DISCRETENESS of MINIMIZING MEASURES

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Discrete energy

$$F : [-1, 1] \longrightarrow \mathbb{R}$$

Let $Z = \{z_1, \dots, z_N\} \subset S^{d-1} \subset \mathbb{R}^d$

Discrete energy:

$$E_F(Z) = \frac{1}{N^2} \sum_{i,j=1}^{N} F(z_i \cdot z_j)$$

Discrete energy and energy integral

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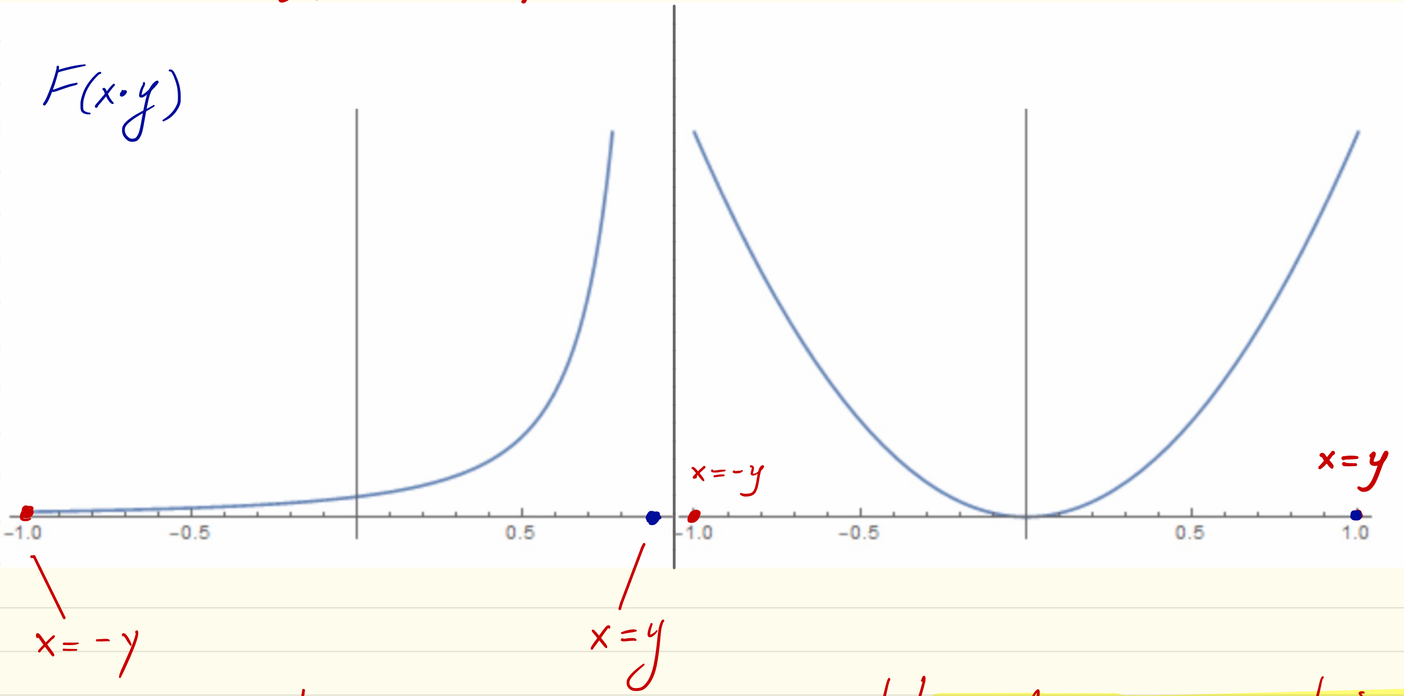
$$E_F(Z) = I_F\left(\frac{1}{N} \sum_{z \in Z} \delta_z\right)$$

