On The Scientific Work of Konstantin Ilyich Oskolkov

D. Bilyk, L. De Carli, A. Petukhov, A. Stokolos and B. D. Wick

Abstract This paper is a brief account of the life and the scientific work of K.I. Oskolkov.

Konstantin Ilyich Oskolkov, or Kostya for his friends and colleagues, was born in Moscow on Feb 17th, 1946. Kostya's father, Ilya Nikolayevich, worked as an engineer at the Research Institute of Cinema and Photography. His mother, Maria Konstantinovna, was a distinguished pediatric cardiology surgeon. Since Maria's father was a priest, during Stalin's purges her parents had to hide away, and for a long time she grew up without them and was forced to hide her background. Kostya's paternal grandfather, Nikolay Innokent'evich Oskolkov, was a famous engineer who built bridges, dams and subways across all of Russia and USSR. At the age of 25, he directed the reconstruction of the famous Borodinsky bridge in Moscow, giving the bridge the look that it still has today. Nikolay Innkont'evich's wife, Anna Vladimirovna Speer, came from the lineage of Karl von Knorre, a famous astronomer, a student of V.Ya. Struve, the founder and director of the Nikolaev branch of the Pulkovo observatory.

The early 70's were a time of scientific bloom in the USSR. Physicists, engineers, and mathematicians were honored members of the society – newspaper articles, movies and TV shows were created about them. It was during this time that Kostya's academic career began. In 1969, Kostya graduated with distinction

Dmitriy Bilyk
University of South Carolina, Columbia, SC 29208, e-mail: bilyk@math.sc.edu

Laura De Carli
Florida International University, Miami, FL 33199, e-mail: decarlil@fiu.edu

Alexander M. Stokolos
Georgia Southern University, Statesboro, GA 30460, e-mail: astokolos@georgiasouthern.edu

Alexander Petukhov
University of Georgia, Athens, GA 30602, e-mail: petukhov@math.uga.edu

Brett D. Wick
Georgia Institute of Technology, Atlanta, GA 30308-0332, e-mail: wick@math.gatech.edu

from the Moscow Institute of Physics and Technology, one of the leading institutions of Soviet higher education specializing in science and technology, with a major in applied mathematics. One of Kostya's professors was Sergey Alexandrovich Telyakovskii, who encouraged Kostya to start graduate school at the Steklov Mathematical Institute of the Academy of Sciences of USSR under his supervision. In 1972, Kostya received the degree of Candidate of Sciences (the equivalent of Ph.D.), and then in 1979 at the same institute he defended the dissertation for the degree of Doctor of Sciences (Dr. Hab.), a nationally recognized scientific degree which was exceptionally hard to achieve.

The beginning of Kostya's scientific work coincided with a revolutionary period of breakthrough results in multidimensional harmonic analysis. In 1971, C. Fefferman [69] proved the duality of the real Hardy space H^1 and BMO. In that same year C. Fefferman [67] constructed an example of a continuous function on the two-dimensional torus whose rectangular Fourier series diverges almost everywhere. In 1972 L. Carleson and P. Sjölin [65] found the sharp region of L^p-convergence of two-dimensional Bochner-Riesz averages. In 1972 C. Fefferman [70] disproved a long-standing "disc multiplier" conjecture by showing that the spherical sums of multidimensional Fourier series converge in the L^p norm only in the trivial case $p = 2$.

In the 70's the Function Theory seminar at Moscow State University was led by D.E. Menshov and P.L. Ulyanov. During that time, an extremely talented group of mathematicians working in harmonic analysis, approximatively of Kostya's age, was active in Moscow. Notable names include S.V. Bochkarev, B.S. Kashin, E.M. Nikishin, and A.M. Olevskii. It was in this academic environment that Kostya began his career. His research activity was also greatly influenced by such well-known Soviet mathematicians as members of the Academy of Sciences S.M. Nikol'skii and L.S. Pontryagin, as well as his Ph.D. advisor S.A. Telyakovskii.

Between 1972 and 1991, Kostya worked at the Steklov Institute. Together with Boris Kashin they led a seminar. The atmosphere of this seminar was extremely welcoming and informal. Both supervisors always tried to encourage the speaker and provide suggestions on how they could improve the results or the presentation (which was not very typical in the Russian academia). He also worked at the Department of Computational Mathematics and Cybernetics of Moscow State University, where he taught one of the main courses on Optimal Control.

Much of Kostya's time and effort was invested into the collaboration between the Academy of Scences of USSR and Hungary. In particular, for a long time he was an editor of the journal "Analysis Mathematica".

Kostya extensively traveled to different cities and towns of the Soviet Union, where he lectured on various topics, served as an opponent in dissertation defenses, and chaired the State Examination Committee. In the former USSR, where much of the scientific activity and potential was concentrated in big centers like Moscow or Leningrad, such visits greatly enriched the mathematical life of other cities. In particular, Kostya often visited Odessa. Numerous mathematicians from Odessa have been inspired by their communication with Kostya. The papers of V. Kolyada, V. Krotov, A. Korenovsky, P. Oswald, and A. Stokolos in the present volume attest to this fact.

At that time Kostya was one of few members of the Steklov Institute who spoke English and German fluently. Because of that, he was constantly involved in receiving frequent foreign visitors to the Institute, which he always did with great pleasure. In particular, he often spoke with L. Carleson, who visited the Institute on several occasions.

The work of L. Carleson profoundly influenced Kostya's mathematical research. From the start of his scientific career, Kostya was very enthusiastic about Carleson's theorem, which establishes the a.e. convergence of Fourier series of L^2 functions (1966). The original proof was so complicated that soon after its publication there appeared more detailed proofs in several books (e.g., C. Mozzochi [93], O. Jørsboe and L. Mejlbro [92]), as well as an alternative proof by C. Fefferman [68]. Lecturing in various parts of the Soviet Union, Kostya often stressed the importance of this proof and attracted attention on this theorem in which he saw great potential for future research. His predictions came true when in the mid-nineties, M. Lacey and C. Thiele (as well as other authors later on) further developed the techniques used in the proof of Carleson's theorem and successfully applied them to problems in multilinear harmonic analysis [91]. In particular, they provided a short proof of Carleson's theorem based on their method of time-frequency analysis of combinatorial model sums.

We now highlight some of Kostya's contributions to mathematics. We choose to violate the chronological order and start with the topic, which we find most interesting and influential (although, this choice inevitably reflects the personal tastes of the authors). The focus of our exposition is on the results in the area of harmonic analysis. The subsequent articles by M. Chakhkiev, V. Kolyada, V. Maiorov and V. Temlyakov give a snapshot of Oskolkov's contribution in the areas of Approximation Theory and Optimal Control.

Kostya's research activity was to a great extent inspired and motivated by his participation in the seminar of Luzin and Men'shov at Moscow State University. For a long time, this seminar was supervised by P.L. Ul'yanov. As a student of N.K. Bari, P.L. Ul'yanov was deeply interested in the finest features of convergence of Fourier series, in particular the problem of finding *spectra of uniform convergence*.

