Name: ______________________

Problem Set 4
Math 4281, Spring 2014
Due: Wednesday, February 19

Equivalence relations

1. Define a relation on $\mathbb{R}$ as follows: $x \sim y$ if and only if $x - y$ is an integer. Prove that $\sim$ is an equivalence relation and describe the set of equivalence classes.

2. Given a function $f\colon S \to T$, consider the following relation on S: $x \sim y \Leftrightarrow f(x) = f(y)$.

   a. Prove that $\sim$ is an equivalence relation.

   b. Prove that if f maps onto T, then there is a one-to-one correspondence between the set of equivalence classes and T.

Rings, domains, and fields

3. Let $a, m \in \mathbb{Z}$ with $m > 0$.

   a. Prove that $\gcd(a, m) = 1$ if and only if $[a] = \overline{a} \in \mathbb{Z}_m$ is a unit.

   b. Prove that if $[a] = \overline{a} \in \mathbb{Z}_m$ is a zero divisor, then $\gcd(a, m) > 1$, and conversely, provided $m \nmid a$.

4. Determine if the set $R = \{a + b\sqrt[3]{3} \mid a, b \in \mathbb{Q}\}$ is a ring with respect to the usual operations of addition and multiplication. If so, is it also a field?

5. Characterize the units in $M_2(\mathbb{Z})$. Then list the units in $M_2(\mathbb{Z}_2)$.

6. Show that $R = \{a + b\sqrt{3}i \mid a, b \in \mathbb{Z}\}$ is a ring.

The complex numbers

7. Prove the following properties of the modulus of a complex number. Let $z, w \in \mathbb{C}$.

   a. $|zw| = |z||w|$

   b. $|\overline{z}| = |z|$

   c. $|z|^2 = z\overline{z}$

Throughout the course of this assignment, I have followed the guidelines of the University of Minnesota Student Conduct Code.

Signed: ______________________