(*April 9, 2024*)

Examples 07

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[07.1] Let k be a field of characteristic 0. Let f be an irreducible polynomial in $k[x]$. Prove that f has no repeated factors, even over an algebraic closure of k.

[07.2] Let K be a finite extension of a field k of characteristic 0. Prove that K is separable over k.

[07.3] Let k be a field of characteristic $p > 0$. Suppose that k is **perfect**, meaning that for any $a \in k$ there exists $b \in k$ such that $b^p = a$. Let $f(x) = \sum_i c_i x^i$ in $k[x]$ be a polynomial such that its (algebraic) derivative

$$f'(x) = \sum_i c_i \, i \, x^{i-1}$$

is the zero polynomial. Show that there is a unique polynomial $g \in k[x]$ such that $f(x) = g(x)^p$.

[07.4] Let k be a perfect field of characteristic $p > 0$, and f an irreducible polynomial in $k[x]$. Show that f has no repeated factors (even over an algebraic closure of k).

[07.5] Show that all finite fields $\mathbb{F}_{p^n}$ with p prime and $1 \leq n \in \mathbb{Z}$ are perfect.

[07.6] Let K be a finite extension of a finite field k. Prove that K is separable over k.

[07.7] Find all fields intermediate between $\mathbb{Q}$ and $\mathbb{Q}(\zeta)$ where ζ is a primitive 17^{th} root of unity.

[07.8] Let f, g be *relatively prime* polynomials in n indeterminates $t_1, \ldots, t_n$, with g not 0. Suppose that the ratio $f(t_1, \ldots, t_n)/g(t_1, \ldots, t_n)$ is invariant under all permutations of the t_i. Show that both f and g are polynomials in the elementary symmetric functions in the t_i.
