Finite element exterior calculus: A new approach to the stability of finite elements

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Outline

1. Motivating the search for stable finite elements: finding the right element for the job
2. The mathematical framework: exterior calculus
3. The star of the show: finite element differential forms
4. Application to elasticity: the holy grail attained?

Steady heat conduction problem: finite elements in H^1

$$-\operatorname{div} C \operatorname{grad} u = f$$

$$\int_\Omega C \operatorname{grad} u \cdot \operatorname{grad} v \, dx = \int_\Omega f \, v \, dx \qquad \forall v$$

$$\int_\Omega \left(\frac{1}{2} C \operatorname{grad} u \cdot \operatorname{grad} u - f \, u \right) \, dx \xrightarrow{u} \text{minimum}$$

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$H^1 : u \in L^2(\Omega)$,
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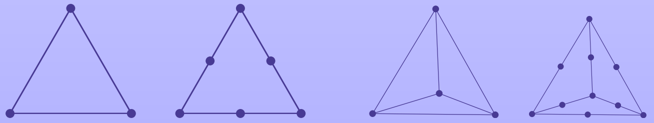
H^1 : $u \in L^2(\Omega)$, grad $u \in L^2(\Omega; \mathbb{R}^n)$

The right FE spaces: Lagrange elements: $\{ v \in H^1(\Omega) \mid v|_T \in \mathcal{P}_r(\Omega) \}$



Elasticity in displacement formulation

$u : \Omega \to \mathbb{R}^n$ displacement field

$$\int_\Omega \left(\frac{1}{2} C\, \epsilon(u) : \epsilon(u)\, dx - f \cdot u \right)\, dx \xrightarrow{u} \text{minimum}$$

$$\int |\,\epsilon(u)|^2\, dx \sim \int |\,\text{grad}\, u|^2\, dx$$ Korn's inequality $u \in [H^1(\Omega)]^n$

Again, Lagrange elements have the right stuff.

First order (mixed) formulations

Thermal $A\,\sigma = \operatorname{grad} u, \quad -div\sigma = f$

$$\int (\frac{1}{2} A\,\sigma \cdot \sigma + \operatorname{div} \sigma\, u + f\, u)\, dx \xrightarrow{\sigma, u} \text{stationary point}$$

$$\sigma \in H(\operatorname{div}, \Omega), \quad u \in L^2(\Omega)$$

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Lagrange elements?

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Lagrange elements? Unstable!

Babuška–Narasimhan

$$\sigma = u', \quad -\sigma' = f \qquad \text{on } (-1,1)$$

$$\frac{1}{2}\int_{-1}^{1}(\sigma^2 + \sigma' u + f\,u)\,dx \xrightarrow[H^1\times L^2]{\sigma,\,u} \text{stationary point}$$

Thermal problem in 1D

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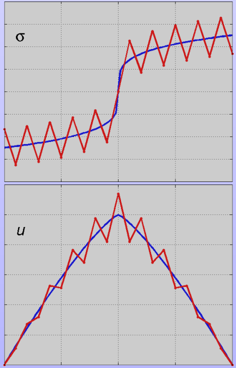

$\mathcal{P}_1$-$\mathcal{P}_1$ (20 elts)

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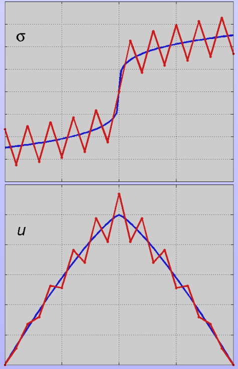

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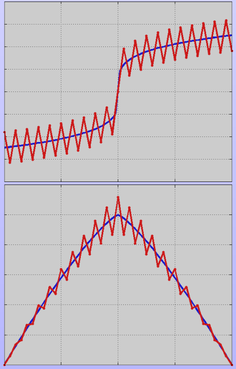

$\mathcal{P}_1$-$\mathcal{P}_1$ (40 elts)

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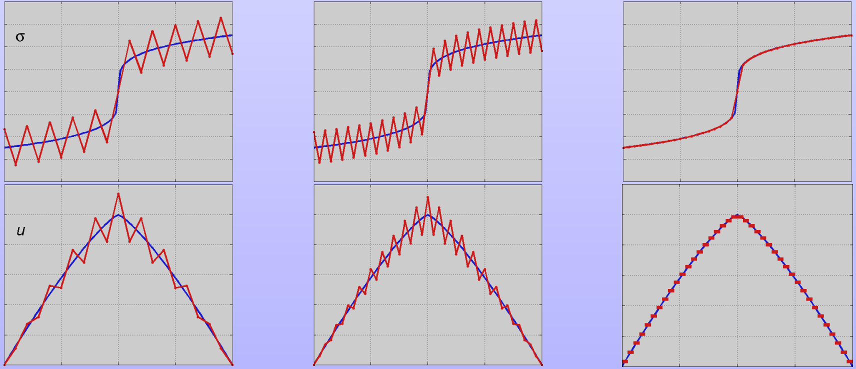