Let us turn to rigorous definitions. Let $\mathscr{K} = \{k_n\}$ be a sequence of pairwise distinct integers. Denote by $\mathscr{C}(\mathscr{K})$ the subspace of continuous 1-periodic functions with uniform norm, whose Fourier spectrum is contained in $\mathscr{K}$, i.e.

$$\mathscr{C}(\mathscr{K}) = \left\{ f(t) : f(t+1) = f(t) \in \mathscr{C}, \hat{f}_k = \int_0^1 f(t) e^{-2\pi i k t} dt = 0, k \notin \mathscr{K} \right\}.$$

Denote

$$S_N f(t) = \sum_{n=0}^{N} \hat{f}_k e^{-2\pi i k_n t}, \qquad L_N(\mathscr{K}) = \sup_{0 \not\equiv f \in \mathscr{C}(\mathscr{K})} \frac{\|S_N f\|}{\|f\|}.$$

The sequence $\mathscr{K}$ is called a *spectrum of uniform convergence* if for any function f in $\mathscr{C}(\mathscr{K})$ the sequence $S_N(f)$ converges to $f(t)$ uniformly in t as $N \to \infty$. The boundedness of the sequence L_N suffices to deduce that $\mathscr{K}$ is a spectrum of uniform convergence, however the main difficulty lies precisely in obtaining good bounds on L_N in terms of the spectrum $\mathscr{K}$.

The classical result of du Bois-Reymond on the existence of a continuous function whose Fourier series diverges at one point shows that the sequence of all integers is not a spectrum of uniform convergence, while all lacunary sequences are spectra of uniform convergence. For a long time it was not known whether the sequence n^2 (or more general polynomial sequences) is a spectrum of uniform convergence. This problem was repeatedly mentioned by P.L. Ulyanov, in particular, in his 1965 survey [101]. In his remarkable publication [30] Kostya gave a negative answer to this question. His proof is very transparent, elegant, short and inspiring, and led to a series of outstanding results.

We shall briefly outline Kostya's approach. If one denotes

$$h_N(P) = \sum_{1 \le |n| \le N} \frac{e^{2\pi i P(n)}}{n},$$

it is then evident that

$$|h_N(P)| \le \sum_{1 \le |n| \le N} \frac{1}{n} \sim 2 \log N \to \infty.$$

This is a trivial bound of h_N. At the same time, any non-trivial estimate of the type $|h_N(P)| \le (\log N)^{1-\varepsilon}$ for all polynomials of degree r would easily imply the bound $L_N \ge (\log N)^{\varepsilon}$, and the growth of the Lebesgue constants would then disprove the uniform convergence. Therefore the question reduces to improving the trivial bounds for the trigonometric sums, which is far from being simple.

Kostya has demonstrated that no power sequence and, more generally, no polynomial sequence can be a spectrum of uniform convergence. In addition, a remarkable lower bound $L_N > a_r (\log N)^{\varepsilon_r}$ for the Lebesgue constants of polynomial spectra has been established. Here $\varepsilon_r = 2^{-r+1}$, the constant a_r is positive and depends only on the degree of the polynomial defining the spectrum, *but not on the polynomial itself*.

Kostya's ingenious insight consisted of applying the method of trigonometric sums to the solution of this problem. His main observation was that the sequence h_N is nothing but the Hilbert transform of the sequence $\{e^{2\pi i P(n)}\}$ and the algebraically regular nature of this sequence allows one to obtain a substantially improved result. For instance, when $r = 1$ and $P(x) = \alpha x$, the following canonical relations hold

$$h(P) \equiv \sum_{n \ne 0} \frac{e^{2\pi i \alpha n}}{n} = 2i \sum_{n=1}^{\infty} \frac{\sin(2\pi i \alpha n)}{n} = 2\pi i \left(\frac{1}{2} - \{\alpha\} \right),$$

where $\{\alpha\}$ is the fractional part of the number α and $\alpha \notin \mathbb{Z}$. Moreover, the supremum of the partial sums is nicely bounded by

$$\sup_{N,\alpha}\left|2i\sum_{n=1}^{N}\frac{\sin(2\pi i\alpha n)}{n}\right|<\infty, \tag{1}$$

as opposed to the aforementioned logarithmic bound, which can be interpreted as boundedness in two parameters: the upper limit of the partial sums and all polynomials of the first degree.

On one hand this estimate demonstrates the applicability of the method of trigonometric sums, on the other hand it shows the type of bound one may expect to obtain by using this method for polynomials of higher degrees.

Consequently, Kostya managed to improve the trivial bound and to deduce the estimate $L_N > a_r(\log N)^{\varepsilon_r}$ with some constant a_r depending on r from the bound

$$|h_N(P)| \le c_r(\log N)^{1-\varepsilon_r}, \tag{2}$$

where P is a polynomial of degree r with real coefficients and $\varepsilon_r = 2^{1-r}$.

The method employed in [30] to prove (2) is elegant and essentially elementary. It is roughly as follows: by squaring out the quantity $|h_N(P)|$, one obtains a double sum

$$|h_N(P)|^2 = \sum_{1\le|n|,}\sum_{|m|\le N}\frac{e^{2\pi i(P(n)-P(m))}}{nm}.$$

Introducing the summation index $\nu = n-m$ and invoking elementary estimates, one obtains a relation of the type

$$|h_N(P)|^2 \le \sum_{1\le|\nu|\le N}\frac{|h_N(P_\nu)|}{\nu}+1$$

where $P_\nu(x) = P(x+\nu)-P(x)$, $(\nu=\pm1,\pm2,\dots)$. Since for each ν the polynomial $P_\nu(x)$ has degree strictly less than r, the proof may be completed by induction on r.

Notice that if $r=1$, inequality (2) turns into (1). Kostya and his coauthor and friend G.I. Arkhipov, came up with the brilliant idea that (2) can be substantially improved; in fact, the logarithmic growth of (2) may be replaced with boundedness, as in (1), for polynomials P of arbitrary degree, not just of degree $r=1$. The proof is not simple, and requires heavy machinery like the Hardy-Littlewood-Vinogradov circle method for trigonometric sums. The following remarkable theorem was proved in [32]:

Theorem A *Let $\mathscr{P}_r$ be the class of algebraic polynomials P of degree r with real coefficients. Then*

$$\sup_N \sup_{\{P\in\mathscr{P}_r\}}\left|\sum_{1\le|n|\le N}\frac{e^{2\pi iP(n)}}{n}\right| \equiv g_r < \infty$$

and for every $P\in\mathscr{P}_r$, the sequence of symmetric partial sums convergences and the sum is bounded uniformly in $\mathscr{P}_r$.

Of course, this stronger bound brought forth new results, that didn't take long to appear. The first application was obtained for the discrete Radon transform. Namely, let $P \in \mathscr{P}_r$ and define

$$Tf(x) = \sum_{j\neq 0} \frac{f(x-P(j))}{j}.$$

Then

$$\widehat{Tf}(n) = \hat{f}(n) \sum_{j\neq 0} \frac{e^{2\pi i n P(j)}}{j},$$

therefore

$$|\widehat{Tf}(n)| \le |\hat{f}(n)| \sup_N \sup_{\{Q\in\mathscr{P}_r\}} \left| \sum_{1\le |j| \le N} \frac{e^{2\pi i Q(j)}}{j} \right| \le g_r |\hat{f}(n)|$$

and

$$T : L^2 \to L^2.$$

In 1990 E. M. Stein and S. Wainger [99] independently proved the boundedness of the discrete Randon transform in the range $3/2 < p < 3$. A. Ionescu and S. Wainger [71] subsequently extended the result to all $1 < p < \infty$. See [96] for a good source of information about the current state of the subject.

