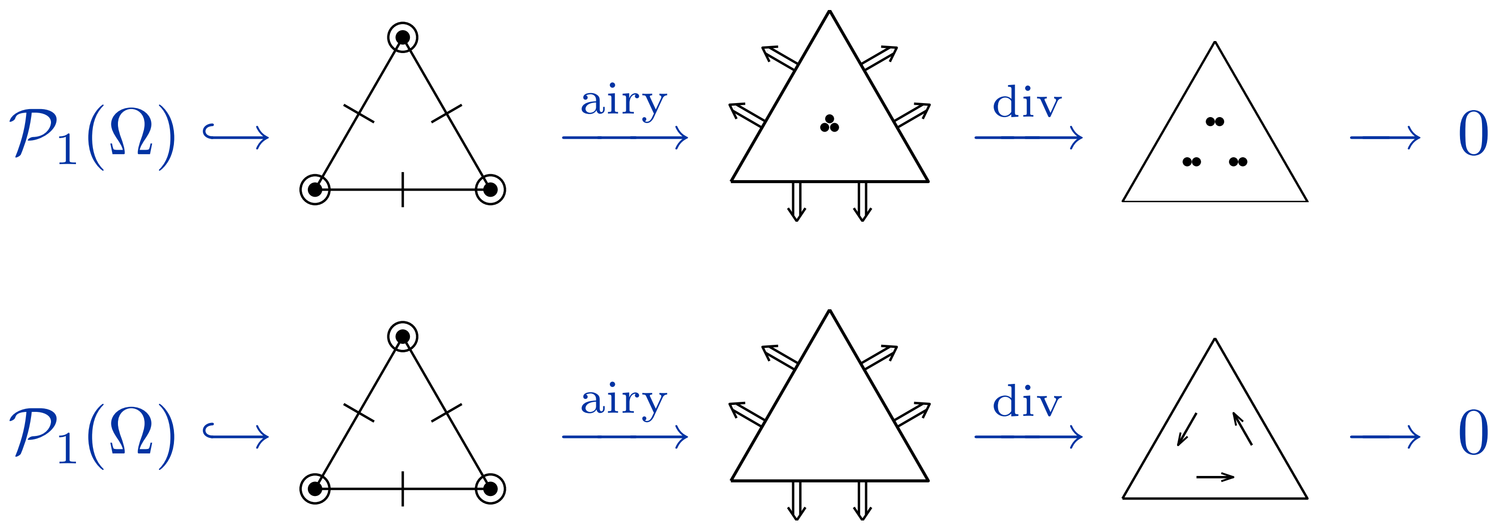
12 stress DOF

$$\|\sigma - \sigma_h\|_{L^2} \le Ch\|u\|_{H^2}$$

$$\|u - u_h\|_{L^2} \le Ch\|u\|_{H^2}$$

The associated H^2 element

The associated H^2 element is a nonconforming element due to Nilsen, Tai, and Winther. The shape functions are quartics which reduce to cubic on the edges, and the DOF are the same as for the HCT elements.



Conclusions

- We have devised a variety of stable mixed finite elements for plane elasticity.
- Conforming elements are of order $3, 4, \ldots$ for stress, $2, 3 \ldots$ for displacement
- A slightly simpler element is of order 2 for stress, 1 for displacement
- Besides composite elements, these are the only ones known to be stable for the stress–displacement formulation of elasticity.
- Vertex DOF are unavoidable for conforming $H(\mathrm{div}, \Omega, \mathbb{S})$ elements with continuous shape functions.
- Two simple nonconforming methods are first order in stress and displacement
- Every element pair is related to an H^2 element and a discrete exact sequence