Math 5385 - Spring 2018
Problem Set 13

Submit solutions to **three** of the following problems.

1. Consider the homogeneous ideal $J = \langle x_0^2 x_1, x_1^3, x_1 x_2 \rangle$ in $\mathbb{C}[x_0, x_1, x_2]$.
   (a) Let I_i for $0 \leq i \leq 2$ be the dehomogenization of J with respect to x_i. Compute each I_i.
   (b) Let J_i be the homogenization of I_i with respect to x_i. Compute each J_i.
   (c) Compute the intersection $J' := J_0 \cap J_1 \cap J_2$. Show that J is a proper subset of J'.
   (d) Show that $(J : \langle x_0, x_1, x_2 \rangle^\infty) \neq J$.

2. When we have a curve parameterized by $t \in \mathbb{A}^1$, there are many situations when we want to let $t \to \infty$. Since $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$, this suggests that we should let our parameter space be $\mathbb{P}^1$. Here are two examples of how this works.
   (a) Consider the parametrization $(x, y) = \left(\frac{1+t^2}{1-t^2}, \frac{2t}{1-t^2} \right)$ of $x^2 - y^2 = 1$ in $\mathbb{A}^2(\mathbb{R})$. To make this projective, we first work in $\mathbb{P}^2$. Identifying $\mathbb{A}^2$ with $U_3 \subset \mathbb{P}^2$, we have
   $$\left(\frac{1+t^2}{1-t^2}, \frac{2t}{1-t^2} \right) = \left[\frac{1+t^2}{1-t^2} : \frac{2t}{1-t^2} : 1 \right] = [1+t^2 : 2t : 1-t^2].$$
   The next step is to make t projective. Given $[a : b] \in \mathbb{P}^1$, we can write it as $[1 : t] = [1 : b/a]$, provided $a \neq 0$. Now substitute $t = b/a$ into the right side and clear denominators. Explain why this gives a well-defined map $\mathbb{P}^1 \to \mathbb{P}^2$.
   (b) The twisted cubic in $\mathbb{A}^3$ is parametrized by (t, t^2, t^3). Apply the method of part (a) to obtain a projective parametrization of $\mathbb{P}^1 \to \mathbb{P}^3$ and show that the image of this map is precisely $X = V(x_2^2 - x_1 x_3, x_1 x_2 - x_0 x_3, x_1^2 - x_0 x_2)$.

3. Consider the ideal $J = \langle x_0 y_0 + x_1 y_1, x_0 y_1 - x_1 y_0 \rangle \subset \mathbb{C}[x_0, x_1, y_0, y_1]$.
   (a) Compute the intersection $J \cap \mathbb{C}[y_0, y_1]$.
   (b) Compute the image $\pi_2(V(J))$, where $V(J) \subset \mathbb{P}^1 \times \mathbb{A}^2$ and $\pi_2 \colon \mathbb{P}^1 \times \mathbb{A}^2 \to \mathbb{A}^2$.
   (c) Since J is bihomogeneous, it determines $X = V(J) \subset \mathbb{P}^1 \times \mathbb{P}^1$. Describe X.

4. Sketch the following curves in $\mathbb{A}^2(\mathbb{R})$ and analyze the multiplicity of all lines meeting these curves at the origin.
   (a) $x^2 = x^4 + y^4$
   (b) $xy = x^6 + y^6$
   (c) $x^3 = y^2 + x^4 + y^4$
   (d) $x^2 y + xy^2 = x^4 + y^4$