Later, Kostya found a new and unexpected method of proof for Theorem A by interlacing the theory of trigonometric sums with PDEs. His key observation was that formal differentiation of the trigonometric sum

$$h(t,x) := (\text{p. v.}) \sum_{|n|\in\mathbb{N}} \frac{e^{\pi i(n^2 t + 2nx)}}{2\pi i n}$$

yields the solution of the Cauchy initial value problem for the Schrödinger equation of a free particle with the initial data $1/2 - \{x\}$

$$\frac{\partial \psi}{\partial t} = \frac{1}{2\pi i} \frac{\partial^2 \psi}{\partial x^2}, \qquad \psi(t,x)\,|_{t=0} = 1/2 - \{x\}.$$

However, one, has to make rigorous sense of this formalism, which is highly non-trivial. For instance, the series $\vartheta(t,x) := \sum_{n\in\mathbb{Z}} e^{\pi i(n^2 t + 2nx)}$, which arises naturally, is not summable by any regular methods for irrational values of t as observed by G.H. Hardy and J.E. Littlewood, see [72].

Using the Green function $\Gamma(t,x) = \sqrt{\frac{i}{t}}\, e^{-\frac{\pi i x^2}{t}}$ and the Poisson summation formula, Kostya established the following identity, which must be understood in the sense of distributions.

$$\vartheta(t,x) = \Gamma(t,x)\,\vartheta\left(-\frac{1}{t}, -\frac{x}{t}\right).$$

This might be viewed as a generalization of the well-known reciprocity of truncated Gauss sums, see [72, p.22]:

$$\sum_{n=1}^{q} e^{\frac{\pi i n^2 p}{q}} = \sqrt{\frac{iq}{p}} \sum_{m=1}^{p} e^{-\frac{\pi i m^2 q}{p}}$$

From this identity, Kostya derives the existence and global boundedness for the discrete oscillatory Hilbert transforms with polynomial phase $h(t,x)$, i.e. a particular case of Theorem A for the polynomials of second degree. The case of higher-degree polynomials, e.g. cubic, requires the analysis of linearized periodic KdV equation. The general case was considered in the remarkable paper [37].

The success achieved by Kostya in the study of the Schrödinger equation of a free particle with the periodic initial data has been developed even further. Z. Ciesielski suggested that Kostya tries to use Jacobi's elliptic ϑ-function as a periodic initial data. This function has lots of internal symmetries and the problem sounded quite promising.

Formally, the problem is the following

$$\frac{\partial \psi}{\partial t} = \frac{1}{2\pi i} \frac{\partial^2 \psi}{\partial x^2}, \qquad \psi(t,x)\,|_{t=0} = \vartheta_\varepsilon(x) = c(\varepsilon) \sum_{m\in\mathbb{Z}} e^{-\frac{\pi(x-m)^2}{\varepsilon}}$$

Here, ε is a small positive parameter which tends to 0 and $c(\varepsilon)$ a positive factor, normalizing the data in the space $L^2(\mathbb{T})$, i. e. on the period.

D. Dix, Kostya's colleague from the University of South Carolina, conducted a series of computer experiments (unpublished) and plotted the 3D-graph of the density function $\rho = \rho(\theta_\varepsilon,t,x) = |\psi(\theta_\varepsilon,t,x)|^2, (t,x) \in \mathbb{R}^2$, for $\varepsilon = 0.01$ The result was astonishing, see Figure 1.

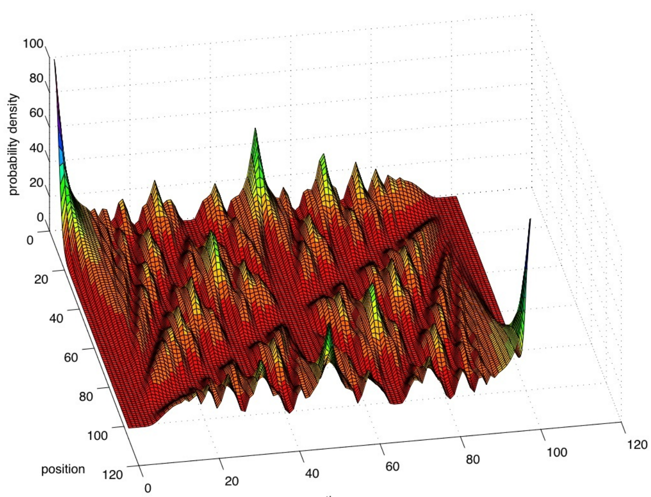

Figure 1. The Schrödinger landscape

Instead of expected chaos, the picture turned out to be well structured. First, the graphs represent a rugged mountain landscape, and second, the landscape is not a completely random combination of "peaks and trenches." In particular, it is criss-crossed by a rather well-organized set of deep rectilinear canyons, or, "the valleys of shadow." The solutions exhibit deep self-similarity features, and complete rational Gauss' sums play the role of scaling factors. Effects of such nature are labeled in the modern physics literature as quantum carpets.

Moreover, Kostya showed that semi-organized and semi-chaotic features, exhibited by the bi-variate Schrödinger densities $|\psi(t,x)|^2$, also occur for a wide class of $\sqrt{\delta}$-type initial data where $\delta = \delta(x)$ denotes the periodic Dirac's delta-function. By definition, $\sqrt{\delta}$ is a family of regular periodic initial data $\{f_\varepsilon(x)\}_{\varepsilon>0}$ such that in the distributional sense $|f_\varepsilon|^2 \to \delta$ for $\varepsilon \to 0$.

These phenomena were mathematically justified by Kostya using the expansions of densities $|\psi_\varepsilon|^2$ into ridge-series (infinite sums of planar waves) consisting of Wigner's functions and by analyzing the distribution of zeros of bi-variate Gauss sums.

Figure 2 below demonstrates Bohm's trajectories – the curves on which the solution ψ conserves the initial value of the phase, i. e. remains real-valued and positive.

