(*January 14, 2009*)

[20.1] Prove the *expansion by minors* formula for determinants, namely, for an n-by-n matrix A with entries a_{ij}, letting A^{ij} be the matrix obtained by deleting the i^{th} row and j^{th} column, for any fixed row index i,

$$\det A = (-1)^i \sum_{j=1}^{n} (-1)^j \, a_{ij} \, \det A^{ij}$$

and symmetrically for expansion along a column.

[iou: prove that this formula is linear in each row/column, and invoke the uniqueness of determinants]

[20.2] Let M and N be free R-modules, where R is a commutative ring with identity. Prove that $M \otimes_R N$ is free and

$$\mathrm{rank}\, M \otimes_R N = \mathrm{rank}\, M \cdot \mathrm{rank}\, N$$

Let M and N be free on generators $i : X \to M$ and $j : Y \to N$. We claim that $M \otimes_R N$ is free on a set map

$$\ell : X \times Y \to M \otimes_R N$$

To verify this, let $\varphi : X \times Y \to Z$ be a set map. For each fixed $y \in Y$, the map $x \to \varphi(x, y)$ factors through a unique R-module map $B_y : M \to Z$. For each $m \in M$, the map $y \to B_y(m)$ gives rise to a unique R-linear map $n \to B(m, n)$ such that

$$B(m, j(y)) = B_y(m)$$

The linearity in the second argument assures that we still have the linearity in the first, since for $n = \sum_t r_t \, j(y_t)$ we have

$$B(m, n) = B(m, \sum_t r_t j(y_t)) = \sum_t r_t B_{y_t}(m)$$

which is a linear combination of linear functions. Thus, there is a unique map to Z induced on the tensor product, showing that the tensor product with set map $i \times j : X \times Y \to M \otimes_R N$ is free. ///

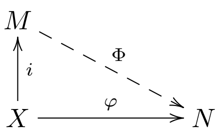
[20.3] Let M be a free R-module of rank r, where R is a commutative ring with identity. Let S be a commutative ring with identity containing R, such that $1_R = 1_S$. Prove that as an S module $M \otimes_R S$ is free of rank r.

We prove a bit more. First, instead of simply an *inclusion* $R \subset S$, we can consider any ring homomorphism $\psi : R \to S$ such that $\psi(1_R) = 1_S$.

Also, we can consider arbitrary sets of generators, and give more details. Let M be free on generators $i : X \to M$, where X is a set. Let $\tau : M \times S \to M \otimes_R S$ be the canonical map. We claim that $M \otimes_R S$ is free on $j : X \to M \otimes_R S$ defined by

$$j(x) = \tau(i(x) \times 1_S)$$

Given an S-module N, we can be a little forgetful and consider N as an R-module via ψ, by $r \cdot n = \psi(r)n$. Then, given a set map $\varphi : X \to N$, since M is free, there is a unique R-module map $\Phi : M \to N$ such that $\varphi = \Phi \circ i$. That is, the diagram



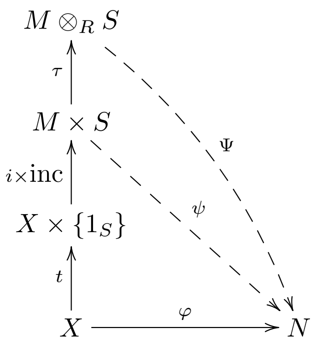
commutes. Then the map

$$\psi : M \times S \to N$$

by

$$\psi(m \times s) = s \cdot \Phi(m)$$

induces (by the defining property of $M \otimes_R S$) a unique $\Psi : M \otimes_R S \to N$ making a commutative diagram



where *inc* is the inclusion map $\{1_S\} \to S$, and where $t : X \to X \times \{1_S\}$ by $x \to x \times 1_S$. Thus, $M \otimes_R S$ is free on the composite $j : X \to M \otimes_R S$ defined to be the composite of the vertical maps in that last diagram. This argument does not depend upon finiteness of the generating set. ///

[20.4] For finite-dimensional vectorspaces V, W over a field k, prove that there is a natural isomorphism

$$(V \otimes_k W)^* \approx V^* \otimes W^*$$

where $X^* = \mathrm{Hom}_k(X, k)$ for a k-vectorspace X.

