

Name: _____

This is an open book/library/notes/web take-home exam, but you are not to collaborate. Your instructor is the only human source you are allowed to consult. Be sure to cite all outside sources you use.

Please submit your solution to each problem on a separate page, and staple this sheet on top of your exam.

This exam is due at 8:40 a.m. in class on November 14, 2018.

Problem	1	2	3	4	Total
Points	24	30	18	28	100
Score					

1. (a) (12 points) What is the order of the group $\mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})$ of 2×2 invertible matrices with entries in $\mathbb{Z}/3\mathbb{Z}$?
- (b) (12 points) What is the order of the group $\mathrm{SL}_3(\mathbb{Z}/3\mathbb{Z})$ of 3×3 matrices with determinant 1 and entries in $\mathbb{Z}/3\mathbb{Z}$?
2. (a) (12 points) A free group $F(x_1, x_2, x_3, \dots)$ of countably infinite rank is not finitely generated.
- (b) (18 points) Prove that the commutator subgroup of the free group $F := F(a, b)$ on 2 generators is not finitely generated.
3. (18 points) Let G be a group of order 56 with a nonnormal Sylow 7-subgroup. Prove that the Sylow 2-subgroup S of G is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
(Hint: Let an element of order 7 act by conjugation on the seven non-identity elements of S and deduce that they all have the same order.)
4. Let V be a finite dimensional \mathbb{C} -vector space, and $\phi, \psi \in \mathrm{End}_{\mathbb{C}}(V)$ with $\phi\psi = \psi\phi$.
 - (a) (10 points) Show that if λ is an eigenvalue of ϕ , then the eigenspace
$$V_\lambda := \{x \in V \mid \phi(x) = \lambda x\}$$
is invariant under ψ .
 - (b) (18 points) Show that ϕ and ψ have at least one common eigenvector (not necessarily belonging to the same eigenvalue).
(Hint: Let V_λ be the λ -eigenspace of ϕ , and consider the linear operator $\psi|_{V_\lambda}$.)