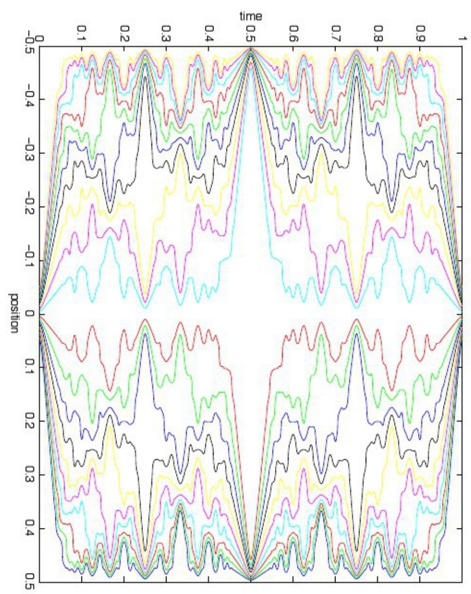

Figure 2. The Bohm trajectories

Figure 2 looks like a typical quantum carpet from the Talbot effect. The Talbot effect phenomenon, discovered in 1836 by W.H.F. Talbot [100], the British inventor of photography, consists of multi-scaled recovery (revival) of the periodic “initial signal” on the grating plane. It occurs on an observation screen positioned parallel to the original plane, at the distances that are rational multiples of the so-called Talbot distance. At the bottom of the figure, the light can be seen diffracting through a grating, and this exact pattern is reproduced at the top of the picture, one Talbot Length away from the grating. Half way down, one sees the image shifted to the side, and at regular fractions of the Talbot Length, the sub-images are clearly seen. A caFul examination of Figure 2 reveals the aforementioned features in this picture.

Kostya suggested the model that explains the Talbot effect mathematically [57]. He established the bridges between the following equations describing the Talbot effect:

$$\text{Wave} \mapsto \text{Helmholtz with small parameter} \mapsto \text{Schrödinger}$$

Subsequently, several theorems concerning the Talbot effect were proved by him, explaining the phenomenon of “the valleys of shadows” – the rectilinear domains of extremely low light intensity in Figure 1.

In particular, it was discovered that there are surprisingly wide and very interesting relations of his results on Vingradov series with many concepts in mathematics, such as the Fresnel integral, continued fractions, Weyl exponential sums, Carlesons theorem on trigonometric Fourier series of L^2 functions, the Riemann ζ-function, shifted truncated Gauss sums – in other words, deep connections exist between the objects of analytic number theory and partial differential equations of Schrödinger type with periodic initial data.

Kostya has explored the complexity features of solutions to the Schrödinger equation which are related to the so-called curlicues studied by M. V. Berry and J. H. Goldberg [77]. Curlicues represent a peculiar class of curves on the complex plane $\mathbb{C}$ resulting from computing and plotting the values of incomplete Gauss sums. In particular, the metric entropy of the Cornu spiral described by the incomplete Fresnel integral equals 4/3. Kostya’s result [43] demonstrates a very remarkable fact that, although the Cauchy initial value problem with periodic initial value $f(x)$ is linear, the solutions may be chaotic even in the case of simple initial data.

These phenomena were enthusiastically received by the mathematical community. In 2010, P. Olver published a paper [95] in the American Math. Monthly attempting to attract the attention of young researchers to the subject.

Kostya also took a different direction of research related to the aforementioned trigonometric sums in [59, 56, 53, 48]. In particular, in [56] he found an answer to S.D. Chowla’s problem, which had been open since 1931. Along the way, Kostya characterized the convergence sets for the series

$$S(t) \sim \sum_{(n,m)\in\mathbb{N}^2} \frac{\sin 2\pi nmt}{nm}, \quad C(t) \sim \sum_{(n,m)\in\mathbb{N}^2} \frac{\cos 2\pi nmt}{nm},$$

as well as for more general double series of the type

$$E(\lambda,t,x,y) \sim \sum_{(n,m)\in\mathbb{N}^2} \lambda_{n,m} \frac{e^{2\pi i(nmt+nx+my)}}{nm},$$

where λ is a bounded "slowly oscillating" multiplier, satisfying, say, the Paley condition, t,x,y - independent real variables. Such series naturally arise in the study of the discrepancy of the distribution of the sequence of fractional parts $\{nt\}$ (mod 1) and Wigner's functions arising from the Schrödinger density $|\psi|^2$.

We now turn our attention to some of Kostya's earlier results, which highlight his versatile contributions to harmonic analysis and approximation theory.

In 1973, E.M. Nikishin and M.Babuh [94] demonstrated that one could construct a function of two variables whose rectangular Fourier series diverges almost everywhere (the existence of such functions was proved by C. Fefferman [67] in 1971) with modulus of continuity $\omega_C(f,\delta) = O\left(\log\frac{1}{\delta}\right)^{-1}$. One year later, Kostya [9] proved that this estimate is close to being sufficient. If $f\in C(\mathbb{T}^2)$ and

$$\omega_C(f,\delta) = o\left(\log\frac{1}{\delta}\log\log\log\frac{1}{\delta}\right)^{-1},$$

then the rectangular Fourier sums converge a.e.; the exact condition is still an open question. Kostya's proof used very delicate estimates of the majorant of the Fourier series of a bounded function of one variable due to R. Hunt. In addition, Kostya suggested a remarkable method for expressing the information about the smoothness of a function in terms of a certain extremal sequence which we shall discuss later. Thus, even Kostya's earliest results are elegant and complete, although very technical and far from trivial.

A natural counterpart of Carleson's theorem is Kolmogorov's example [85] of an L^1 function whose Fourier series diverges almost everywhere. Finding the optimal integrability class in Kolmogorov's theorem is an important open question. The first step in this direction was made in 1966 by V.I. Prohorenko [97]. The best result known today was obtained by S.V. Konyagin [90] in 1998. In his paper Konyagin wrote, *"The author expresses his sincere thanks to K. I. Oskolkov for a very fruitful scientific discussion during his (the author's) visit to the University of South Carolina, which stimulated the results of the present paper."*

One of Kostya's earliest research interests was the quest for a.e. analogues of estimates written in terms of norms. We shall take the liberty of drawing a parallel to the Diophantine approximation. The classical Dirichlet-Hurwitz estimate

$$\left|x-\frac{p}{q}\right| \le \frac{1}{q^2\sqrt{5}}$$

holds for for all real x and for infinitely many values of p and q with $(p,q)=1$. Moreover, for some values of x, (such as the "golden ratio" $\frac{\sqrt{5}-1}{2}$), the constant $\sqrt{5}$ cannot be increased. At the same time, as shown by A. Khinchin for almost all x, the order of approximation can be greatly improved. For example, for almost all x

there exist infinitely many p, q with $(p,q)=1$ such that

$$\left|x-\frac{p}{q}\right| \le \frac{1}{q^2\log q}.$$

More generally, instead of $\log q$, one can use any increasing function $\varphi(q)$, where the series $\sum \frac{1}{q\varphi(q)}$ diverges. The divergence condition is sharp, which easily follows from the Borel-Cantelli Lemma. Therefore, the Dirichlet-Hurwitz esimate can be improved by a logarithmic factor almost everywhere.