TWO TYPES of POTENTIALS



repulsive

attractive - repulsive

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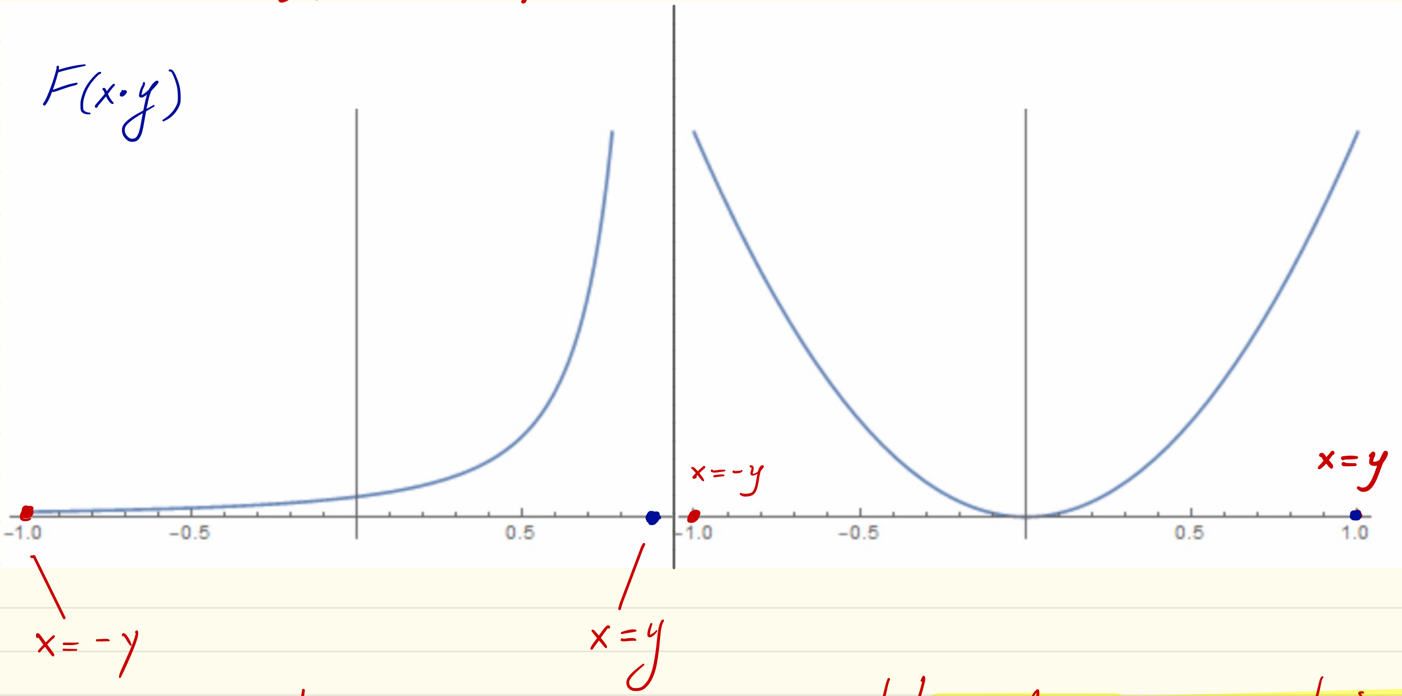

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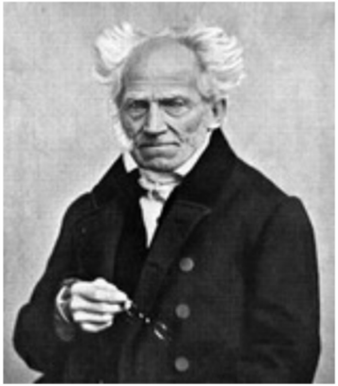
"gregarious"

"solitarious"

