Math 5385 - Spring 2018
Problem Set 1

Submit solutions to **four** of the following problems.

1. This exercise from the textbook: [IVA] §1.1 #2 (p. 5)

2. Let $p \in \mathbb{N}$ be a prime and let $\mathbb{F}_p$ be a finite field with p elements. Prove that $x^p - x$ is a nonzero polynomial in $\mathbb{F}_p[x]$ which vanishes at every point of $\mathbb{F}_p$.

3. Use MathSciNet (through a UMN proxy), the arXiv, and MathOverflow to answer the following questions:
   (a) Estimate the number of journal articles published with the words "Gröbner basis" in their title.
   (b) How many algebraic geometry preprints were added to the e-print archives in November 2017?
   (c) Estimate the number of research level math questions tagged with `ag.algebraic-geometry`.

4. (a) Show that the polynomial
$$\binom{x}{d} := \frac{x(x-1)\cdots(x-d+1)}{d!}$$
   takes integer values for all integers x.
   (b) Show that every polynomial of degree d which takes integer values for all integers can be written as a unique integer linear combination of $\binom{x}{d}, \binom{x}{d-1}, \ldots, \binom{x}{0}$.

5. Given a sequence $a_n, a_n - 1, \ldots, a_m$ of real numbers with $n > m$ and $a_n \neq 0$, the *number of sign changes* is defined as follows: count one sign change if $a_i a_k < 0$ with $k = i - 1$ or $k < i - 1$ and $a_j = 0$ for every j satisfying $k < j < i$.
   (a) Prove Descarte's rule of signs: If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_m x^m \in \mathbb{R}[x]$ with $a_n a_m \neq 0$, then the number of positive roots of f is less than or equal to the number of sign changes in the sequence of coefficients $a_n, a_n - 1, \ldots, a_m$.
   Hint. Proceed by induction, consider the derivative, and use Rolle's Theorem.
   (b) Verify that $x^{11} + x^8 - 3x^5 + x^4 + x^3 - 2x^2 + x - 2$ has at most 5 positive and 2 negative roots. Deduce that it has at least 4 nonreal roots.