Bull. London Math. Soc. 37 (2005) 427–434 © 2005 London Mathematical Society
doi:10.1112/S0024609304004035

INTERPLAY BETWEEN DISTRIBUTIONAL ESTIMATES AND BOUNDEDNESS IN HARMONIC ANALYSIS

DMITRIY BILYK AND LOUKAS GRAFAKOS

Abstract

In this paper, a proof is given that certain boundedness properties of operators yield distributional estimates that have exponential decay at infinity. Such distributional estimates imply local exponential integrability, and apply to many operators such as m-linear Calderón–Zygmund operators and their maximal counterparts.

1. *Introduction and the main result*

It is a classical result that the Hilbert transform H and its maximal counterpart H_*, defined for functions f on the line by the identities

$$H(f)(x) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{|t|\geqslant\varepsilon} f(x-t)\,\frac{dt}{t}, \qquad H_*(f)(x) = \sup_{\varepsilon>0} \frac{1}{\pi}\left| \int_{|t|\geqslant\varepsilon} f(x-t)\,\frac{dt}{t}\right|,$$

satisfy, for all measurable sets F of finite measure, the distributional estimates

$$\big|\{|H(\chi_F)|>\lambda\}\big| + \big|\{|H_*(\chi_F)|>\lambda\}\big| \leqslant C\,|F| \begin{cases} \lambda^{-1} & \text{when } \lambda<1, \\ e^{-c\lambda} & \text{when } \lambda\geqslant 1, \end{cases} \tag{1}$$

for a pair of constants C and c. For a proof of this result we refer to the book by Garsia [**1**], in which explicit properties of the kernel $1/t$ of H are exploited.

In this paper, we show that distributional estimates of the type (1) hold for a variety of linear (and sublinear) operators that may not have the rich structure of the Hilbert transform. In fact, we prove that any linear operator of restricted weak type $(1,1)$, whose adjoint is also of restricted weak type $(1,1)$, must satisfy the distributional estimate (1), provided that it has a bounded kernel, or can be written as a pointwise limit of linear operators with bounded kernels. Our results also apply to m-linear operators that are of restricted weak type $(1,\dots,1,1/m)$ and whose adjoints have the same property. Extensions of this result to certain maximal operators are also obtained.

We will be working with a multilinear operator T, defined on the m-fold product of spaces of measurable functions on measure spaces (X_j,μ_j) that contain the simple functions. We assume that T takes values in the set of measurable functions on another measure space (X,μ). We denote by T^{*j}, the adjoint with respect to the jth variable, where $j\in\{1,2,\dots,m\}$. The operator T^{*j} satisfies

$$\int g\,T(f_1,\dots,f_m)\,d\mu = \int f_j\,T^{*j}(\dots,f_{j-1},g,f_{j+1},\dots)\,d\mu_j$$

Received 21 October 2003; revised 18 May 2004.

2000 *Mathematics Subject Classification* 46B70, 42B99.

Both authors were supported by the National Science Foundation under grant DMS 0099881.

for all functions $f_1, \ldots, f_m$, g in the corresponding domains; an implicit assumption is that all integrals converge absolutely. We also set $T^{*0} = T$.

We say that a multilinear operator T is of *restricted weak type* $(p_1, \ldots, p_m, q)$ if there is a positive constant A such that for all measurable sets $F_1, \ldots, F_m$ of finite measure, we have

$$\left\|T(\chi_{F_1}, \ldots, \chi_{F_m})\right\|_{L^{q,\infty}} \leqslant A\mu_1(F_1)^{1/p_1} \ldots \mu_m(F_m)^{1/p_m}. \tag{2}$$

Here, $\|g\|_{L^{q,\infty}} = \sup_{\lambda>0} \lambda|\{|g| > \lambda\}|^{1/q}$, and the smallest constant A such that (2) is satisfied for all sets $F_1, \ldots, F_m$ is called the *restricted weak type* $(p_1, \ldots, p_m, q)$ *constant of* T.

We have the following result.

