Math 5385 - Spring 2018
Problem Set 2

Submit solutions to **four** of the following problems.

1. (a) Show that $X = \{(x,x) \mid x \in \mathbb{R}, x \neq 1\} \subset \mathbb{R}^2$ is not an affine variety.
   Hint. If $f \in \mathbb{R}[x,y]$ vanishes on X, then prove that $f(1,1) = 0$. Consider $g(t) := f(t,t)$.
   (b) Show that $Y = \{(x,y) \in \mathbb{R}^2 \mid y > 0\} \subset \mathbb{R}^2$ is not an affine variety.

2. Consider the set $U(1) := \{z \in \mathbb{C} \mid z\bar{z} = 1\}$.
   (a) If we identify $\mathbb{C}$ with $\mathbb{R}^2$, then show that $U(1)$ is an affine subvariety of $\mathbb{R}^2$.
   (b) Prove that $U(1)$ is not an affine subvariety of $\mathbb{C}^1$.

3. Consider the map $\sigma\colon \mathbb{A}^3(\Bbbk) \to \mathbb{A}^6(\Bbbk)$ defined by $(x,y,z) \mapsto (x^2, xy, xz, y^2, yz, z^2)$. Let a,b,c,d,e,f denote the corresponding coordinates on $\mathbb{A}^6(\Bbbk)$.
   (a) Show that the image of σ satisfies the equations given by the 2-minors of the symmetric matrix
   $$\Omega = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}.$$
   (b) Compute the dimension of the vector space V in $S = \Bbbk[a,b,c,d,e,f]$ spanned by these 2-minors.
   (c) Show that every homogeneous polynomial of degree 2 in S vanishing on the image of σ is contained in V.

4. Consider the curve, called a *strophoid*, with the trigonometric parametrization given by
   $$x = a\sin(t) \qquad y = a\tan(t)\,(1+\sin(t)),$$
   where a is a constant.
   (a) Find the implicit equation in x and y that describes the strophoid.
   (b) Find a rational parametrization of the strophoid.

5. (a) Prove the equality of the ideals $\langle x+xy, y+xy, x^2, y^2\rangle = \langle x, y\rangle$.
   (b) Prove that $V(x+xy, y+xy, x^2, y^2) = V(x,y)$.

6. An ideal $I \subseteq k[x_1,\ldots,x_n]$ is said to be *radical* if for any $f \in k[x_1,\ldots,x_n]$, whenever $f^m \in I$, then also $f \in I$.
   (a) Prove that for an affine variety $V \subseteq k^n$, $I(V)$ is always a radical ideal.
   (b) Prove that $\langle x^2, y^2\rangle$ is not a radical ideal. This implies that $\langle x^2, y^2\rangle \neq I(V)$ for any variety $V \subseteq k^2$.