Name: ________________________

This is an open book/library/notes/web take-home exam, but you are not to collaborate. Your instructor is the only human source you are allowed to consult. Be sure to cite all outside sources you use.

Please submit your solution to each problem on a **separate page**.

This exam is due by **email** (cberkesc@umn.edu) at **10:00 a.m. on Monday, December 17, 2018**. Your solutions do not need to be typed. You are permitted to send scans of handwritten work.

Problem	1	2	3	4	5	6	Total
Points	15	15	15	15	20	20	100
Score							

1. (15 points) Let a finite group G act on a finite set S, with $|S| \geq 2$. Suppose that G acts transitively on S. Show that there is an element $g \in G$ which does not have a fixed point on S.

2. (15 points) Let F be the free group on the set X, and let $Y \subset X$. Prove that if H is the smallest normal subgroup containing Y, then F/H is a free group.

3. (15 points) Let $p \neq q$ be odd primes. Show that any group G of order $2pq$ is solvable.

4. (15 points) Show that any two commuting operators $\phi, \psi \colon V \to V$ on a finite-dimensional $\mathbb{C}$-vector space V can be simultaneously triangularized, that is, there exists a basis for V in which the matrices that represent ϕ, ψ are both upper triangular.

5. (20 points) Let F be a field and V an F-vector space with $\dim_F V = n$. Given a linear operator $V \to V$, let c_k be the coefficient of t^k in its characteristic polynomial:

$$\det(t \cdot 1_V - \phi) = c_0 + c_1 t^1 + c_2 t^2 + \cdots + c_{n-1} t^{n-1} + c_n t^n.$$

   (a) Prove that $c_n = 1$, $c_{n-1} = -\mathrm{tr}(\phi)$, and $c_0 = (-1)^n \det(\phi)$.

   (b) Prove more generally that $c_{n-k} = (-1)^k \mathrm{tr}(\bigwedge^k \phi)$, where $\bigwedge^k \phi$ is defined by sending $v_1 \wedge \cdots \wedge v_k \mapsto \phi(v_1) \wedge \cdots \wedge \phi(v_k)$.
   (If you do part (b) correctly, there is no need to do part (a) separately.)

6. (20 points) Let R be a commutative ring with $1 \neq 0$, and consider the polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in R[x],$$

   Assume there exists a nonzero polynomial $q(x) = b_m x^m + \cdots + b_0 \in R[x]$ such that $q(x)p(x) = 0$. Prove or disprove that there exists a nonzero element $b \in R$ such that $bp(x) = 0$.