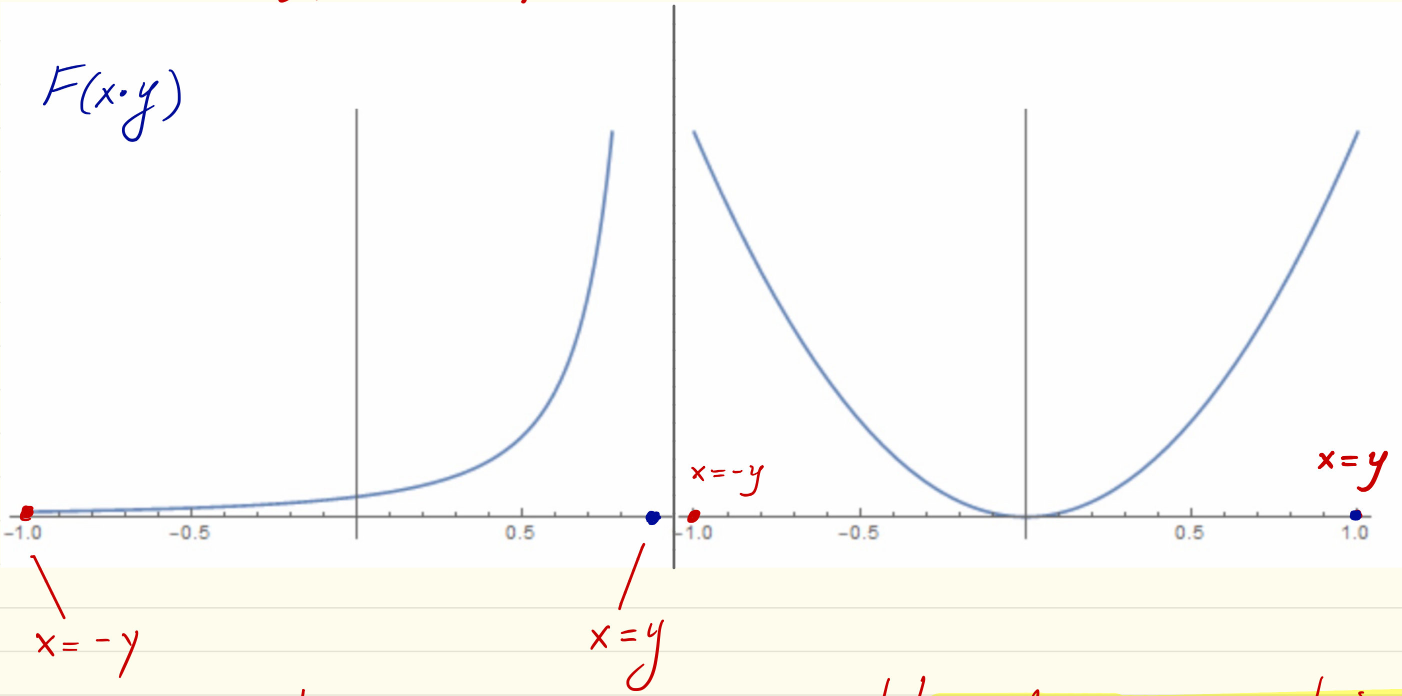
Arthur Schopenhauer > Quotes



"A number of porcupines huddled together for warmth on a cold day in winter; but, as they began to prick one another with their quills, they were obliged to disperse. However the cold drove them together again, when just the same thing happened. At last, after many turns of huddling and dispersing, they discovered that they would be best off by remaining at a little distance from one another. In the same way the need of society drives the human porcupines together, only to be mutually repelled by the many prickly and disagreeable qualities of their nature. The moderate distance which they at last discover to be the only tolerable condition of intercourse, is the code of politeness and fine manners; and those who transgress it are roughly told—in the English phrase—to keep their distance. By this arrangement the mutual need of warmth is only very moderately satisfied; but then people do not get pricked. A man who has some heat in himself prefers to remain outside, where he will neither prick other people nor get pricked himself."

― **Arthur Schopenhauer, Parerga and Paralipomena**

TWO TYPES of POTENTIALS



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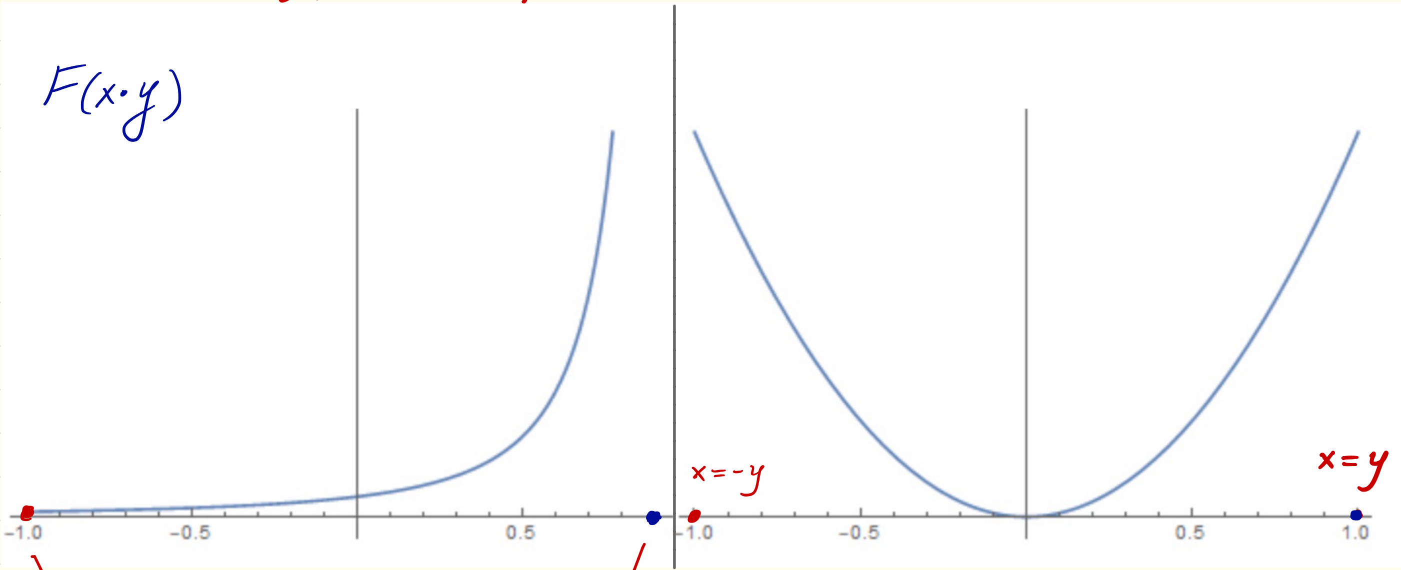