In the same spirit, Kostya improved Lebesgue's result on the approximation of continuous functions with the partial sums of Fourier series. Unifrom estimates may be substantially strengthened in the a.e. sense. More precisely, Lebesgue's Theorem [74] implies that if $f \in \mathrm{Lip}_\alpha$, $0<\alpha<1$ then the following uniform estimate of the rate of approximation is valid

$$|f(x)-S_nf(x)| \le C\frac{\log n}{n^\alpha},$$

and there is a function $f \in \mathrm{Lip}_\alpha$ such that

$$\limsup_{n\to\infty} \frac{n^\alpha}{\log n}|f(0)-S_nf(0)| > 0.$$

In [14], using the exponential estimates on the majorants of the Fourier sums of a bounded function due to R. Hunt [84], Kostya showed that for almost all $x \in \mathbb{T}$, where $\mathbb{T}=[0,2\pi)$, the estimate can be improved to

$$|f(x)-S_nf(x)| \le C_x\frac{\log\log n}{n^\alpha}$$

and there is a function $f \in \mathrm{Lip}_\alpha$ such that for almost all $x \in \mathbb{T}$

$$\limsup_{n\to\infty} \frac{n^\alpha}{\log\log n}|f(x)-S_nf(x)| > 0.$$

We would like to mention that the parallel with the Diophantine approximation is more than just formal. In his later works, Kostya used continued fractions, the main tool of Diophantine approximation, to obtain convergence theorems for trigonometric series. See for example [48, 53, 56, 59].

The proof of the aforementioned metric version of Lebesgue's theorem was based on a remarkable sequence δ_k, defined for a modulus of continuity $\omega(\delta)$ by the following rule

$$\delta_0=1,\ \delta_{k+1}=\min\left\{\delta:\ \max\left(\frac{\omega(\delta)}{\omega(\delta_k)};\ \frac{\delta\,\omega(\delta_k)}{\delta_k\omega(\delta)}\right)\le\frac{1}{2}\right\},\ k=0,1,\dots$$

One can view this sequence as a discrete K-functional. Namely, it is well known, that the modulus of continuity $\omega(\delta)$ controls the rate of convergence while the ratio

$\omega(\delta)/\delta$ controls the growth of the derivative of a smooth approximation process when $\delta \to 0$. So, the δ_k system controls both, which is similar to the idea of the K-functional.

The idea of such partitions was already in the air, probably since the work of S.B. Stechkin [75] in the early fifties. Simultaneous partition of a modulus of continuity $\omega(\delta)$ and the function $\delta/\omega(\delta)$ apparently was first used by V.A. Andrienko, [63]. As in the work of Stechkin, Andrienko used such partitions to construct counterexamples.

Kostya however was the first who wrote this sequence explicitly and employed it to obtain positive results. Amazingly, this sequence turns out to be very useful in the description of phenomena that are either close to or seemingly far from the rate of a.e. approximations. For instance, the classical Bari-Stechkin-Zygmund condition on the modulus of continuity just means that δ_k/δ_{k+1} is bounded. Later on, this method was widely used by many authors, see for example [86, 87].

Another example of application of δ_k sequence is the a.e. form of a Jackson-type theorems from constructive approximation theory. Namely, let $f \in L^p(\mathbb{T})$, $1 \leq p < \infty$; let $\omega_p(f,\delta)$ denote the L^p-modulus of continuity of a function f and let $S_\delta(f)(x) = \delta^{-1}\int_x^{x+\delta} f(y)dy$. Then

$$\|f - S_\delta(f)\|_p \leq C_p \omega_p(f,\delta).$$

In [14] Kostya suggested an a.e. version of the above theorem. Let $\omega(t)/t$, $w(t)$ and $\omega(t)/w(t)$ be increasing, and assume also that

$$\sum_{k=0}^{\infty} \left(\frac{\omega(\delta_k)}{w(\delta_k)}\right)^p < \infty. \tag{3}$$

If $\omega_p(f,\delta) = O(\omega(\delta))$, then

$$f(x) - S_\delta(f)(x) = O_x(w(\delta)) \qquad \text{a.e. on } \mathbb{T}.$$

If (1) diverges, then there is a function f such that $\omega_p(f,\delta) = O(\omega(\delta))$ and

$$\limsup_{\delta\to 0+} \frac{f(x) - S_\delta(f)(x)}{w(\delta)} = \infty \qquad \text{a.e. on } \mathbb{T}.$$

Further applications of the sequence δ_k include a quantitative characterization of the Luzin C-property. By Luzin Theorem, an integrable function is continuous if restricted to a proper subset of the domain whose complement has arbitrarily small measure. It is then natural to ask the following: if the function has some smoothness in the integral metric, what can be concluded about the uniform smoothness of this restriction?

Kostya [25, 17] suggested the following sharp statement: let $\omega(\delta)$ be a modulus of continuity, and let f be such that $\omega_p(f,\delta) \leq \omega(\delta)$. Let another modulus of continuity $w(\delta)$ be as above (see (3)). Then for some measurable function $C(t) \in L^{p,\infty}$

$$|f(x)-f(y)| \leq (C(x)+C(y))\,w(|x-y|).$$

The convergence of the series in (1) is a sharp condition. Since any $L^{p,\infty}$ function is bounded modulo a proper set of arbitrary small measure, the above inequality provides the quantitative version of the Luzin C-property.

Later, that property was generalized to functions in H^p, $0<p\leq 1$ and in L^p, $p\geq 0$ by A. A. Solyanik [76]. Also V. G. Krotov and his collaborators have studied the C-property in more general settings (see his paper in this volume).

Kostya's interest in the convergence of Fourier series lead him to consider the question of the best approximation of a continuous function f with trigonometric polynomials. This problem has a long history and tradition, especially in the Russian school. Here Kostya again used a combination of deep and simple ideas and obtained optimal results.

To be specific, let f be a continuous periodic function with Fourier sums $S_n(f)$, and let $E_n(f)=E_n$ be the best approximation of f by trigonometric polynomials of order n. Classic estimates due to Lebesgue state that

$$\|f-S_n(f)\| \leq (L_n+1)E_n(f),$$

where L_n are Lebesgue constants. From this inequality it follows that

$$\|f-S_n(f)\| \leq C(\log n)E_n(f).$$

This inequality is sharp in many function classes defined in terms of a slowly decreasing majorant of best approximations. But the inequality is not sharp if the best approximations decrease quickly.

The following estimate was proved by Kostya in [11] :

$$\|f-S_n(f)\| \leq C\sum_{k=n}^{2n}\frac{E_k(f)}{n-k+1}.$$

Here, C is an absolute constant and $\|\cdot\|$ is a norm in the space of continuous functions. This estimate sharpens Lebesgue's classical inequality for fast decreasing E_k. The sharpness of this estimate is proved for an arbitrary class of functions having a given majorant of best approximation. Kostya also investigated the sharpness of the corresponding estimate for the rate of almost everywhere convergence of Fourier series. See the note by V. Kolyada in this volume.

When f is continuous with no extra regularity assumptions, the partial Fourier sums may not provide a good approximation of f. In a paper with D. Offin, [39] Kostya constructed a simple and explicit orthonormal trigonometric polynomial basis in the space of continuous periodic functions by simply periodizing a well-known wavelet on the real line. They obtained trigonometric polynomials whose degrees have optimal order of growth if their indices are powers of 2. Also, Fourier sums with respect to this polynomial basis have almost best approximation properties.

More recently, Kostya wrote an interesting series of papers on the approximation of multivariate functions. He became interested in the *ridge approximation* (approximation by finite linear combination of planar waves) and the algorithms used to generate such approximations. His interest in these problems was motivated by the connections between the ridge approximation and optimal quadrature formulas for trigonometric polynomials, which are discussed in [42]. In this paper Kostya also studied the best ridge approximation of L^2 radial functions in the unit ball of $\mathbb{R}^2$ and showed that the orthogonal projections on the set of algebraic polynomials of degree k are linear and optimal with respect to degree n ridge approximation. The proof of this result uses, in particular, the inverse Radon transform and Fourier-Chebyshev analysis.