THEOREM 1.1. *Suppose that for some $p_k \geqslant 1$ $(k = 1, \ldots, m)$, the operator T is of restricted weak type $(p_1, \ldots, p_m, q)$, where q satisfies $1/q = 1/p_1 + \ldots + 1/p_m$.*

Suppose also that for all $j = 1, \ldots, m$, the operator T^{*j} is of restricted weak type $(p_{1,j}, \ldots, p_{m,j}, q_j)$, where $1/q_j = 1/p_{1,j} + \ldots + 1/p_{m,j}$, $p_{k,j} \geqslant 1$, and $p_{j,j} = 1$.

Finally, suppose that T maps $L^{\alpha p_1} \times \ldots \times L^{\alpha p_m}$ to $L^{\alpha q}$ for some $\alpha \geqslant q^{-1}$.

Then there are constants C and c (depending on the previous indices, T and m) such that for all measurable sets $F_1, \ldots, F_m$ of finite measure, we have

$$\mu\big(\{\left|T\big(\chi_{F_1}, \ldots, \chi_{F_m}\big)\right| > \lambda\}\big) \\ \leqslant C\big(\mu_1(F_1)^{1/p_1} \ldots \mu_m(F_m)^{1/p_m}\big)^q \begin{cases} \lambda^{-q} & \text{when } \lambda < 1, \\ e^{-c\lambda} & \text{when } \lambda \geqslant 1. \end{cases} \tag{3}$$

REMARK 1.2. In many cases, the assumption that T maps $L^{\alpha p_1} \times \ldots \times L^{\alpha p_m}$ into $L^{\alpha q}$ can be removed. For example, if T has a bounded kernel, this condition follows from the restricted weak type conditions on T^{*j} via the multilinear interpolation theorem given in [**5**]. The same conclusion follows for operators that can be written as a limit of a sequence of operators with bounded kernels.

Theorem 1.1 is motivated by the properties of the bilinear Hilbert transform that satisfies the restricted weak type assumptions modulo some logarithmic factors. A similar conclusion is valid for this operator as well (see the subsequent work of the authors). Setting all exponents p_k, $p_{k,j}$ equal to 1, we obtain the following important corollary.

COROLLARY 1.3. *Suppose that for $j = 0, 1, \ldots m$, the operator T^{*j} is of restricted weak type $(1, \ldots, 1, 1/m)$. Suppose also that T maps $L^{mq} \times \ldots \times L^{mq}$ to L^q for some $q \geqslant 1$. Then there are constants C and c, such that for all measurable sets $F_1, \ldots, F_m$ of finite measure, we have*

$$\mu\big(\{\left|T\big(\chi_{F_1}, \ldots, \chi_{F_m}\big)\right| > \lambda\}\big) \\ \leqslant C\big(\mu_1(F_1) \ldots \mu_m(F_m)\big)^{1/m} \begin{cases} \lambda^{-1/m} & \text{when } \lambda < 1, \\ e^{-c\lambda} & \text{when } \lambda \geqslant 1. \end{cases} \tag{4}$$

REMARK 1.4. As in the proof of Theorem 1.1, the assumption that T maps $L^{mq} \times \ldots \times L^{mq}$ to L^q can be dropped if T is assumed to have a bounded kernel, or if it can be written as a limit of a sequence of operators with bounded kernels; see [**5**].

The distributional estimates (4) imply that T^{*j} is bounded from $L^{pm} \times \ldots \times L^{pm}$ to L^p for all indices $1/m < p < \infty$. It follows by duality and standard multilinear interpolation [**2**, **6**], that T is bounded from $L^{p_1} \times \ldots \times L^{p_m}$ to L^p whenever $1 < p_1, \ldots, p_m < \infty$ and $p^{-1} = p_1^{-1} + \ldots + p_m^{-1}$. In this case, therefore, Corollary 1.3 recovers and strengthens the result given in [**5**].

In particular, in the linear case (that is, where $m = 1$), Corollary 1.3 implies that estimate (4) holds for the Hilbert transform and other self-adjoint (or skew adjoint) singular integrals that are of weak type $(1,1)$. In Section 3, we extend Corollary 1.3 for certain maximal singular integral operators.