$x=-y$ $\qquad$ $x=y$

repulsive

attractive - repulsive

"orthogonalizing"

$$\min F(t) = F(0)$$

$$F(t) = F(-t)$$

When does the uniform surface measure σ minimize I_F ???

$F \in C[-1,1]$ is positive definite on $\mathbb{S}^{d-1}$

iff

(i) For any $Z = \{z_1, \dots, z_N\} \subset \mathbb{S}^{d-1}$ the matrix $[F(z_i \cdot z_j)]_{i,j=1}^{N}$ is positive semidefinite.

(ii) For any signed Borel measure ν: $I_F(\nu) \geq 0$

Schoenberg: (iii)
$$F(t) = \sum_{n=0}^{\infty} a_n \underbrace{C_n^{\frac{d-2}{2}}(t)}_{\text{Gegenbauer polynomials}}, \quad a_n \geq 0$$

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(iv) For each Borel probability measure μ:

$$I_F(\mu) \geq I_F(\sigma) \geq 0$$

Positive definiteness and energy minimization

Lemma: Let $I_F(\mu_{\min}) = \min I_F(\mu) \geq 0$.

Then F is positive definite on $\operatorname{supp} \mu_{\min}$.

Positive definiteness and energy minimization

Lemma: Let $I_F(\mu_{MIN}) = \min I_F(\mu) \geq 0$.

Then F is positive definite on $\operatorname{supp} \mu_{MIN}$.

Corollary: If σ does NOT minimize I_F on $\mathbb{S}^{d-1}$,

then $\operatorname{supp} \mu_{MIN} \subsetneq \mathbb{S}^{d-1}$.

Gegenbauer coefficients and energy minimization

$$F(t) = \sum_{n=0}^{\infty} a_n \underbrace{C_n^{\frac{d-2}{2}}(t)}_{\text{Gegenbauer polynomials}},$$

- $a_n \geq 0$ for all $n \geq 1$ $\iff$ σ is a minimizer
- $a_n > 0$ for all $n \geq 1$ $\iff$ σ is the UNIQUE minimizer
- $a_n \leq 0$ for all $n \geq 1$ $\implies$ δ_z is a minimizer
  discrete
- $(-1)^{n+1} a_n \geq 0$ $\implies$ $\frac{1}{2}(\delta_z + \delta_{-z})$ is a minimizer
- $a_{2n} = 0$, $a_{2n+1} \geq 0$ $\implies$ every centrally symmetric measure is a minimizer (there exist discrete minimizers)
- $a_n \geq 0$ and $a_n = 0$ for $n \geq n_0$ $\implies$ there exist discrete minimizers: (weighted) spherical designs
  (F is a p.d. polynomial)

EXAMPLES in $\mathbb{R}^d$

$$\mathcal{E}_W(\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(\|x-y\|)\, d\mu(x)\, d\mu(y)$$

Carillo – Figalli – Patacchini (2017):

Let $W(0) = 0$; $\left.\begin{array}{ll} W(r) < 0 & \text{for } 0 < r < R \\ W(r) > 0 & \text{for } r > R \end{array}\right\}$ attractive-repulsive

Let $W(r) \approx -r^\alpha$ for r-small, $\alpha > 2$ (weak repulsion)

If μ_{MIN} a global minimizer of $\mathcal{E}_W$,

then μ_{MIN} is discrete (finite support).

