Math 5385 - Spring 2018
Problem Set 8

Submit solutions to **three** of the following problems.

1. The purpose of this exercise is to show that, if $\mathbb{k}$ is any field which is not algebraically closed, then any affine variety $X \subseteq \mathbb{A}^n(\mathbb{k})$ can be defined by a single equation.
   (a) For a polynomial $f := a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ of degree m in x, define the *homogenization* to be $f^h := a_m x^m + a_{m-1}x^{m-1}y + \cdots + a_1 x y^{m-1} + a_0 y^m$. Show that f has a root in k if and only if there is $(p, q) \in \mathbb{A}^2(\mathbb{k})$ such that $(p, q) \neq (0, 0)$ and $f^h(p, q) = 0$.
   (b) If $\mathbb{k}$ is not algebraically closed, show that there exists $h \in \mathbb{k}[x, y]$ such that the variety defined by $h = 0$ consists of just the origin.
   (c) If $\mathbb{k}$ is not algebraically closed, show that for each integer $n > 0$ there exists an element $f \in \mathbb{k}[x_1, \ldots, x_n]$ such that the only solution of $f = 0$ is the origin.
   (d) If $X = V(g_1, \ldots, g_r)$ is any affine variety in $\mathbb{A}^n(\mathbb{k})$ where $\mathbb{k}$ is not algebraically closed, then show that X can be defined by a single equation.

2. Solve this problem WITHOUT the use of a computer algebra system.
   (a) Find the minimal Gröbner basis for
   $$\sqrt{\langle x^5 - 2x^4 + 2x^2 - x, x^5 - x^4 - 2x^3 + 2x^2 + x - 1\rangle} \subseteq \mathbb{Q}[x].$$
   (b) Let $J = \langle xy, (x - y)x\rangle$. Describe $V(J)$ and show that $\sqrt{J} = \langle x\rangle$.

3. Solve this problem WITHOUT the use of a computer algebra system.
   Determine whether the following polynomials lie in the given radical ideals. What is the smallest power of the polynomial that lies in the ideal?
   (a) Is $x + y$ in $\sqrt{\langle x^3, y^2, xy(x + y)\rangle}$?
   (b) Is $x^2 + 3xz$ in $\sqrt{\langle x + z, x^2y, x - z^2\rangle}$?

4. If I is an ideal in $S := \mathbb{k}[x_1, \ldots, x_n]$ and $f \in S$, then the *saturation* of I with respect to f is the set
   $$(I : f^\infty) := \{g \in S \mid f^m g \in I \text{ for some } m > 0\}.$$
   (a) Prove that $(I : f^\infty)$ is an ideal.
   (b) Prove that there is an ascending chain of ideals $(I : f) \subseteq (I : f^2) \subseteq (I : f^3) \subseteq \cdots$.
   (c) Prove that $(I : f^\infty) = (I : f^m)$ if and only if $(I : f^m) = (I : f^{m+1})$.

5. A subset $U \subseteq S := \mathbb{k}[x_1, \ldots, x_n]$ is multiplicatively closed if any product of elements of U is also in U (including the empty product 1).
   (a) Let U be a multiplicatively closed subset of S. If I is an ideal in S maximal with respect to inclusion among all ideals not meeting U, then show that I is prime.
   (b) Let J be any proper ideal in S. Show that the radical ideal J is the intersection of all prime ideals containing J.