References

Scientific Articles by K.I. Oskolkov

1. K. I. Oskolkov, Convergence of a trigonometric series to a function of bounded variation, Mat. Zametki 8 (1970), 47-58. (Russian)
2. K. I. Oskolkov, S. B. Steckin, and S. A. Teljakovskii, Petr Vasil'evic Galkin, Mat. Zametki 10 (1971), 597-600. MR 44 No 6436 (Russian)
3. K. I. Oskolkov, The norm of a certain polynomial operator, Sibirsk Mat. Z. 12 (1971), 1151-1157. MR 45 No 4021 (Russian)
4. K. I. Oskolkov and S. A. Teljakovskii, On the estimates of P. L. Ul'janov for integral moduli of continuity, Izv. Akad. Nauk Armjan. SSR Ser. Mat. 6 (1971), 406-411. MR 45 No 8782 (Russian)
5. K. I. Oskolkov, The sharpness of the Lebesgue estimate for the approximation of functions with prescribed modulus of continuity by Fourier sumes, Trudy Mat. Inst. Steklov. 112 (1971), 337-345, 389, Collection of articles dedicated to Academician Ivan Matveevic Vinogradov on his 80th birthday, I. MR 49 No No 970 (Russian)
6. K. I. Oskolkov, Generalized variation, the Banach indicatrix and the uniform convergence of Fourier series, Mat. Zametki 12 (1972), 313-324. MR 47 No 5507 (Russian)
7. K. I. Oskolkov, Subsequences of Fourier sums of functions with a prescribed modulus of continuity, Mat. Sb. (N.S.) 88(130) (1972), 447-469. MR 48 No 11874 (Russian)
8. K. I. Oskolkov, Fourier sums for the Banach indicatrix, Mat. Zametki 15 (1974), 527-532. MR 50 No 10177 (Russian)
9. K. I. Oskolkov, Estimation of the rate of approximation of a continuous function and its conjugate by Fourier sums on a set of full measure Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974), 1393-1407. MR 50 No 10663 (Russian)
10. K. I. Oskolkov, An estimate for the approximation of continuous functions by sequences of Fourier sums, Trudy Mat. Inst. Steklov. 134 (1975), 240-253, 410, Theory of functions and its applications (collection of articles dedicated to Sergei Mihailovic Nikolskii on the occasion of his 70th birthday). MR 53 No 6203 (Russian)
11. K. I. Oskolkov, Lebesgue's inequality in the uniform metric and on a set of full measure, Mat. Zametki 18 (1975), 515-526. MR 54 No 833 (Russian)
12. K. I. Oskolkov, On strong summability of Fourier series and differentiability of functions, Anal. Math. 2 (1976), 41-47. MR 53 No 6210 (English, with Russian summary)
13. K. I. Oskolkov, The uniform modulus of continuity of summable functions on sets of positive measure, Dokl. Akad. Nauk SSSR 229 (1976), 304-306. MR 57 No 9917 (Russian)
14. K. I. Oskolkov, Approximation properties of integrable functions on sets of full measure, Mat. Sb. (N.S.) 103(145) (1977), 563-589, 631. MR 57 No 13343 (Russian)
15. K. I. Oskolkov, Sequences of norms of Fourier sums of bounded functions, Trudy Mat. Inst. Steklov. 142 (1977), 129-142, 210, Analytic number theory, mathematical analysis and their applications (dedicated to I. M. Vinogradov on his 85th birthday).

16. K. I. Oskolkov, Polygonal approximation of functions of two variables, Mat. Sb. (N.S.) 107(149) (1978), 601-612, 639. MR 81j:41020 (Russian)
17. K. I. Oskolkov, Quantitative estimates of N. N. Luzin's C-property for classes of integrable functions, Approximation Theory (Papers, VIth Semester, Stefan Banach Internat. Math. Center, Warsaw, 1975), Banach Center Publ. 4 PWN, Warsaw (1979), 185-196 MR 81a:26003
18. K. I. Oskolkov, Optimality of a quadrature formula with equidistant nodes on classes of periodic functions, Dokl. Akad. Nauk SSSR 249 (1979), 49-52. MR 81b:41077 (Russian)
19. K. I. Oskolkov, Lebesgue's inequality in the mean, Mat. Zametki 25 (1979), 551-555, 636. MR 81c:42005 (Russian)
20. K. I. Oskolkov, The upper bound of the norms of orthogonal projections onto subspaces of polygonals, Approximation Theory (Papers, VIth Semester, Sefan Banach Internat. Math. Center, Warsaw, 1975), Banach Center Publ., 4, PWN, Warsaw (1979), 177-183. MR 82e:41013
21. K. I. Oskolkov, Approximate properties of classes of periodic functions Mat. Zametki 27 (1980), 651-666. MR 81j:42011 (Russian)
22. K. I. Oskolkov, Partial sums of the Taylor series of a bounded analytic function, Trudy Mat. Inst. Steklov. 157 (1981), 153-160, 236, Number Theory, mathematical analysis and their applications. MR 83c:300004 (Russian)
23. K. I. Oskolkov, On optimal quadrature formulas on certain classes of periodic functions, Appl. Math. Optim. 8 (1982), 245-263. MR 83h:41032
24. K. I. Oskolkov, On exponential polynomials of the least Lp-norm, Constructive Function Theory '81 (Varna, 1981), Publ. House Bulgar. Acad. Sci., Sofia (1983), 464-467. MR 85a:41022
25. K. I. Oskolkov, Luzin's C-property for a conjugate function, Trudy Mat. Inst. Steklov. 164 (1983), 124-135. Orthogonal series and approximations of functions. MR 86e:42019 (Russian)
26. Z. Ciselskii and K. I. Oskolkov, Approximation by algebraic polynomials on simplexes, Uspekhi Mat. Nauk 40 (1985), 212-214, Translated from the English by K. I. Oskolkov. 807 760 (Russian)
27. K. I. Oskolkov and K. Tandor, Systems of signs, Uspekhi Mat. Nauk 40 (1985), 105-108, Translated from the German by K. I. Oskolkov; International conference on current problems in algebra and analysis (Moscow-Leningrad, 1984). 804 790 (Russian)
28. K. I. Oskolkov, Strong summability of Fourier series Trudy Mat. Inst. Steklov. 172 (1985), 280-290, 355, Studies in the theory of functions of several real variables and the approximation of functions. MR 87a:42021 (Russian)
29. K. I. Oskolkov, A subsequence of Fourier sums of integrable functions, Trudy Mat. Inst. Steklov 167 (1985), 239-260, 278, Current problems in mathematics. Mathematical analysis, algebra, topology. MR 87i:42008 (Russian)
30. K. I. Oskolkov, Spectra of uniform convergence, Dokl. Akad. Nauk. SSSR 288 (1986), 54-58. MR 88e:42012 (Russian)
31. K. I. Oskolkov, Inequalities of the "large size" type and applicatiojns to problems of trigonometric approximation, Anal. Math. 12 (1986), 143-166. MR 88i:42002 (English, with Russian summary)
32. G. I. Arkhipov and K. I. Oskolkov, A special trigonometric series and its applications, Mat. Sb. (N.S.) 134(176) (1987), 147-157, 287. MR 89a:42010 (Russian)
33. K. I. Oskolkov, Continuous functions with polynomial spectra, Investigations in the theory of the approximation of functions (Russian), Akad. Nauk SSSR Bashkir. Filial Otdel Fiz. Mat., Ufa, (1987), 187-200. MR 90b:42013 (Russian)
34. K. I. Oskolkov, Properties of a class of I. M. Vinogradov series, Dokl. Akad. Nauk SSSR 300 (1988), 803-807. MR 89f:11117 (Russian)
35. K. I. Oskolkov, I. M. Vinogradov series and integrals and their applications, Trudy Mat. Inst. Steklov. 190 (1989), 186-221, Translated in Proc. Steklov Math. 1992, no. 1, 193-229; Theory of functions (Russian) (Amberd, 1987). MR 90g:11112 (Russian)
36. K. I. Oskolkov, On functional properties of incomplete Gaussian sums, Canad. J. Math. 43 (1991), 182-212. MR 92e:11083

