Constructions of virtual resolutions

Christine Berkesch

Let X be a smooth toric variety over a field $\Bbbk$ with $\mathrm{Pic}(X)$-graded Cox ring S and irrelevant ideal B. For example, when X is a product of projective spaces $\mathbb{P}^{\boldsymbol{n}} := \mathbb{P}^{\boldsymbol{n}_1} \times \mathbb{P}^{\boldsymbol{n}_2} \times \cdots \times \mathbb{P}^{\boldsymbol{n}_r}$, then its Cox ring is $\Bbbk[x_{i,j} \mid 1 \leq i \leq r, 0 \leq j \leq n_i]$ and its irrelevant ideal is $\bigcap_{i=1}^{r} \langle x_{i,0}, x_{i,1}, \ldots, x_{i,n_i} \rangle$; in this case, we identify $\mathbb{Z}^r \cong \mathrm{Pic}(\mathbb{P}^{\boldsymbol{n}})$ via the standard basis and write $\boldsymbol{n} := (\boldsymbol{n}_1, \ldots, \boldsymbol{n}_r) \in \mathbb{Z}^r$, so $\mathrm{Cox}(\mathbb{P}^{\boldsymbol{n}})$ has a natural $\mathbb{Z}^r$-multigrading.

Definition 1. *A free complex $F := [F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow \cdots]$ of $\mathrm{Pic}(X)$-graded S-modules is called a* virtual resolution *of a $\mathrm{Pic}(X)$-graded S-module M if the corresponding complex $\widetilde{F}$ of vector bundles on X is a locally free resolution of the sheaf $\widetilde{M}$.*

In other words, a virtual resolution is a free complex of S-modules whose higher homology groups are supported on the irrelevant ideal of X.

Our results in [2] for products of projective spaces provide strong evidence that, when seeking to understand geometric properties through a homological lens, virtual resolutions for smooth toric varieties provide are the proper analogue of minimal free resolutions for projective space. Here we survey a number of constructions of virtual resolutions.

Resolution of the diagonal. The first construction arises in the proof of a Hilbert Syzygy type theorem for virtual resolutions over a product of projective spaces.

Theorem 2. [2] *If M is a B-saturated finitely generated graded module over $\mathrm{Cox}(\mathbb{P}^{\boldsymbol{n}})$, then there is a virtual resolution of M of length at most $\dim(\mathbb{P}^{\boldsymbol{n}}) = |\boldsymbol{n}|$.*

This proof is based on a minor variation of Beilinson's resolution of the diagonal, which uses $\Omega^{\boldsymbol{a}}_{\mathbb{P}^{\boldsymbol{n}}} := \Omega^{a_1}_{\mathbb{P}^{n_1}} \boxtimes \Omega^{a_2}_{\mathbb{P}^{n_2}} \boxtimes \cdots \boxtimes \Omega^{a_r}_{\mathbb{P}^{n_r}}$, the external tensor product of the exterior powers of the cotangent bundles on the factors of $\mathbb{P}^{\boldsymbol{n}}$. Choose $\boldsymbol{d} \in \mathbb{Z}^r$ such that for any $\boldsymbol{u} \in \mathbb{Z}^r$, $\Omega^{\boldsymbol{u}}_{\mathbb{P}^{\boldsymbol{n}}}(\boldsymbol{u} + \boldsymbol{d}) \otimes \widetilde{M}$ has no higher cohomology; such a $\boldsymbol{d}$ exists by the Fujita Vanishing Theorem [4, Theorem 1]. Then our proof produces a virtual resolution of M, in which the i-th module is

$$\bigoplus_{\substack{\boldsymbol{0} \leq \boldsymbol{u} \leq \boldsymbol{n} \\ |\boldsymbol{u}| = i}} S(-\boldsymbol{u}) \otimes_{\Bbbk} H^q\big(\mathbb{P}^{\boldsymbol{n}}, \Omega^{\boldsymbol{u}}_{\mathbb{P}^{\boldsymbol{n}}} \otimes \widetilde{M}(\boldsymbol{u} + \boldsymbol{d})\big).$$

While such a virtual resolution has the benefit of being somewhat short in homological length, it also has a large number of multidegrees appearing in each homological degree.

Monomial ideals. For this construction, we return to working over an arbitrary smooth complete toric variety X, and again observe that there is a Hilbert Syzygy type result in the monomial ideal case.