$\mathcal{P}_1$-$\mathcal{P}_1$ (20 elts) $\mathcal{P}_1$-$\mathcal{P}_1$ (40 elts) $\mathcal{P}_1$-$\mathcal{P}_0$ (40 elts)

Thermal problem in 2D

$$\sigma = \operatorname{grad} u, \quad -\operatorname{div} \sigma = f$$

$$\int_\Omega \left(\frac{1}{2}|\sigma|^2 + \operatorname{div} \sigma \, u + f \, u\right) dx \xrightarrow[H(div)\times L^2]{\sigma,\, u} \text{stationary point}$$

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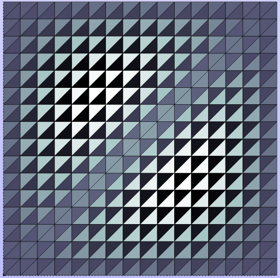
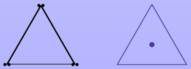


$\mathcal{P}_1$-$\mathcal{P}_0$



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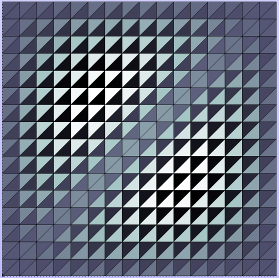

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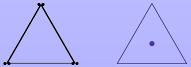
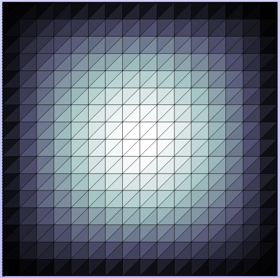
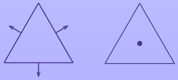


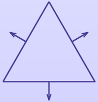


Raviart–Thomas - $\mathcal{P}_0$



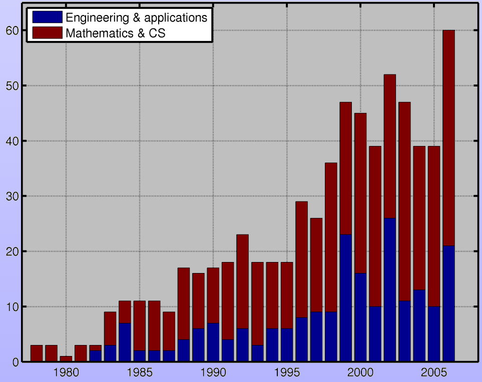
Raviart–Thomas elements

A mixed FEM for 2nd order elliptic problems, Proc. conf. Math'l Aspects of the FEM, Rome 1975. Springer Lect. Notes in Math #606, 1977.

Shape functions: $\binom{1}{0}$, $\binom{0}{1}$, $\binom{x}{y}$ DOFs: 

Generalizes to all degrees, and all dimensions ($n = 3$: Nédélec '80)

Citations to Raviart-Thomas 1977



Math & CS

SIAM J. Numerical Analysis
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RAIRO – M^2AN
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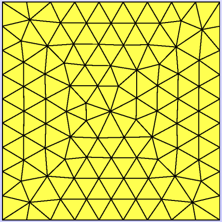
IJNME
COMPEL

Maxwell eigenvalue problem, unstructured mesh

$$\int_\Omega \mu^{-1} \operatorname{curl} E \cdot \operatorname{curl} \tilde{E} = \omega^2 \int_\Omega \epsilon E \cdot \tilde{E} \quad \forall \tilde{E}$$



Right space is $H(\text{curl})$

$$\lambda = m^2 + n^2 = 0, 1, 1, 2, 4, 4, 5, 5, 8, \dots$$

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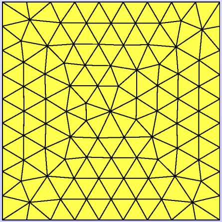

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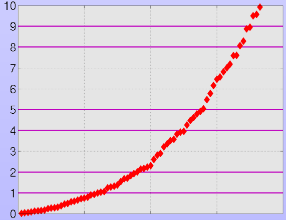

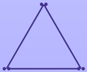

$(\text{Lag.}\mathcal{P}_1)^2$

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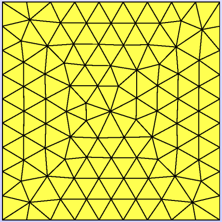

