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Examples 03

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[03.1] Find a polynomial $P \in \mathbb{Q}[x]$ so that $P(\sqrt{2}+\sqrt{3})=0$.

[03.2] Find a polynomial $P \in \mathbb{Q}[x]$ so that $P(\sqrt{2}+\sqrt[3]{5})=0$.

[03.3] Let α be a root of $x^2+\sqrt{2}\,x+\sqrt{3}=0$ in an algebraic closure of $\mathbb{Q}$. Find $P \in \mathbb{Q}[x]$ so that $P(\alpha)=0$.

[03.4] Let α be a root of $x^5-x+1=0$ in an algebraic closure of $\mathbb{Q}$. Find $P \in \mathbb{Q}[x]$ so that $P(\alpha+\sqrt{2})=0$.

[03.5] Gracefully verify that the octic $x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1$ factors properly in $\mathbb{Q}[x]$.

[03.6] Gracefully verify that the quartic $x^4+x^3+x^2+x+1$ is irreducible in $\mathbb{F}_2[x]$.

[03.7] Gracefully verify that the sextic $x^6+x^5+x^4+x^3+x^2+x+1$ is irreducible in $\mathbb{F}_3[x]$.

[03.8] Gracefully verify that the quartic $x^4+x^3+x^2+x+1$ factors into irreducible quadratics in $\mathbb{F}_{19}[x]$.

[03.9] Let $f(x)=x^6-x^3+1$. Find primes p with each of the following behaviors: f is irreducible in $\mathbb{F}_p[x]$, f factors into irreducible quadratic factors in $\mathbb{F}_p[x]$, f factors into irreducible cubic factors in $\mathbb{F}_p[x]$, f factors into linear factors in $\mathbb{F}_p[x]$.

[03.10] Explain why x^4+1 properly factors in $\mathbb{F}_p[x]$ for any prime p.

[03.11] Explain why $x^8-x^7+x^5-x^4+x^3-x+1$ properly factors in $\mathbb{F}_p[x]$ for any prime p. (*Hint:* It factors either into linear factors, irreducible quadratics, or irreducible quartics.)

[03.12] Why is x^4-2 irreducible in $\mathbb{F}_5[x]$?

[03.13] Why is x^5-2 irreducible in $\mathbb{F}_{11}[x]$?

[03.14] Let k be a field. Determine the units and ideals in the formal power series ring

$$k[[x]] \;=\; \{\sum_{n\ge 0} c_n\, x^n :\ \text{arbitrary } c_n \in k\}$$

[03.15] Let k be a field. Show that the field of fractions of the formal power series ring $k[[x]$ is the collection of *finite-nosed* formal Laurent series

$$k((x)) \;=\; \{\sum_{n\ge -N} c_n\, x^n : \text{arbitrary } c_n \in k,\ \text{arbitrary } N \in \mathbb{Z}\}$$