Theorem 3. [9] *If $I \neq S$ is a B-saturated monomial ideal, then there is a virtual resolution of S/I of length at most* $\dim(X)$.

In fact, the construction in the proof of this theorem produces a monomial ideal J in S with $I = J : B^\infty$, so that the projective dimension of S/J is at most $\dim(X)$. As such, the free resolution of S/J is a virtual resolution of S/I with the desired short length.

Returning to products of projective spaces, a recent result provides a case in which a squarefree monomial ideal I is "virtually Cohen–Macaulay."

Theorem 4. [5] *Let Δ be a simplicial complex that is pure and balanced, meaning that each of the vertices of every facet correspond to a different projective space in $\mathbb{P}^{\boldsymbol{n}}$. If I_Δ is a Stanley–Riesner squarefree monomial ideal in $S = \mathrm{Cox}(\mathbb{P}^{\boldsymbol{n}})$, then there is a virtual resolution of S/I_Δ of length equal to the codimension of S/I_Δ.*

As in the previous result, the virtual resolution constructed in this proof is again a free resolution. This time, it resolves another squarefree monomial ideal $I_{\Delta'}$, where $\Delta' \supseteq \Delta$ is a pure, balanced simplicial complex that is shown to be shellable, and hence Cohen–Macaulay.

Virtual resolution of a pair. The definition of multigraded regularity, as defined in [7], provides an invariant that detects several virtual resolutions that are subcomplexes of a free resolution.

Theorem 5. [2] *Let $X = \mathbb{P}^{\boldsymbol{n}}$, fix a degree $\boldsymbol{d} \in \mathbb{Z}^r$, and let M be a B-saturated graded finitely generated S-module that is $\boldsymbol{d}$-regular. If $C(M, \boldsymbol{d}+\boldsymbol{n})$ is the subcomplex of a minimal free resolution of M consisting of all free summands of degree at most $\boldsymbol{d}+\boldsymbol{n}$, then $C(M, \boldsymbol{d}+\boldsymbol{n})$ is a virtual resolution for M.*

The complex $C(M, \boldsymbol{d}+\boldsymbol{n})$ is called the *virtual resolution of the pair* $(M, \boldsymbol{d}+\boldsymbol{n})$. Note that it can be computed without first computing an entire minimal free resolution of M; simply dispose of generators not within the degree bound at each homological degree.

Mapping cone construction. Again working over an arbitrary smooth complete toric variety X, let M be a finitely generated graded S-module, and suppose that $F\colon F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} \cdots \xleftarrow{\varphi_t} F_t \leftarrow 0$ is a virtual resolution of M and $\mathrm{Ext}^t(M,S)^\sim = 0$. If G^* is a free resolution of φ_t^*, that is shifted with indexing reversed, as in the diagram below, then there is an induced map α^* to G^* from $\mathrm{Hom}_S(F,S)$, i.e., from the projective to the acyclic complex.

$$
\begin{array}{ccccccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & F_0^* & \xrightarrow{\varphi_1^*} & F_1^* & \xrightarrow{\varphi_2^*} & \cdots & \xrightarrow{\varphi_{t-2}^*} & F_{t-2}^* & \xrightarrow{\varphi_{t-1}^*} & F_{t-1}^* & \xrightarrow{\varphi_t^*} & F_t^* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_0^*} & & \downarrow{\scriptstyle \alpha_1^*} & & \downarrow{\scriptstyle \alpha_2^*} & & & & \downarrow{\scriptstyle \alpha_{t-2}^*} & & \downarrow{\scriptstyle \alpha_{t-1}^*} & & \downarrow{\scriptstyle \alpha_t^*} & & \\
\cdots & \longrightarrow & G_{-1}^* & \xrightarrow[\psi_0^*]{} & G_0^* & \xrightarrow[\psi_1^*]{} & G_1^* & \xrightarrow[\psi_2^*]{} & \cdots & \xrightarrow[\psi_{t-2}^*]{} & G_{t-2}^* & \xrightarrow[\psi_{t-1}^*]{} & G_{t-1}^* & \xrightarrow[\psi_t^*=\varphi_t^*]{} & G_t^* & \longrightarrow & 0
\end{array}
$$