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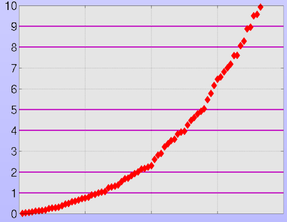
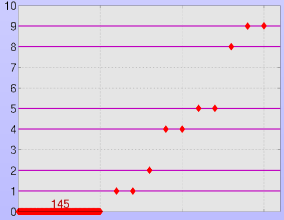
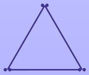
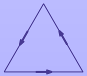






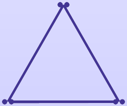
$(\text{Lag}.\mathcal{P}_1)^2$

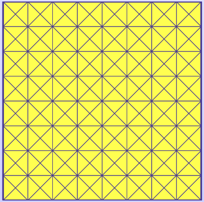


$\mathcal{P}_1^- \Lambda^1$

Maxwell eigenvalue problem, regular mesh

$\lambda = m^2 + n^2 = 1, 1, 2, 4, 4, 5, 5, 8, \ldots$





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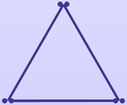
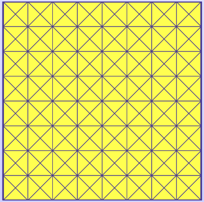
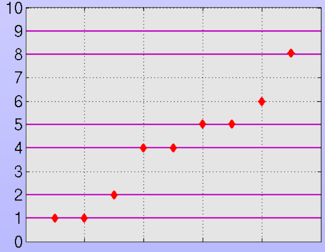






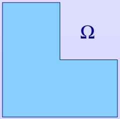
254	574	1022	1598
1.0043	1.0019	1.0011	1.0007
1.0043	1.0019	1.0011	1.0007
2.0171	2.0076	2.0043	2.0027
4.0680	4.0304	4.0171	4.0110
4.0680	4.0304	4.0171	4.0110
5.1063	5.0475	5.0267	5.0171
5.1063	5.0475	5.0267	5.0171
5.9229	5.9658	5.9807	5.9877
8.2713	8.1215	8.0685	8.0438

Boffi-Brezzi-Gastaldi '99

Vector Laplacian

$$\operatorname{curl}\operatorname{curl} u - \operatorname{grad}\operatorname{div} u = f \text{ in } \Omega$$

$$u \cdot t = 0, \quad \operatorname{div} u = 0 \text{ on } \partial\Omega$$



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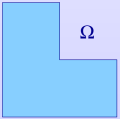

$$\int_\Omega \frac{1}{2}(|\operatorname{curl} u|^2 + |\operatorname{div} u|^2) - f \cdot v \xrightarrow{u} \text{minimum}$$

Lagrange finite elements will converge

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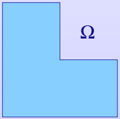

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Lagrange finite elements will converge **to the wrong solution!**

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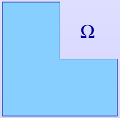

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Lagrange finite elements will converge **to the wrong solution!**
In fact same holds for any conforming finite elements.

Vector Laplacian

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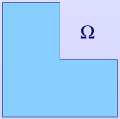

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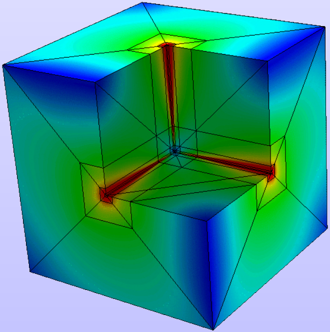
Lagrange finite elements will converge **to the wrong solution!** In fact same holds for any conforming finite elements.

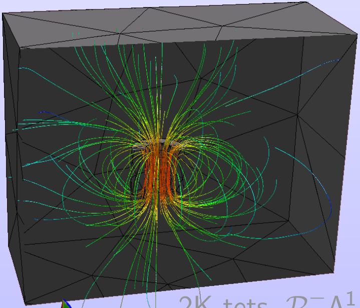
A mixed formulation *based on appropriate finite elements* works just fine

$$\int_\Omega (\frac{1}{2}|\sigma^2| - \operatorname{curl}\sigma \cdot u - \frac{1}{2}|\operatorname{div} u|^2 - f\ v)\, dx \xrightarrow[H(\operatorname{curl}) \times H(\operatorname{div})]{\sigma, u} \text{stationary point}$$

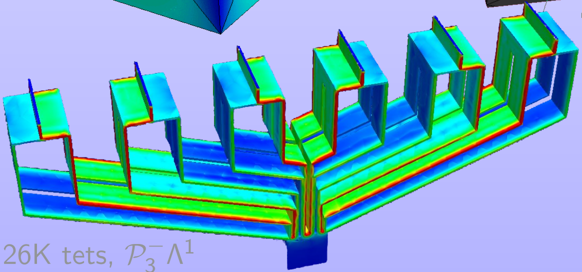
EM calculations based on the generalized RT elements

Schöberl, Zaglmayr 2006, NGSolve





2K tets, $\mathcal{P}_6^- \Lambda^1$



26K tets, $\mathcal{P}_3^- \Lambda^1$

Also: White EMSolve,
Demkowicz 3Dhp90,
Durufle Montjoie,. . .