2. *The proof of Theorem* 1.1

For simplicity, we denote the measure of any set S that appears in what follows by $|S|$, on the understanding that this may be either $\mu_j(S)$ or $\mu(S)$, depending on the context. Let T be as in the statement of Theorem 1.1. We first prove the following lemma.

Lemma 2.1. *There is a constant C_1 such that for all measurable sets of finite measure $E, F_1, \ldots, F_m$, there is a subset S of E such that $|S| \geqslant \frac{1}{2}|E|$ and*

$$\left| \int_S T(\chi_{F_1}, \ldots, \chi_{F_m})\, d\mu \right| \leqslant C_1\, |E|^{1-1/q} |F_1|^{1/p_1} \ldots |F_m|^{1/p_m}.$$

Proof. Define

$$\Omega = \left\{ x : |T(\chi_{F_1}, \ldots, \chi_{F_m})(x)| > 2^{1/q} A \frac{|F_1|^{1/p_1} \ldots |F_m|^{1/p_m}}{|E|^{1/q}} \right\},$$

where A is the restricted weak $(p_1, \ldots, p_m, q)$ constant of T. Then $|\Omega| < \frac{1}{2}|E|$. Now we set $S = E \setminus \Omega$. We have $|S| \geqslant \frac{1}{2}|E|$, and also

$$\begin{aligned} \left| \int_S T(\chi_{F_1}, \ldots, \chi_{F_m})\, d\mu \right| &\leqslant 2^{1/q} A \frac{|F_1|^{1/p_1} \ldots |F_m|^{1/p_m}}{|E|^{1/q}}\, |E| \\ &= C_1\, |E|^{1-1/q} |F_1|^{1/p_1} \ldots |F_m|^{1/p_m}. \end{aligned}$$

□

Lemma 2.2. *There is a constant C_2 such that for all measurable sets of finite measure $E, F_1, \ldots, F_m$ that satisfy $|E|^{1/q} \leqslant |F_1|^{1/p_1} \ldots |F_m|^{1/p_m}$, we have*

$$\left| \int_E T(\chi_{F_1}, \ldots, \chi_{F_m})\, d\mu \right| \leqslant C_2\, |E| \left(1 + \log \frac{|F_1|^{1/p_1} \ldots |F_m|^{1/p_m}}{|E|^{1/q}} \right).$$

Proof. Let us denote $F_i^{(0)} = F_i$ for $i = 1, \ldots, m$. We now proceed inductively. At the jth step, we choose the index k_j such that $|F_{k_j}^{(j)}| = \max(|F_1^{(j)}|, \ldots, |F_m^{(j)}|)$. By Lemma 2.1 applied to T^{*k_j} for exponents $p_{1,k_j}, \ldots, p_{m,k_j}, q_{k_j}$ with the roles of E and F_{k_j} interchanged, we can choose $S_{k_j}^{(j)} \subset F_{k_j}^{(j)}$ such that $|S_{k_j}^{(j)}| \geqslant \frac{1}{2}|F_{k_j}^{(j)}|$ and

$$\begin{aligned} \left| \int_{S_{k_j}^{(j)}} T^{*k_j}(\chi_{F_1}, \ldots, \chi_E, \ldots \chi_{F_m})\, d\mu_{k_j} \right| &\leqslant C\, \frac{|E| \prod_{i \neq k_j} |F_i^{(j)}|^{1/p_{i,k_j}}}{|F_{k_j}^{(j)}|^{1/q_{k_j} - 1}} \\ &\leqslant C\, |E|. \end{aligned}$$

We now define

$$F_i^{(j+1)} = F_i^{(j)} \setminus S_i^{(j)} \qquad \text{for all } i = 1, \dots, m,$$

where we set $S_i^{(j)} = \emptyset$ for $i \neq k_j$. We proceed by induction, and we stop at the first integer n such that $|E|^{1/q} \geqslant |F_1^{(n)}|^{1/p_1} \dots |F_m^{(n)}|^{1/p_m}$. (Such an integer always exists, since the quantity $|F_1^{(j)}|^{1/p_1} \dots |F_m^{(j)}|^{1/p_m}$ gets smaller by at least a factor of $(\frac{1}{2})^{1/\max p_i}$ when j is replaced by $j+1$.) Obviously, the number of steps n is at most

