Math 5385 - Spring 2018
Problem Set 7

Submit solutions to **three** of the following problems.

1. If $f = a_\ell x^\ell + \cdots + a_0 \in \Bbbk[x]$, where $a_\ell \neq 0$ and $\ell > 0$, then the *discriminant* of f is defined to be
$$\operatorname{disc}(f) = \frac{(-1)^{\ell(\ell-1)/2}}{a_\ell} \operatorname{Res}(f, f'; x).$$
   (a) Prove that f has a multiple factor if and only if $\operatorname{disc}(f) = 0$.
   (b) Does $6x^4 - 23x^3 + 32x^2 - 19x + 4$ have a multiple root in $\mathbb{C}$?
   (c) Compute the discriminant of the quadratic polynomial $f = ax^2 + bx + c$. Explain how your answer relates to the quadratic formula.

2. In $\mathbb{Q}[x, y]$, consider $f = x^2y - 3xy^2 + x^2 - 3xy$ and $g = x^3y + x^3 - 4y^2 - 3y + 1$.
   (a) Compute $\operatorname{Res}(f, g; x)$.
   (b) Compute $\operatorname{Res}(f, g; y)$.
   (c) What does the result in part (b) imply about f and g?

3. Consider $f, g \in \mathbb{Q}[x, y]$ and let $J := \langle f, g\rangle \cap \mathbb{Q}[y]$.
   (a) If $f = xy - 1$ and $g = x^2 + y^2 - 4$, then prove that $\operatorname{Res}(f, g; x)$ generates J.
   (b) If $f = xy - 1$ and $g = yx^2 + y^2 - 4$, then prove that $\operatorname{Res}(f, g; x)$ does not generate the ideal J.

4. Let $f = x^2y + x - 1$ and $g = x^2y + x + y^2 - 4$. If $h := \operatorname{Res}(f, g; x) \in \mathbb{C}[y]$, then show that $h(0) = 0$. However, if we substitute $y = 0$ into f and g, we get $x - 1$ and $x - 4$, respectively. Show that these polynomials have a nonzero resultant. In particular, we conclude that $h(0)$ is not a resultant.

5. Suppose that $f, g \in \mathbb{C}[x]$ are monic polynomials of positive degree.
   (a) Show that $\gamma \in \mathbb{C}$ is a root of $\operatorname{Res}(f(x), g(y-x); x)$ if and only if we have $\gamma = \alpha + \beta$, where $f(\alpha) = 0 = g(\beta)$.
   (b) Show that $\gamma \in \mathbb{C}$ is a root of $\operatorname{Res}(f(x), g(y/x) \cdot x^{\deg(g)}; x)$ if and only if we have $\gamma = \alpha \cdot \beta$, where $f(\alpha) = 0 = g(\beta)$.