The Mathematical Framework:

Exterior Calculus

Differential forms

An algebraic k-form F on $\mathbb{R}^n$ is a skew-symmetric k-linear form: it takes k vectors and delivers a number.

$$\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k} \to \mathbb{R}, \qquad (v_1, \ldots, v_k) \mapsto F(v_1, \ldots, v_k)$$

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For example if $\mathbf{u} = \left(u_x, u_y, u_z\right)$ denotes a vector

$dx(\mathbf{u}) := u_x$ is a 1-form, $\quad dx{\wedge}dy\,(\mathbf{u}, \mathbf{v}) := u_x v_y - u_y v_x$ is a 2-form

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A differential k-form on $\Omega \subset \mathbb{R}^n$ is a field of algebraic k-forms:

$$(v_1, \ldots, v_k) \mapsto \omega_x(v_1, \ldots, v_k) \in \mathbb{R} \quad \forall x \in \Omega.$$

0-form = function, 1-form = covector field $\qquad f(x, y)dx + g(x, \mathrm{y})\mathrm{dy}$

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$$\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k} \to \mathbb{R}, \qquad (v_1, \ldots, v_k) \mapsto F(v_1, \ldots, v_k)$$

For example if $\mathbf{u} = \big(u_x, u_y, u_z\big)$ denotes a vector

$dx(\mathbf{u}) := u_x$ is a 1-form, $\quad dx{\wedge}dy\,(\mathbf{u}, \mathbf{v}) := u_x v_y - u_y v_x$ is a 2-form

A differential k-form on $\Omega \subset \mathbb{R}^n$ is a field of algebraic k-forms:

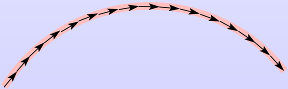
$$(v_1, \ldots, v_k) \mapsto \omega_x(v_1, \ldots, v_k) \in \mathbb{R} \quad \forall x \in \Omega.$$

0-form = function, 1-form = covector field $\qquad f(x, y)dx + g(x, y)\mathrm{d}y$

- 0-forms: temperature, electric field potential
- 1-forms: electric field, magnetic field
- 2-forms: electric flux, magnetic flux, heat flux
- 3-forms: charge density, heat density, mass density

Exterior calculus and the de Rham complex

- A k-form ω can be naturally integrated over a k-dimensional surface: $\int_S \omega \in \mathbb{R}$



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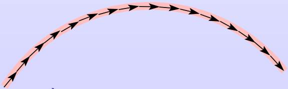

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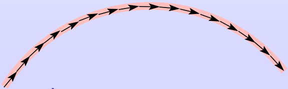

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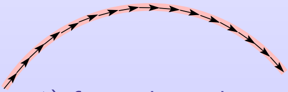

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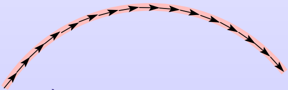

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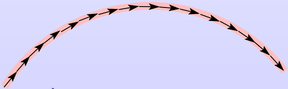

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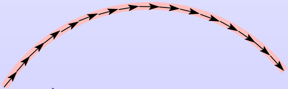

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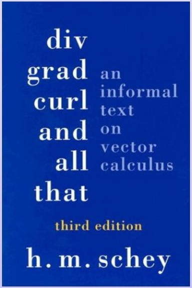

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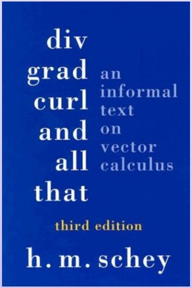

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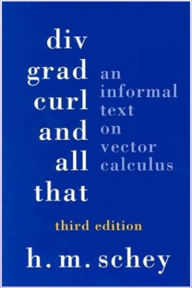

$$\dim(\ker/\text{range}) = \begin{cases} \#\ \text{components of } \Omega, & k = 0 \\ \#\ \text{tunnels thru } \Omega, & k = 1 \\ \#\ \text{voids in } \Omega, & k = 2 \end{cases}$$

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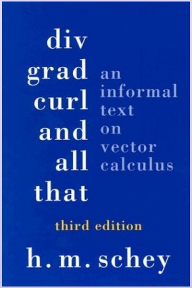

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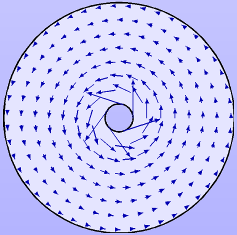

PDEs closely connected to the de Rham sequence

- $-\operatorname{div}\operatorname{grad} u = f$
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To get a stable numerical method, our discretization must capture the essential structure of the de Rham complex, in particular the cohomology.