$$C\left(1 + \log \frac{|F_1|^{1/p_1} \dots |F_m|^{1/p_m}}{|E|^{1/q}}\right).$$

We now have the sequence of estimates

$$\begin{aligned}
&\left|\int_E T\big(\chi_{F_1}, \dots, \chi_{F_m}\big)\, d\mu\right| \\
&\qquad = \left|\int_E T\big(\dots, \chi_{S_{k_0}^{(0)}} + \chi_{F_{k_0}^{(1)}}, \dots\big)\, d\mu\right| \\
&\qquad \leqslant \left|\int_{S_{k_0}^{(0)}} T^{*k_0}(\dots, \chi_E, \dots)\, d\mu_{k_0}\right| + \left|\int_E T\big(\chi_{F_1^{(1)}}, \dots, \chi_{F_m^{(1)}}\big)\, d\mu\right| \\
&\qquad \leqslant C\,|E| + \left|\int_E T\big(\chi_{F_1^{(1)}}, \dots, \chi_{F_m^{(1)}}\big)\, d\mu\right|.
\end{aligned}$$

Writing $\chi_{F_{k_1}^{(1)}}$ as $\chi_{S_{k_1}^{(1)}} + \chi_{F_{k_1}^{(2)}}$ and applying this argument $n-1$ more times, we see that the previous expression is controlled by

$$\begin{aligned}
&n\,C\,|E| + \left|\int_E T\big(\chi_{F_1^{(n)}}, \dots, \chi_{F_m^{(n)}}\big)\, d\mu\right| \\
&\qquad \leqslant C\,|E|\left(1 + \log \frac{|F_1|^{1/p_1} \dots |F_m|^{1/p_m}}{|E|^{1/q}}\right) \\
&\qquad\quad + \|T\|_{(\alpha p_1, \dots, \alpha p_m, \alpha q)} \left|F_1^{(n)}\right|^{1/\alpha p_1} \dots \left|F_m^{(n)}\right|^{1/\alpha p_m} |E|^{1/(\alpha q)'} \\
&\qquad \leqslant C_2\,|E|\left(1 + \log \frac{|F_1|^{1/p_1} \dots |F_m|^{1/p_m}}{|E|^{1/q}}\right),
\end{aligned}$$

where in the second and third lines from the bottom we have used the Hölder inequality and the fact that T is of strong type $(\alpha p_1, \dots, \alpha p_m, \alpha q)$. $\square$

Combining Lemmata 2.1 and 2.2, we obtain the following corollary.

Corollary 2.3. *There is a constant C_3 such that for all $E, F_1, \dots, F_m$ measurable sets of finite measure, there is a subset $S = S_{E,F_1,\dots,F_m}$ of E with $|S| \geqslant \frac{1}{2}|E|$ such that*

$$\begin{aligned}
&\left|\int_S T(\chi_{F_1}, \dots, \chi_{F_m})\, d\mu\right| \\
&\qquad \leqslant C_3\,|E| \min\left(1, \frac{|F_1|^{1/p_1} \dots |F_m|^{1/p_m}}{|E|^{1/q}}\right)\left(1 + \log^+ \frac{|F_1|^{1/p_1} \dots |F_m|^{1/p_m}}{|E|^{1/q}}\right).
\end{aligned}$$

We are now ready to prove the distributional estimate (3).

For a given $\lambda > 0$, we set

$$
\begin{aligned}
E_\lambda^1 &= \{\mathrm{Re}\, T(\chi_{F_1}, \dots, \chi_{F_m}) > \lambda\};\\
E_\lambda^2 &= \{\mathrm{Re}\, T(\chi_{F_1}, \dots, \chi_{F_m}) < -\lambda\};\\
E_\lambda^3 &= \{\mathrm{Im}\, T(\chi_{F_1}, \dots, \chi_{F_m}) > \lambda\};\\
E_\lambda^4 &= \{\mathrm{Im}\, T(\chi_{F_1}, \dots, \chi_{F_m}) < -\lambda\}.
\end{aligned}
$$