EXAMPLES in $\mathbb{R}^d$

$$\mathcal{E}_W(\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(\|x-y\|)\, d\mu(x)\, d\mu(y)$$

Lennart – Jones potential

$$W(r) = \frac{r^\beta}{\beta} - \frac{r^\alpha}{\alpha}, \quad \beta \geq \alpha$$

(attractive – repulsive)

Lim – McCann, 2019:

Let $\alpha \geq 2$.
For all $\beta \geq \alpha$ sufficiently large

the only minimizers of $\mathcal{E}_W$ are of the form $\mu_{MIN} = \frac{1}{d+1} \sum_{z \in C_d} \delta_z$,
where C_d is a unit **d-simplex**.

$$F(x \cdot y) = \|x - y\|^{\alpha}, \quad \underline{\alpha > 0}$$

Euclidean distance

Björck:

	α	μ_{MAX}
(i)	$0 < \alpha < 2$	σ
(ii)	$\alpha = 2$	any baricentric measure
(iii)	$\alpha > 2$	$\frac{1}{2}(\delta_p + \delta_{-p})$

$$F(x \cdot y) = \underbrace{(d(x,y))}_{\text{geodesic distance}}{}^{\alpha}, \quad \underline{\alpha > 0}$$

DB, Dai, Matzke: α $\qquad\qquad$ μ_{MAX}

(i)	$0 < \alpha < 1$	σ
(ii)	$\alpha = 1$	any centrally symmetric measure
(iii)	$\alpha > 1$	$\frac{1}{2}(\delta_p + \delta_{-p})$

SPHERICAL DESIGNS

If $\quad F(t) = \sum_{n=0}^{m} a_n \, C_n^{\frac{d-2}{2}}(t), \qquad a_n \geq 0,$

is a polynomial, positive definite on S^{d-1},

then there exist discrete minimizers:

if $Z = \{z_1, \dots, z_N\}$ - spherical m-design

then $\quad I_F\left(\frac{1}{N}\sum \delta_{z_i}\right) = I_F(\sigma) = \min I_F(\mu)$

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More generally, discrete minimizers are exactly weighted m- designs, if $a_n > 0$,

i.e. $\quad I_F\left(\sum w_i \delta_{z_i}\right) = I_F(\sigma)$

Generalization

Theorem: DB, Glazyrin, Matzke, Park, Vlasiuk

Let $F(t) = \sum_{n=0}^{\infty} a_n C_n^{\frac{d-2}{2}}(t)$ with only finitely many $\underline{a_n > 0}$.

Then there exists a discrete minimizer μ_{MIN} of $I_F(\mu)$ with

$$\boxed{\#(\operatorname{supp} \mu_{MIN}) \leq 1 + \sum_{\substack{n \geq 1 \\ a_n > 0}} \dim \mathcal{H}_n^d}$$

the space of spherical harmonics of degree n.

FRAME POTENTIAL: $F(x\cdot y) = |x\cdot y|^2$

Benedetto-Fickus: $Z = \{z_1, \dots, z_N\} \subset S^{d-1}$

is a (local) MINIMIZER of $E_F(Z)$ with

$F(t) = |t|^2$ if and only if

Z is a unit norm tight frame (UNTF)

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Z is a unit norm tight frame (UNTF)

i.e. $\forall x \in \mathbb{R}^d$:

$$x = \frac{d}{N} \sum_i \langle x, z_i \rangle z_i$$

$$\Updownarrow$$

$$\|x\|^2 = \frac{d}{N} \sum |\langle x, z_i \rangle|^2$$

FRAME POTENTIAL: $F(x\cdot y) = |x\cdot y|^2$

$$I_F(\mu) = \int_{S^{d-1}} \int_{S^{d-1}} |x\cdot y|^2 \, d\mu(x)\, d\mu(y)$$

MINIMIZERS:

- uniform measure σ
- isotropic measures:

$$\int_{S^{d-1}} |x\cdot y|^2 \, d\mu(y) = \frac{1}{d} \|x\|^2, \quad \forall x \in \mathbb{R}^d$$

- in particular, all UNTF

  i.e. $\mu = \frac{1}{N} \sum_{i=1}^{N} \delta_{z_i}$, where $Z = \{z_1, \dots, z_N\}$ is a UNTF

- e.g. <u>ONB</u>

p - FRAME ENERGY : $F(t) = |t|^p$

$p \in (0,2)$

The only minimizer of I_F (up to symmetries) is the ONB / crosspolytope:

$$\mu = \frac{1}{2d} \sum_{i,\pm} \delta_{\pm e_i}$$

(Ehler - Okoudjou)

(i.e. only discrete)

p - FRAME ENERGY : $F(t) = |t|^p = t^{2k}$

$p \in 2\mathbb{N}$. $p = 2k$

MINIMIZERS:

- uniform surface measure σ

discrete $\longrightarrow$

- projective k-designs (spherical $2k$-designs)

p - FRAME ENERGY : $F(t) = |t|^p$

0 – 2	2	2 – 4	4	4 – 6	6	> 6
only discrete (ONB)	$\mathcal{G}$ + discrete (UNTFs, ONB)	???	$\mathcal{G}$ + discrete (proj. 2-designs, when exist → {max ETF's, (d+1) MUB's})	???	$\mathcal{G}$ + discrete (projective 3-designs)	???