Design principles for discretizing PDEs related to de Rham

- Treat things for what they are: treat 1-forms as 1-forms, 2-forms as 2-forms, . . .
- A finite element subspace Λ_h^k of some $H\Lambda^k$ should fit together with finite element subspaces of all $H\Lambda^j$
- $d\Lambda_h^{k-1}$ should be contained in Λ_h^k so we get a discrete de Rham subcomplex

$$\cdots \longrightarrow \Lambda_h^{k-1} \xrightarrow{d} \Lambda_h^k \xrightarrow{d} \Lambda_h^{k+1} \longrightarrow \cdots$$

- The subcomplex should relate to the full complex via *commuting projections*

$$\begin{array}{ccccccc} \cdots \longrightarrow & \Lambda^{k-1} & \xrightarrow{d^{k-1}} & \Lambda^k & \longrightarrow \cdots \\ & \downarrow \pi_h^{k-1} & & \downarrow \pi_h^k & \\ \cdots \longrightarrow & \Lambda_h^{k-1} & \xrightarrow{d^{k-1}} & \Lambda_h^k & \longrightarrow \cdots \end{array}$$

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$$\pi_h d = d\pi_h$$

The payoff

A finite element method based on these principles generally captures all the essential structure:

- dimension of the cohomology spaces
- the cohomology classes
- Hodge decomposition (Helmoltz decomposition)
- Poincaré inequality

If the continuous problem is well-posed, the discretization inherits this, i.e., is stable.

The Star of the Show:

Finite Element Differential Forms

Constructing spaces of finite element differential forms

To construct a finite element space of differential forms, we have to specify for a given simplex $T \subset \mathbb{R}^n$:

- Shape functions: a finite dimensional space of polynomial forms on the simplex
- Degrees of freedom: grouped into subspaces associated to the subsimplices

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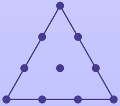
Prototypical case: Lagrange finite elements

Shape functions: $V(T) = \mathcal{P}_r(T)$

DOFs associated to a subsimplex f:



$$W(T,f) = \{\, u \mapsto \int_f \operatorname{tr}_{T,f} u\, v\, dx \;:\; v \in \mathcal{P}_{r-1-\dim f}(f) \,\}$$

The assembled space is then precisely

$$\{\, u \in H^1(\Omega) \;:\; u|_T \in V(T) \quad \forall T \,\}$$

The spaces $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$

For general form degree k there are *two* families of spaces of polynomial differential forms, $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$, which, when assembled lead to *the* natural finite element subspaces of $H\Lambda^k(\Omega)$.

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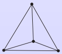
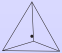
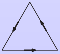
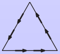
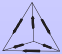
Special cases:

- $\mathcal{P}_r\Lambda^0 = \mathcal{P}_r^-\Lambda^0$, the Lagrange finite elements
- $\mathcal{P}_r\Lambda^n(\mathcal{T}) = \mathcal{P}_{r+1}^-\Lambda^n$, all piecewise polynomials of degree r
- $\mathcal{P}_1^-\Lambda^k(\mathcal{T})$ is the space of Whitney k-forms (1 DOF per k-face)

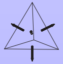
Whitney, 1957

Finite element differential forms and classical mixed FEM

- $\mathcal{P}_r^-\Lambda^0(\mathcal{T}) = \mathcal{P}_r\Lambda^0(\mathcal{T}) \subset H^1$ Lagrange elts 
- $\mathcal{P}_r^-\Lambda^n(\mathcal{T}) = \mathcal{P}_{r-1}\Lambda^n(\mathcal{T}) \subset L^2$ discontinuous elts 
- $n = 2$: $\mathcal{P}_r^-\Lambda^1(\mathcal{T}) \subset H(\text{curl})$ Raviart–Thomas elts 

  $\mathcal{P}_r\Lambda^1(\mathcal{T}) \subset H(\text{curl})$ Brezzi–Douglas–Marini elts 
- $n = 3$: $\mathcal{P}_r^-\Lambda^1(\mathcal{T}) \subset H(\text{curl})$ Nedelec 1st kind edge elts 

  $\mathcal{P}_r\Lambda^1(\mathcal{T}) \subset H(\text{curl})$ Nedelec 2nd kind edge elts 

  $\mathcal{P}_r^-\Lambda^2(\mathcal{T}) \subset H(\text{div})$ Nedelec 1st kind face elts 