We shall prove the required estimate for a fixed E_λ^j. Suppose that $|E_\lambda^j|^{1/q} \geqslant |F_1|^{1/p_1} \dots |F_m|^{1/p_m}$. Then by Corollary 2.3 there is a subset S_λ^j of E_λ^j of at least half its measure, so that

$$
\frac{\lambda}{2}|E_\lambda^j| \leqslant \lambda |S_\lambda^j| \leqslant \left| \int_{S_\lambda^j} T\big(\chi_{F_1}, \dots, \chi_{F_m}\big)\, d\mu \right| \leqslant C_3 \,\frac{|F_1|^{1/p_1} \dots |F_m|^{1/p_m}}{\big|E_\lambda^j\big|^{1/q-1}},
$$

which implies that

$$
\big|E_\lambda^j\big| \leqslant (2C_3)^q \big(|F_1|^{1/p_1} \dots |F_m|^{1/p_m}\big)^q \lambda^{-q}.
$$

This in turn implies that if $\lambda > 2C_3$, we must have $|E_\lambda^j|^{1/q} \leqslant |F_1|^{1/p_1} \dots |F_m|^{1/p_m}$. In this case, Corollary 2.3 gives

$$
\frac{\lambda}{2}\big|E_\lambda^j\big| \leqslant C_3 \big|E_\lambda^j\big| \left(1 + \log \frac{|F_1|^{1/p_1} \dots |F_m|^{1/p_m}}{\big|E_\lambda^j\big|^{1/q}}\right),
$$

from which one easily deduces that $|E_\lambda^j| \leqslant C\, e^{-c\lambda}(|F_1|^{1/p_1} \dots |F_m|^{1/p_m})^q$. Summing over $j = 1, 2, 3, 4$, we deduce the required conclusion with a constant four times as large.

3. *Extensions to (multi)sublinear operators*

Next, we prove the following extension of Corollary 1.3 for operators that may be sublinear in each variable. Our setting here will be $\mathbb{R}^n$ (endowed with Lebesgue measure), and M will denote the Hardy–Littlewood maximal operator.

THEOREM 3.1. *Suppose that a positive sublinear operator T_* satisfies the following Cotlar-type inequality:*

$$
T_*(f_1, \dots, f_m) \leqslant A\left[M(T(f_1, \dots, f_m)) + \prod_{j=1}^m M(f_j)\right]
$$

for some operator T that satisfies estimate (3). *Then there exist constants $C_*, c_* > 0$ such that for $\lambda > 1$,*

$$
\big|\big\{T_*\big(\chi_{F_1}, \dots, \chi_{F_m}\big) > \lambda\big\}\big| \leqslant C_*\, e^{-c_*\lambda}\big(|F_1|^{1/p_1} \dots |F_m|^{1/p_m}\big)^q.
$$

Proof. Obviously, it is enough to show that $M(T(\chi_{F_1}, \dots, \chi_{F_m}))$ satisfies the required distributional estimate, since $M(\chi_{F_j}) \leqslant 1$. We denote

$$
\Omega_\lambda = \big\{M\big(T\big(\chi_{F_1}, \dots, \chi_{F_m}\big)\big) > \lambda\big\} \qquad \text{and} \qquad f = T\big(\chi_{F_1}, \dots, \chi_{F_m}\big),
$$

and we set

$$
E_j = \{x : 2^{j-1}\lambda < |f(x)| \leqslant 2^j \lambda\} \qquad \text{for } j \geqslant 0.
$$

By our assumption, we have

$$
|E_j| \leqslant C\, e^{-c2^j\lambda}\big(|F_1|^{1/p_1} \dots |F_m|^{1/p_m}\big)^q.
$$

We claim that there exists a constant $B > 0$, such that for all $x \in \Omega_\lambda$ there exist an integer $k \geqslant 0$ and a ball I containing x with the property that

$$|I \cap E_k| \geqslant B2^{-2k}|I|. \tag{5}$$

Indeed, if this were not the case, then for any ball I containing x, we would have

$$\begin{aligned}\frac{1}{|I|}\int_I |f(z)|dz &= \frac{1}{|I|}\int_{I\cap\{|f|\leqslant\lambda/2\}} |f(z)|dz + \sum_{j=0}^{\infty}\frac{1}{|I|}\int_{I\cap E_j}|f(z)|dz \\ &\leqslant \frac{\lambda}{2} + B\sum_{j=0}^{\infty} 2^{-2j}2^{j+1}\lambda \\ &= \lambda\big(\tfrac{1}{2}+4B\big) \\ &< \tfrac{3}{4}\lambda\end{aligned}$$

for $B < \frac{1}{16}$. However, this would imply that $M(f)(x) < \lambda$ and that $x \notin \Omega_\lambda$, a contradiction.