37. K. I. Oskolkov, I. M. Vinogradov series in the Cauchy problem for Schrödinger-type equations, Trudy Mat. Inst. Steklov. 200 (1991), 265-288. MR 93b:11104 (Russian)
38. K. I. Oskolkov, A class of I. M. Vinogradov's series and its applications in harmonic analysis, Progress in Approximation Theory (Tampa, FL, 1990), Springer Ser. Comput. Math., 19, Springer, New York (1992), 353-402. MR 94m:42016
39. D. Offin and K. I. Oskolkov, A note on orthonormal polynomial bases and wavelets, Constr. Approx. 9 (1993), 319-325. MR 94f:42047
40. A. Andreev, V. I. Berdyshev, B. Bojanov, B. S. Kashin, S. V. Konyagin, S. M. Nikol'skii, K. I. Oskolkov, P. Petrushev, B. Sendov, S. A. Telyakovskii, and V. N. Temlyakov, In memory of Sergei Borisovich Stechkin [1920-1995], East J. Approx. 2 (1996), 131-133. 1 407 059
41. Ronald A. DeVore, Konstantin I. Oskolkov, and Pencho P. Petrushev, Approximation by feed-forward neural networks, Ann. Numer. Math. 4 (1997), 261-287, The heritage of P.L. Chebyshev: A Festschrift in honor of the 70th birthday of T. J. Rivlin. MR 97i:41043
42. K. I. Oskolkov, Ridge approximation, Fourier-Chebyshev analysis, and optimal quadrature formulas, Tr. Mat. Inst. Steklov 219 (1997), 269-285. MR 99j:41036 (Russian)
43. K. I. Oskolkov, Schrödinger equation and oscillatory Hilbert transforms of second degree, J. Fourier Anal. Appl. 4 (1998), 341-356. MR 99j:42004
44. K. I. Oskolkov, Ridge approximation and the Kolmogorov-Nikolskii problem, Dokl. Akad. Nauk 368 (1999), 445-448. MR 2001b:41024 (Russian)
45. K. I. Oskolkov, Linear and nonlinear methods for ridge approximation, Metric theory of functions and related problems in analysis (Russian), Izd. Nauchno-Issled. Aktuarno-Finans. Tsentra (AFTs), Moscow (1999), 165-195. MR 2001i:41039 (Russian, with Russian summary)
46. V. E. Maiorov, K. Oskolkov, and V. N. Temlyakov, Gridge approximation and Radon compass, Approximation Theory: a Volume-Dedicated to Blagovest Sendov (B. Bojanov, ed.), DARBA, Sofia (2002), 284-309.
47. K. Oskolkov, On representations of algebraic polynomials by superpositions of plane waves, Serdica Math. J. 28 (2002), 379-390, Dedicated to the memory of Vassil Popov on the occasion of his 60th birthday.
48. K. Oskolkov, *Continued fractions and the convergence of a double trigonometric series*, East J. Approx. 9 (2003), 375-383.
49. K. Oskolkov, *On a result of Telyakovskii and multiple Hilbert transforms with polynomial phases*, Mat. Zametki 74 (2003), 242-256.
50. DeVore, Ronald A.; Oskolkov, Konstantin I.; Petrushev, Pencho P. *Approximation by feed-forward neural networks.* [J] Ann. Numer. Math. 4, No.1-4, 261-287 (1997). ISSN 1021-2655
51. Oskolkov, K. *Schrödinger equation and oscillatory Hilbert transforms of second degree.*
52. Oskolkov, K. I.*Ridge approximations and the Kolmogorov-Nikol'skij problem.* [J] Dokl. Math. 60, No. 2, 209-212 (1999); translation from Dokl. Akad. Nauk, Ross. Akad. Nauk 368, No. 4, 445-448 (1999). ISSN 1064-5624; ISSN 1531-8362
53. Oskolkov, K.I. *Continued fractions and the convergence of a double trigonometric series.* [J] East J. Approx. 9, No. 3, 375-383 (2003). ISSN 1310-6236
54. Oskolkov, K. I. *On representations of algebraic polynomials by superpositions of plane waves.* [J] Serdica Math. J. 28, No. 4, 379-390 (2002). ISSN 0204-4110
55. Maiorov, V. E.; Oskolkov, K. I.; Temlyakov, V. N. *Gridge approximation and Radon compass.* [A] Bojanov, B. D. (ed.), Approximation theory. A volume dedicated to Blagovest Sendov. Sofia: DARBA. 284-309 (2002). ISBN 954-90126-5-4/hbk
56. Oskolkov, K.I., *The series* $\sum \frac{e^{2\pi i n m x}}{mn}$ *and Chowla's problem.* Problem of Chowla, Proc. Steklov Inst. Math., 248 (2005), 197215.
57. Oskolkov, K.I., *The Schrödinger density and the Talbot effect.* [A] Figiel, Tadeuz (ed.) et al., Approximation and probability. Papers of the conference held on the occasion of the 70th anniversary of Prof. Zbigniew Ciesielski, Bedlewo, Poland, September 20–24, 2004. Warsaw: Polish Academy of Sciences, Institute of Mathematics. Banach Center Publications 72, 189-219 (2006).

58. Oskolkov, K.I., *Linear and nonlinear methods of relief approximation.* (English. Russian original) J. Math. Sci., New York 155, No. 1, 129-152 (2008); translation from Sovrem. Mat., Fundam. Napravl. 25, 126–148 (2007). ISSN 1072-3374; ISSN 1573-8795
59. Oskolkov, K.I.; Chakhkiev, M.A., *On Riemann "nondifferentiable" function and Schrödinger equation.* Proc. Steklov Inst. Math. 269, 186-196 (2010); translation from Trudy Mat. Inst. Steklova 269, 193-203 (2010).

