[15b.1] Let f, g be *relatively prime* polynomials in n indeterminates $t_1, \ldots, t_n$, with g not 0. Suppose that the ratio $f(t_1, \ldots, t_n)/g(t_1, \ldots, t_n)$ is invariant under all permutations of the t_i. Show that both f and g are polynomials in the elementary symmetric functions in the t_i.

Let s_i be the i^{th} elementary symmetric function in the t_j's. Earlier we showed that $k(t_1, \ldots, t_n)$ has Galois group S_n (the symmetric group on n letters) over $k(s_1, \ldots, s_n)$. Thus, the given ratio lies in $k(s_1, \ldots, s_n)$. Thus, it is *expressible* as a ratio

$$\frac{f(t_1, \ldots, t_n)}{g(t_1, \ldots, t_n)} = \frac{F(s_1, \ldots, s_n)}{G(s_1, \ldots, s_n)}$$

of polynomials F, G in the s_i.

To prove the stronger result that the original f and g were themselves literally polynomials in the t_i, we seem to need the characteristic of k to be not 2, and we certainly must use the unique factorization in $k[t_1, \ldots, t_n]$.

Write

$$f(t_1, \ldots, t_n) = p_1^{e_1} \ldots p_m^{e_m}$$

where the e_i are positive integers and the p_i are irreducibles. Similarly, write

$$g(t_1, \ldots, t_n) = q_1^{f_1} \ldots q_m^{f_n}$$

where the f_i are positive integers and the q_i are irreducibles. The relative primeness says that none of the q_i are *associate* to any of the p_i. The invariance gives, for any permutation π of

$$\pi\left(\frac{p_1^{e_1} \ldots p_m^{e_m}}{q_1^{f_1} \ldots q_m^{f_n}}\right) = \frac{p_1^{e_1} \ldots p_m^{e_m}}{q_1^{f_1} \ldots q_m^{f_n}}$$

Multiplying out,

$$\prod_i \pi(p_i^{e_i}) \cdot \prod_i q_i^{f_i} = \prod_i p_i^{e_i} \cdot \prod_i \pi(q_i^{f_i})$$

By the relative prime-ness, each p_i divides some one of the $\pi(p_j)$. These ring automorphisms preserve irreducibility, and $\gcd(a, b) = 1$ implies $\gcd(\pi a, \pi b) = 1$, so, symmetrically, the $\pi(p_j)$'s divide the p_i's. And similarly for the q_i's. That is, permuting the t_i's must permute the irreducible factors of f (up to units $k^\times$ in $k[t_1, \ldots, t_n]$) among themselves, and likewise for the irreducible factors of g.

If all permutations *literally* permuted the irreducible factors of f (and of g), rather than merely up to *units*, then f and g would be symmetric. However, at this point we can only be confident that they are permuted *up to constants.*

What we have, then, is that for a permutation π

$$\pi(f) = \alpha_\pi \cdot f$$

for some $\alpha \in k^\times$. For another permutation τ, certainly $\tau(\pi(f)) = (\tau\pi)f$. And $\tau(\alpha_\pi f) = \alpha_\pi \cdot \tau(f)$, since permutations of the indeterminates have no effect on elements of k. Thus, we have

$$\alpha_{\tau\pi} = \alpha_\tau \cdot \alpha_\pi$$

That is, $\pi \to \alpha_\pi$ is a group homomorphism $S_n \to k^\times$.

It is very useful to know that the alternating group A_n is the *commutator subgroup* of S_n. Thus, if f is not actually invariant under S_n, in any case the group homomorphism $S_n \to k^\times$ factors through the quotient S_n/A_n, so is the *sign function* $\pi \to \sigma(\pi)$ that is $+1$ for $\pi \in A_n$ and -1 otherwise. That is, f is **equivariant** under S_n by the sign function, in the sense that $\pi f = \sigma(\pi) \cdot f$.

Now we claim that if $\pi f = \sigma(\pi) \cdot f$ then the square root

$$\delta = \sqrt{\Delta} = \prod_{i<j} (t_i - t_j)$$

of the discriminant Δ divides f. To see this, let s_{ij} be the 2-cycle which interchanges t_i and t_j, for $i \neq j$. Then

$$s_{ij} f = -f$$

Under any homomorphism which sends $t_i - t_j$ to 0, since the characteristic is not 2, f is sent to 0. That is, $t_i - t_j$ divides f in $k[t_1, \ldots, t_n]$. By unique factorization, since no two of the monomials $t_i - t_j$ are associate (for distinct pairs $i < j$), we see that the square root δ of the discriminant must divide f.

That is, for f with $\pi f = \sigma(\pi) \cdot f$ we know that $\delta | f$. For f/g to be invariant under S_n, it must be that also $\pi g = \sigma(\pi) \cdot g$. But then $\delta | g$ also, contradicting the assumed relative primeness. Thus, in fact, it must have been that both f and g were *invariant* under S_n, not merely equivariant by the sign function. ///