For each $x \in \Omega_\lambda$, we denote by k_x the smallest k for which (5) holds, and we set

$$\Omega_\lambda^k = \{x \in \Omega_\lambda \colon k_x = k\}.$$

It is easy to see that

$$\Omega_\lambda^k \subset \{M(\chi_{E_k}) \geqslant B2^{-2k}\}.$$

Thus, using weak type (1,1) property of M, we obtain

$$\big|\Omega_\lambda^k\big| \leqslant B'2^{2k}|E_k| \leqslant B''2^{2k-c'2^k\lambda}\big(|F_1|^{1/p_1}\dots|F_m|^{1/p_m}\big)^q.$$

Now the required estimate for $\lambda \geqslant 1$ is obtained by summing the series

$$|\Omega_\lambda| = \sum_{j=0}^{\infty}|\Omega_\lambda^j|. \qquad \square$$

4. *Applications to m-linear Calderón–Zygmund operators*

Bounded m-linear operators from $L^{p_1}(\mathbb{R}^n)\times\dots\times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ (for some exponents $1 < p_1,\dots,p_m < \infty$ with $p^{-1} = p_1^{-1}+\dots+p_m^{-1}$) are called *multilinear Calderón–Zygmund* if they have the form

$$T(f_1,\dots,f_m)(x) = \int_{(\mathbb{R}^n)^m} K(x,y_1,\dots,y_m)f_1(y_1)\dots f_m(y_m)\,dy_1\dots dy_m, \tag{6}$$

for some distributional kernel $K(x,y_1,\dots,y_m)$ that coincides with a function defined away from the diagonal $x = y_1 = y_2 = \dots = y_m$ that satisfies the size estimate

$$|K(y_0,y_1,\dots,y_m)| \leqslant \frac{A}{\big(\sum_{k,l=0}^m |y_k-y_l|\big)^{mn}}, \tag{7}$$

and, for some $\epsilon > 0$, the regularity condition

$$|K(y_0,\dots,y_j,\dots,y_m) - K(y_0,\dots,y_j',\dots,y_m)| \leqslant \frac{A|y_j-y_j'|^\epsilon}{\big(\sum_{k,l=0}^m |y_k-y_l|\big)^{mn+\epsilon}}, \tag{8}$$

whenever $0 \leqslant j \leqslant m$ and $|y_j - y_j'| \leqslant \frac{1}{2}\max_{0\leqslant k\leqslant m}|y_j-y_k|$.

In view of a result given in [**3**], an m-linear Calderón–Zygmund operator T must be bounded from the product $L^1 \times \ldots \times L^1$ to $L^{1/m,\infty}$. As the properties of the kernel K are symmetric in all variables, it follows that for any j between 1 and m we also have $T^{*j} \colon L^1 \times \ldots \times L^1 \longrightarrow L^{1/m,\infty}$. Thus multilinear Calderón–Zygmund operators satisfy the hypotheses of Corollary 1.3. It follows that they must also satisfy the distributional estimates (4).

Next we show that the maximal multilinear Calderón–Zygmund operators also satisfy the distributional estimates (4). We define the *maximal truncated operator* as

$$T_*(f_1, \ldots, f_m) = \sup_{\delta>0} |T_\delta(f_1, \ldots, f_m)|,$$

where we set

$$\begin{aligned} &T_\delta(f_1, \ldots, f_m)(x) \\ &\qquad = \int_{|x-y_1|^2+\ldots+|x-y_m|^2<\delta^2} K(x, y_1, \ldots, y_m) f_1(y_1) \ldots f_m(y_m)\, dy_1 \ldots dy_m. \end{aligned}$$

It was proved in [**4**] that T_* satisfies the pointwise estimate

$$T_*(f_1, \ldots, f_m) \leqslant C_\eta \left[(M(|T(f_1, \ldots, f_m)|^\eta))^{1/\eta} + \prod_{j=1}^m M(f_j) \right] \tag{9}$$

for some $C_\eta > 0$, whenever $0 < \eta < \infty$.