Books Translated or Edited by K.I. Oskolkov

60. B. Sendov and V. Popov, Usrednennye moduli gladkosti, "Mir", Moscow, 1988, Translated from the Bulgarian and with a preface by Yu. A. Kuznetsov and K.I. Oskolkov. (Russian)
61. A. Brensted, Vvedenie v teoriyu vypuklykh mnogogrannikov, "Mir", Moscow, 1988,Translated from the English by K. I. Oskolkov; Translation edited and with a preface by B. S. Kashin. (Russian)
62. I. M. Vinogradov, A. A. Karacuba, K. I. Oskolkov, and A. N. Parsin, Trudy mezhdunarodnoi konferenctsii po teorii chisel (Moskva, 14-18 sentyabrya 1971 g.), Izdat. "Nauka", Moscow, 1973, With an introductory address by M. V. Keldys; Trudy Mat. Inst. Steklov. 132 (1973). (Russian)

List of other references

63. V.A. Andrienko, On imbeddings of certain classes of functions, Izv. Akad. Nauk SSSR, Ser. Mat. 31 (1967), 1311-1326.
64. Lennart Carleson, On convergence and growth of partial sums of Fourier series, Acta Math. 116 (1966) 135157.
65. Carleson, Lennart; Sjölin, P. Oscillatory integrals and a multiplier problem for the disc. Stud. Math. 44, 287-299 (1972).
66. R.DeVore and R. Sharpley, Maximal Functions Measuring Smoothness, Memoirs of AMS, Volume 293, 1984.
67. C. Fefferman, On the divergence of multiple Fourier series. Bull. Amer. Math. Soc. 77 1971 191-195.
68. C. Fefferman, Pointwise convergence of Fourier series. Ann. of Math. (2) 98 (1973), 551-571.
69. C. Fefferman, Characterizations of bounded mean oscillation. [J] Bull. Am. Math. Soc. 77, 587-588 (1971).
70. C. Fefferman, The multiplier problem for the ball. Ann. Math. (2) 94, 330-336 (1971).
71. A. Ionescu ans S. Wainger, L^p boundedness of discrete singular Radon transforms. [J] J. Am. Math. Soc. 19, No. 2, 357-383 (2006).
72. G.H. Hardy, *Collected papers of G.H. Hardy*, 1 Oxford: Clarendon Press.
73. Hunt R. A., On the convergence of Fourier series, Orthogonal expansions and their continuous analogues, Proc. Conf. S. 111. Univ., Edwardsville, 1967. SIU Press, Carbondale, Illinois, 1968
74. H. Lebesgue. Sur la représentation trigonometrique approchée des fonctions satisfaisant à une condition de Lipschitz. Bull. Soc. Math. France, 1910, 38, 184-210
75. S.B. Stechkin, On absolute convergence of Fourier series, Izv. AN SSSR, Ser. matem. 17 (1953), 87-98
76. A. A. Soljanik, "Approximation of Functions from Hardy Classes" , Ph.D. thesis, Odessa, 1986
77. M. V. Berry and J. H. Goldberg, Renormalisation of curlicues, Nonlinearity **1** (1988), no. 1, 1–26.
78. M. V. Berry. *Quantum fractals in boxes.* J. Physics A: Math. Gen **29**(1996), pp. 6617– 6629.
79. Michael Berry, Irene Marzoli and Wolfgang Schleich. *Quantum carpets, carpets of light.* Physics World, June 2001, pp. 1 - 6.
80. M. V. Berry and S. Klein. *Integer, fractional and fractal Talbot effects.* J. Mod. Optics **43**(1996), pp. 2139 – 2164.
81. S.D. Chowla. Some problems of diophantine approximation (I). *Mathematische Zeitschrift,* **33**(1931), pp. 544 – 563.
82. M. Christ, A. Nagel,E.M. Stein, S. Wainger. Singular and maximal Radon transforms: analysis and geometry. Ann. of Math. (2) 150 (1999), no. 2, 489–577.

83. R. DeVore and R. Sharpley "Maximal functions measuring smoothness" , Memoirs of AMS, Volume 293, 1984.
84. G.H. Hardy and J.E. Littlewood. *Some problems of Diophantine approximation: The analytic character of the sum of a Dirichlet's series considered by Hecke.* Abhandlungen aus der Mathematischen Seminar der Hamburgischen Universität, **3**(1923), pp. 57 – 68.
85. Andrey Kolmogorov, Une série de FourierLebesgue divergente presque partout, Fundamenta math. 4 (1923), 324-328.
86. V.I. Kolyada, Estimates of rearrangements and imbedding theorems. Math. USSR, Sb. 64, No.1, 1-21 (1989)
87. V.I. Kolyada, On relations between moduli of continuity in different metrics. Proc. Steklov Inst. Math. 181, 127-148 (1989)
88. V.I. Kolyada, Estimates of maximal functions connected with local smoothness. Sov. Math., Dokl. 35, 345-348 (1987)
89. V.I. Kolyada, Estimates of maximal functions measuring local smoothness. Anal. Math. 25, No.4, 277-300 (1999)
90. S. V. Konyagin, On everywhere divergence of trigonometric Fourier series, Sb. Math., 191:1 (2000), 97-120
91. Lacey, Michael; Thiele, Christoph A proof of boundedness of the Carleson operator. Math. Res. Lett. 7, No.4, 361-370 (2000).
92. O. Jørsboe, L. Mejlbro, (1982), The Carleson-Hunt theorem on Fourier series, Lecture Notes in Mathematics, 911, Berlin, New York: Springer-Verlag
93. Mozzochi, Charles J. On the pointwise convergence of Fourier series. Lecture Notes in Mathematics, Vol. 199. Springer-Verlag, Berlin-New York, 1971.
94. E.M. Nikishin i M.Babuh, On convergence of double Fourier series of continuous functions, (Russian), Sib. Math. Zh., 11, No 6 (1973), 1189-1199
95. Olver, P.J., Dispersive quantization, Amer. Math. Monthly 117 (2010) 599-610.
96. Lillian B. Pierce. *Discrete fractional Radon transforms and quadratic forms*, Duke Math. J. Volume 161, Number 1 (2012), 69-106.
97. V. I. Prohorenko, Divergent Fourier series, USSR Sb. 4(1968), 167
98. Sjölin, Per (1971), "Convergence almost everywhere of certain singular integrals and multiple Fourier series", Arkiv für Matematik 9 (1-2): 65-90,
99. E.M. Stein, S. Wainger. *Discrete analogies of singular Radon transforms*, Bull. Amer. Math. Soc **23** (1990), 537-534.
100. W.H.F. Talbot. *Facts relating to optical sciences.* No. IV, Philosophical Magazine **9**, no. 56 (1836), pp. 401 – 407.
101. P.L. Ulyanov. *Some problems in the theory of orthogonal and biorthogonal series.* Izv. Akad. Nauk Azerb. SSR, Ser. Fiz.-Tekhn. Mat. Nauk (1965), no. 6, pp. 11 – 13.
102. I.M. Vinogradov. *The method of trigonometric sums in the theory of numbers.* Translated by A. Davenport and K.F. Roth. Interscience, London, 1954.