Dualize this diagram to get:

$$\begin{array}{cccccccccccccccc}
\cdots & \longleftarrow & 0 & \longleftarrow & F_0 & \xleftarrow{\varphi_1} & F_1 & \xleftarrow{\varphi_2} & \cdots & \xleftarrow{\varphi_{t-2}} & F_{t-2} & \xleftarrow{\varphi_{t-1}} & F_{t-1} & \xleftarrow{\varphi_t} & F_t & \longleftarrow 0 \\
 & & \uparrow{\scriptstyle \alpha_{-1}} & & \uparrow{\scriptstyle \alpha_0} & & \uparrow{\scriptstyle \alpha_1} & & & & \uparrow{\scriptstyle \alpha_{t-2}} & & \uparrow{\scriptstyle \alpha_{t-1}} & & \uparrow{\scriptstyle \alpha_t} & \\
\cdots & \xleftarrow[\psi_{-1}]{} & G_{-1} & \xleftarrow[\psi_0]{} & G_0 & \xleftarrow[\psi_1]{} & G_1 & \xleftarrow[\psi_2]{} & \cdots & \xleftarrow[\psi_{t-2}]{} & G_{t-2} & \xleftarrow[\psi_{t-1}]{} & G_{t-1} & \xleftarrow[\psi_t=\varphi_t]{} & G_t & \longleftarrow 0
\end{array}$$

When $G_{-2} = 0$, then the mapping cone cone(α) can be partially minimized to the following virtual resolution of M, which is shorter than F:

$$\begin{matrix} F_0 \\ \oplus \\ G_{-1} \end{matrix} \xleftarrow{\partial_1} \begin{matrix} F_1 \\ \oplus \\ G_0 \end{matrix} \xleftarrow{\partial_2} \begin{matrix} F_2 \\ \oplus \\ G_1 \end{matrix} \xleftarrow{\partial_3} \cdots \leftarrow \begin{matrix} F_{t-2} \\ \oplus \\ G_{t-3} \end{matrix} \leftarrow G_{t-2} \leftarrow 0.$$

What makes a complex virtual. We conclude with a result that generalizes to virtual resolutions the well-known exactness criterion of Buchsbaum and Eisenbud.

Theorem 6. [6] *A graded free chain complex* $F \colon F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} \cdots \xleftarrow{\varphi_t} F_t \leftarrow 0$ *over the Cox ring of a smooth complete toric variety is a virtual resolution if and only if both of the following conditions are satisfied:*

(a) $\operatorname{rank}(\varphi_i) + \operatorname{rank}(\varphi_{i+1}) = \operatorname{rank}(F_i)$ *for each* $i = 1, 2, \ldots, t$, *and*
(b) $\operatorname{depth}(I(\varphi_i) : B^\infty) \geq i$.

A number of theorems here have been implemented in the `VirtualResolutions` pacakage of Macaulay2 [8]. See [1] for more details.

References

[1] Ayah Almousa, Juliette Bruce, Michael C. Loper, and Mahrud Sayrafi, *The Virtual Resolutions package for Macaulay2*, 7 pages. arXiv:math.AG/1905.07022.

[2] Christine Berkesch, Daniel Erman, and Gregory G. Smith, *Virtual resolutions for a product of projective spaces*, to appear in *Alg. Geom.*, 22 pages. arXiv:math.AC/1703.07631

[3] David Buchsbaum and David Eisenbud, *What makes a complex exact?* J. Algebra **25** (1973), 259–268.

[4] Takao Fujita, *Vanishing theorems for semipositive line bundles*, Lecture Notes in Math. **1016**, 1983, 519–528.

[5] Nathan Kenshur, Feiyang Lin, Sean McNally, Zixuan Xu, and Teresa Yu, *On virtually Cohen–Macaulay simplicial complexes*, 9 pages.

[6] Michael C. Loper, *What Makes a Complex Virtual*, 11 pages. arXiv:math.AC/1904.05994.

[7] Diane Maclagan and Gregory G. Smith, *Multigraded Castelnuovo-Mumford regularity*, J. Reine Angew. Math. **571** (2004), 179–212.

[8] Daniel R. Grayson and Michael E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, Available at `http://www.math.uiuc.edu/Macaulay2/`.

[9] Jay Yang, *Virtual resolutions of monomial ideals on toric varieties*, 12 pages. arXiv:math.AG/1906.00508