Using Theorem 3.1, we therefore deduce the following conclusion.

Proposition 4.1. *If T is a multilinear Calderón–Zygmund operator, then T_* satisfies the distributional estimate* (4).

Proof. The estimate for $\lambda < 1$ follows from the weak type $(1, \ldots, 1, 1/m)$ property of T_* (see [**4**]); the estimate for $\lambda > 1$ follows from Theorem 3.1 and inequality (9) with $\eta = 1$. □

The next corollary is an immediate consequence of the results just obtained. Naturally, the same conclusion applies to any operator that satisfies the hypotheses of Theorem 1.1 or Theorem 3.1 accordingly.

Corollary 4.2. *There is a constant $c_1 > 0$ so that for any multilinear Calderón–Zygmund operator T, for any ball B, and for any tuple of measurable sets $F_1, \ldots, F_m$ of finite measure, we have*

$$\int_B e^{c_1 T(\chi_{F_1}, \ldots, \chi_{F_m})}\, dx + \int_B e^{c_1 T_*(\chi_{F_1}, \ldots, \chi_{F_m})}\, dx < \infty.$$

5. *Concluding remarks*

One may wonder whether the conclusion of Theorem 1.1 would still be valid if it were assumed that T and its adjoints are bounded on some product of Lebesgue spaces $L^{p_1} \times \ldots \times L^{p_m}$ with all $p_j > 1$. We show that this is not the case, even when $m = 1$.

Consider a linear operator T that maps L^q into $L^{q,\infty}$ for some $1 < q < 2$, and suppose that T^* has the same property. Then both T and T^* are L^p bounded for all $p \in (q, q')$. Suppose, furthermore, that T does not map L^r into itself for any $r \notin [q, q']$. Following the same procedure as that discussed in the proof of Theorem 1.1, we deduce that there is a constant C such that for all sets F of finite measure we have

$$|\{|T(\chi_F)| > \lambda\}| \leqslant C\,|F| \begin{cases} \lambda^{-q} & \text{when } \lambda < 1, \\ \lambda^{-q'}(1 + \log\lambda)^{q'} & \text{when } \lambda \geqslant 1, \end{cases} \tag{10}$$

where $q' = q/(q-1)$. It is clear that the term $\lambda^{-q'}(1 + \log\lambda)^{q'}$ cannot be replaced by a term of the form $e^{-c\lambda}$, as this would imply that T is bounded on L^p for all $p > q$, which we assume is not the case.

References

1. A. Garsia, *Topics in almost everywhere convergence*, Lectures in Adv. Math. 4 (Markham Publishing Co., Chicago, IL, 1970).

2. L. Grafakos and N. Kalton, 'Some remarks on multilinear maps and interpolation', *Math. Ann.* 319 (2001) 151–180.

3. L. Grafakos and R. Torres, 'Multilinear Calderón–Zygmund theory', *Adv. Math.* 165 (2002) 124–164.

4. L. Grafakos and R. Torres, 'Maximal operator and weighted norm inequalities for multilinear singular integrals', *Indiana Univ. Math. J.* 51 (2002) 1261–1276.

5. L. Grafakos and T. Tao, 'Multilinear interpolation between adjoint operators', *J. Funct. Anal.* 199 (2003) 379–385.

6. S. Janson, 'On interpolation of multi-linear operators', *Function spaces and applications*, Proceedings, Lund 1986, Lecture Notes in Math. 1302 (ed. M. Cwikel, J. Peetre, Y. Sagher and H. Wallin, Springer, New York, 1988) 290–302.

Dmitriy Bilyk and Loukas Grafakos
Department of Mathematics
University of Missouri
Columbia, MO 65211
USA

bilykd@math.missouri.edu
loukas@math.missouri.edu