CONJECTURE :

If $p \notin 2\mathbb{N}$, then all minimizers of I_F with $F(t) = |t|^p$ are

DISCRETE

REMARK: setting $r = \|x-y\|$, $\|x\|=\|y\|=1$

$$|x\cdot y|^p = \left(1-\frac{1}{2}\|x-y\|^2\right)^p$$

$$\simeq 1-\frac{p}{2}\underline{r^2}$$

(So it is in the endpoint case of the Carillo-Figalli-Patacchini: $\alpha = 2$)

KNOWN RESULTS:

Theorem: (DB, Glazyrin, Matzke, Park, Vlasiuk) If there exists a tight spherical $(2t+1)$-design $Z \subset S^{d-1}$, then it is a minimizer of the p-frame energy I_F for

$$2t-2 \leq p \leq 2t.$$

Moreover, in this case, for $p \in (2t-2, 2)$ ALL MINIMIZERS ARE TIGHT DESIGNS.

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Def. $Z \subset S^{d-1}$ is a tight $(2t+1)$-design iff

- symmetric $(2t+1)$ design
- $(t+1)$ distances between distinct points.

KNOWN TIGHT SPHERICAL DESIGNS of odd strength

$d-1$	$\#Z$	Strength	configuration/origin
$d-1$	$2d$	3	cross polytope in $\mathbb{R}^{d+1}$
1	$2k$	$2k-1$	regular polygon
2	12	5	icosahedron
3	120	11	600-cell
6	56	5	kissing configuration of E_8
7	240	7	E_8 root system
22	552	5	equiangular lines
22	4600	7	kissing configuration of Leech lattice
23	196560	11	Leech lattice minimal vectors

KNOWN RESULTS:

Theorem: DB, Glazyrin, Matzke, Park, Vlasiuk

Let $p \notin 2\mathbb{N}$, $p > 0$.

Then EVERY MINIMIZER μ_{MIN} of the p-frame energy I_F satisfies

$$\left(\operatorname{supp} \mu_{MIN}\right)^{\circ} = \emptyset.$$

KNOWN RESULTS:

Theorem: DB, Glazyrin, Matzke, Park, Vlasiuk

Let F be real-analytic, BUT **NOT** positive definite on S^{d-1}

Then EVERY MINIMIZER μ_{MIN} of the energy I_F

satisfies

$$(\operatorname{supp} \mu_{MIN})^{\circ} = \emptyset .$$

— on S^1, minimizers are discrete.

KNOWN RESULTS:

POLYNOMIALS:

Theorem: DB, Glazyrin, Matzke, Park, Vlasiuk

Let F be a polynomial

$$F(t) = \sum_{n=0}^{m} a_n \, C_n^{\frac{d-2}{2}}(t)$$

(i) There exists a discrete minimizer with

$$\#\,(\operatorname{supp} \mu_{MIN}) \leq 1 + \sum_{\substack{n=1 \\ a_n > 0}}^{m} \dim \mathcal{H}_n^d$$

(ii) If σ is NOT a minimizer of I_F
(there exists $n > 0$: $a_n < 0$)

then every minimizer satisfies

$$(\operatorname{supp} \mu_{MIN})^{\circ} = \emptyset.$$

CAUSAL VARIATIONAL PRINCIPLE

$$F(t) = \max\{0,\ 2\tau^2(1+t)(2-\tau^2(1-t))\}$$

Conjecture:

Finster, Schiefeneder

- There exist discrete minimizers when $\tau \geq 1$
- All minimizers are discrete when $\tau \geq 2$.

RESULTS:

- True for two values of τ:
  - CROSSPOLYTOPE
  - ICOSAHEDRON

  (DB, Glazyrin, Matzke, Park, Vlasiuk)
- $(\operatorname{supp} \mu_{min})^{\circ} = \emptyset$

  (FINSTER, Schiefeneder)