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The Koszul complex

The key to the construction is the Koszul differential $\kappa : \Lambda^k \to \Lambda^{k-1}$:

$$(\kappa\omega)_x(v^1, \ldots, v^{k-1}) = \omega_x(X, v^1, \ldots, v^{k-1}), \qquad X = x - x_0$$

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For $\Omega \subset \mathbb{R}^3$

$$0 \leftarrow \mathcal{P}_r(\Omega) \xleftarrow{\bullet X} \mathcal{P}_{r-1}(\Omega;\mathbb{R}^3) \xleftarrow{\times X} \mathcal{P}_{r-2}(\Omega;\mathbb{R}^3) \xleftarrow{X} \mathcal{P}_{r-3}(\Omega) \leftarrow 0$$

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$$0 \leftarrow \mathcal{P}_r(\Omega) \xleftarrow{\bullet X} \mathcal{P}_{r-1}(\Omega;\mathbb{R}^3) \xleftarrow{\times X} \mathcal{P}_{r-2}(\Omega;\mathbb{R}^3) \xleftarrow{X} \mathcal{P}_{r-3}(\Omega) \leftarrow 0$$

Key relation: $(d\kappa + \kappa d)\omega = (r+k)\omega \quad \forall\omega \in \mathcal{H}_r\Lambda^k$ (homogeneous polys)

The Koszul complex

The key to the construction is the Koszul differential $\kappa : \Lambda^k \to \Lambda^{k-1}$:

$$(\kappa\omega)_x(v^1, \dots, v^{k-1}) = \omega_x(X, v^1, \dots, v^{k-1}), \qquad X = x - x_0$$

$$0 \longleftarrow \mathcal{P}_r\Lambda^0 \xleftarrow{\kappa} \mathcal{P}_{r-1}\Lambda^1 \xleftarrow{\kappa} \cdots \xleftarrow{\kappa} \mathcal{P}_{r-n}\Lambda^n \longleftarrow 0$$

C.f., the polynomial de Rham complex

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$$\mathcal{H}_r\Lambda^k = d\mathcal{H}_{r+1}\Lambda^{k-1} \oplus \kappa\mathcal{H}_{r-1}\Lambda^{k+1}$$

Definition of $\mathcal{P}_r^- \Lambda^k$

Using this decomposition, we define the space $\mathcal{P}_r^- \Lambda^k$ contained between $\mathcal{P}_r \Lambda^k$ and $\mathcal{P}_{r-1} \Lambda^k$:

$$\mathcal{P}_r \, \Lambda^k := \mathcal{P}_{r-1} \Lambda^k + \kappa \mathcal{H}_{r-1} \Lambda^{k+1} + d \mathcal{H}_{r+1} \Lambda^{k-1}$$

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Note

$$\mathcal{P}_r^- \Lambda^0 = \mathcal{P}_r \Lambda^0$$

$$\mathcal{P}_r^- \Lambda^n = \mathcal{P}_{r-1} \Lambda^n$$

$\mathcal{P}_{r-1} \Lambda^k \subsetneq \mathcal{P}_r^- \Lambda^k \subsetneq \mathcal{P}_r \Lambda^k$ otherwise

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God made $\mathcal{P}_r \Lambda^k$ and $\mathcal{P}_r^- \Lambda^k$,
all the rest is the work of man.

Degrees of freedom

The other ingredient of a finite element space are the *degrees of freedom*, i.e., a decomposition of the dual spaces $(\mathcal{P}_r\Lambda^k(T))^*$ and $(\mathcal{P}_r^-\Lambda^k(T))^*$, into subspaces associated to subsimplices f of T.

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DOF for $\mathcal{P}_r\Lambda^k(T)$: to a subsimplex f of dim. $d \geq k$ we associate

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The resulting FE spaces have exactly the continuity required by $H\Lambda^k$:

Theorem. $\mathcal{P}_r\Lambda^k(\mathcal{T}) = \{\, \omega \in H\Lambda^k(\Omega) \;:\; \omega|_T \in \mathcal{P}_r\Lambda^k(T) \quad \forall T \in \mathcal{T} \,\}$.

Similarly for $\mathcal{P}_r^-$.

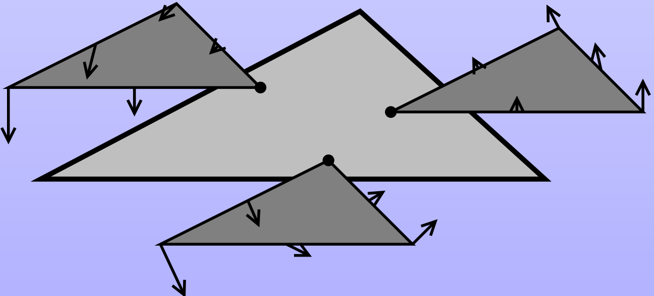
Bases for $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$

As a basis for $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ we may take the dual basis to the degrees of freedom.

For $k = 0$ this is the standard Lagrange basis.

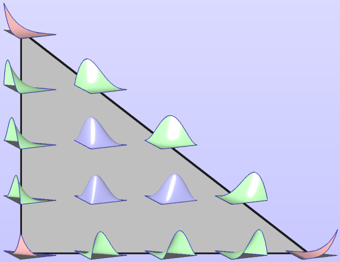
For $\mathcal{P}_1^-\Lambda^k(\mathcal{T})$ there is one basis element for each k-simplex, the Whitney form

$$\phi_{\sigma_0\cdots\sigma_k} := \sum_{i=0}^{k}(-1)^i\lambda_{\sigma_i}d\lambda_{\sigma_0}\wedge\cdots\wedge\widehat{d\lambda_{\sigma_i}}\wedge\cdots\wedge d\lambda_{\sigma_k}$$



Geometric bases

A useful alternative to the Lagrange basis for the Lagrange finite elements is the *Bernstein basis*, given by monomials in the barycentric coords.



$$\mathcal{P}_r(T) = \bigoplus_{f \text{ subsimplex}} \mathcal{P}_r(T,f)$$

$$\mathcal{P}_r(T,f) \xrightarrow[\textit{trace}]{\cong} \mathring{\mathcal{P}}_r(f) \cong \mathcal{P}_{r-\dim f-1}(f)$$

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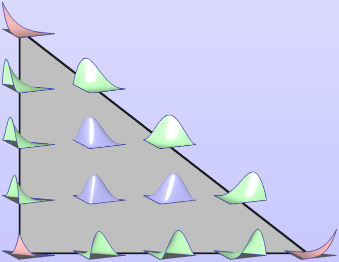

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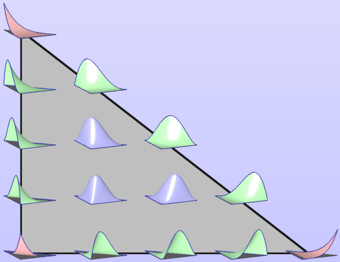

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Finite element de Rham subcomplexes

We don't only want spaces, we also want them to fit together into discrete de Rham complexes.

- One such FEdR subcomplex uses the $\mathcal{P}_r^- \Lambda^k$ spaces of constant degree r:

$$0 \to \mathcal{P}_r^- \Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{P}_r^- \Lambda^1(\mathcal{T}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^- \Lambda^n(\mathcal{T}) \to 0$$

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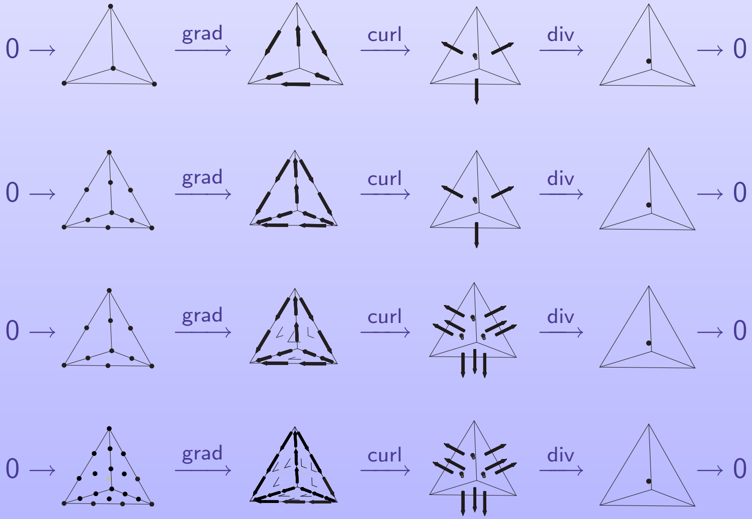
- These are extreme cases. For every r there are 2^{n-1} such FEdR subcomplexes.

The 4 FEdR subcomplexes ending with $\mathcal{P}_0\Lambda^3$ in 3D



Application to Elasticity:

The Holy Grail Attained?

Stress–displacement mixed finite elements for elasticity

The search for such elements dates back to Fraeijs de Veubeke, Pian, Watwood and Hartz, Zienkiewicz, . . . in the 1960's.

> It is, of course, possible to derive elements that exhibit complete continuity of the appropriate components along interfaces and indeed this was achieved by Raviart and Thomas in the case of the heat conduction problem discussed previously. Extension to the full stress problem is difficult and as yet such elements have not been successfully noted.
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Thanks to FEEC, it is time to retire that statement!

Mixed formulation with weak symmetry

Idea goes back to Fraeijs de Veubeke 1975, Amara–Thomas 1979

In the classical Hellinger–Reissner principle, symmetry of the stress tensor (balance of angular momentum) is assumed to hold exactly.

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$$\int_\Omega \left(\frac{1}{2} A\sigma : \sigma + \operatorname{div}\sigma \cdot u + f \cdot u\right) dx \xrightarrow[H(\operatorname{div};\mathbb{S})\times L^2(\mathbb{R}^n)]{\sigma, u} \text{stationary point}$$

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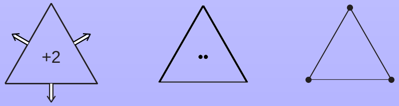
Arnold–Brezzi–Douglas '84: PEERS element



The elasticity complex

There is a complex for elasticity analogous to the de Rham complex. It has versions both for strong symmetry and weak symmetry.

$$\begin{array}{ccccc} \text{displacement} & \text{rotation} & & \text{strain} & \\ \downarrow & \downarrow & & \downarrow & \\ 0 \to H^1(\Omega;\mathbb{R}^3) \times & L^2(\Omega,\mathbb{K}) & \xrightarrow{(\text{grad},-I)} & H(J,\Omega;\mathbb{M}) & \xrightarrow{J} \end{array}$$

$$\begin{array}{ccccc} \xrightarrow{J} & H(\text{div},\Omega;\mathbb{M}) & \xrightarrow{\begin{pmatrix}\text{div}\\ \text{skw}\end{pmatrix}} & L^2(\Omega;\mathbb{R}^3) \times & L^2(\Omega;\mathbb{K}) \to 0 \\ & \uparrow & & \uparrow & \uparrow \\ & \text{stress} & & \text{load} & \text{couple} \end{array}$$

J is second order!

New mixed finite elements for elasticity

The elasticity complex can be derived from the de Rham complex by an intricate construction. Mimicking this construction on the discrete level we have derived stable mixed finite elements for elasticity. (Arnold-Falk-Winther 2006, 2007).

Main result

Choose *two* discretizations of the de Rham complex:

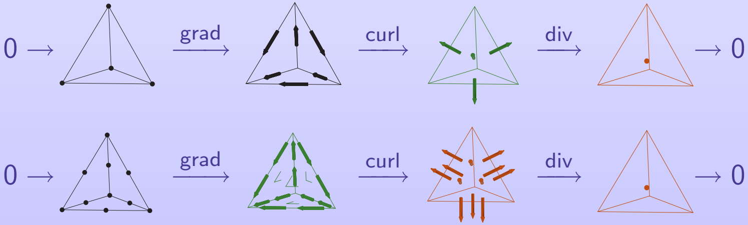
$$0 \longrightarrow \Lambda_h^0 \xrightarrow{\text{grad}} \Lambda_h^1 \xrightarrow{\text{curl}} \Lambda_h^2 \xrightarrow{\text{div}} \Lambda_h^3 \longrightarrow 0$$

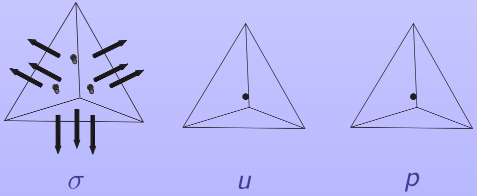
$$0 \longrightarrow \tilde{\Lambda}_h^0 \xrightarrow{\text{grad}} \tilde{\Lambda}_h^1 \xrightarrow{\text{curl}} \tilde{\Lambda}_h^2 \xrightarrow{\text{div}} \tilde{\Lambda}_h^3 \longrightarrow 0$$

Surjectivity Hypothesis: (roughly) for each DOF of Λ_h^2 there is a corresponding DOF of $\tilde{\Lambda}_h^1$.

$$\text{Then} \quad \left\{ \begin{array}{ll} \text{stress:} & \tilde{\Lambda}_h^2(\mathbb{R}^3) \\ \text{displacement:} & \tilde{\Lambda}_h^3(\mathbb{R}^3) \\ \text{rotation:} & \Lambda_h^3(\mathbb{K}) \end{array} \right\} \text{ is a stable element choice.}$$

The simplest choice





Features of the new mixed elements

- Based on HR formulation with weak symmetry; very natural
- Lowest degree element is very simple: full $\mathcal{P}_1$ for stress, $\mathcal{P}_0$ for displacement and rotation
- Works for every polynomial degree
- Works the same in 2 and 3 (or more) dimensions
- Robust to material constraints like incompressibility
- Provably stable and convergent

Conclusions

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- The $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$ spaces are the natural finite element discretizations for differential forms and the de Rham complex.
- Through FEEC we believe we have completed the long search for "just the right" mixed finite elements for elasticity.

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II. The elasticity complex
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