For finite-dimensional V and W, since $V \otimes_k W$ is free on the cartesian product of the generators for V and W, the dimensions of the two sides match. We make an isomorphism from right to left. Create a bilinear map

$$V^* \times W^* \to (V \otimes_k W)^*$$

as follows. Given $\lambda \in V^*$ and $\mu \in W^*$, as usual make $\Lambda_{\lambda,\mu} \in (V \otimes_k W)^*$ from the bilinear map

$$B_{\lambda,\mu} : V \times W \to k$$

defined by

$$B_{\lambda,\mu}(v, w) = \lambda(v) \cdot \mu(w)$$

This induces a unique functional $\Lambda_{\lambda,\mu}$ on the tensor product. This induces a unique linear map

$$V^* \otimes W^* \to (V \otimes_k W)^*$$

as desired.

Since everything is finite-dimensional, bijectivity will follow from injectivity. Let $e_1, \ldots, e_m$ be a basis for V, $f_1, \ldots, f_n$ a basis for W, and $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_n$ corresponding dual bases. We have shown that a basis of a tensor product of free modules is free on the cartesian product of the generators. Suppose that $\sum_{ij} c_{ij} \lambda_i \otimes \mu_j$ gives the 0 functional on $V \otimes W$, for some scalars c_{ij}. Then, for every pair of indices s, t, the function is 0 on $e_s \otimes f_t$. That is,

$$0 = \sum_{ij} c_{ij} \lambda_i(e_s)\, \lambda_j(f_t) = c_{st}$$

Thus, all constants c_{ij} are 0, proving that the map is injective. Then a dimension count proves the isomorphism. ///

[20.5] For a finite-dimensional k-vectorspace V, prove that the bilinear map

$$B : V \times V^* \to \mathrm{End}_k(V)$$

by

$$B(v \times \lambda)(x) = \lambda(x) \cdot v$$

gives an isomorphism $V \otimes_k V^* \to \mathrm{End}_k(V)$. Further, show that the composition of endormorphisms is the same as the map induced from the map on

$$(V \otimes V^*) \times (V \otimes V^*) \to V \otimes V^*$$

given by

$$(v \otimes \lambda) \times (w \otimes \mu) \to \lambda(w) v \otimes \mu$$

The bilinear map $v \times \lambda \to T_{v,\lambda}$ given by

$$T_{v,\lambda}(w) = \lambda(w) \cdot v$$

induces a *unique* linear map $j : V \otimes V^* \to \mathrm{End}_k(V)$.

To prove that j is injective, we may use the fact that a basis of a tensor product of free modules is free on the cartesian product of the generators. Thus, let $e_1, \ldots, e_n$ be a basis for V, and $\lambda_1, \ldots, \lambda_n$ a dual basis for V^*. Suppose that

$$\sum_{i,j=1}^{n} c_{ij}\, e_i \otimes \lambda_j \to 0 \mathrm{End}_k(V)$$

That is, for every e_ℓ,

$$\sum_{ij} c_{ij} \lambda_j(e_\ell) e_i = 0 \in V$$

This is

$$\sum_i c_{ij} e_i = 0 \qquad \text{(for all } j\text{)}$$

Since the e_is are linearly independent, all the c_{ij}s are 0. Thus, the map j is injective. Then counting k-dimensions shows that this j is a k-linear isomorphism.

Composition of endomorphisms is a bilinear map

$$\mathrm{End}_k(V) \times \mathrm{End}_k(V) \xrightarrow{\circ} \mathrm{End}_k(V)$$

by

$$S \times T \to S \circ T$$

Denote by

$$c : (v \otimes \lambda) \times (w \otimes \mu) \to \lambda(w) v \otimes \mu$$

the allegedly corresonding map on the tensor products. The induced map on $(V \otimes V^*) \otimes (V \otimes V^*)$ is an example of a **contraction map** on tensors. We want to show that the diagram

$$\begin{array}{ccc}
\mathrm{End}_k(V) \times \mathrm{End}_k(V) & \xrightarrow{\circ} & \mathrm{End}_k(V) \\
{\scriptstyle j \times j} \uparrow & & \uparrow {\scriptstyle j} \\
(V \otimes_k V^*) \times (V \otimes_k V^*) & \xrightarrow{c} & V \otimes_k V^*
\end{array}$$

commutes. It suffices to check this starting with $(v \otimes \lambda) \times (w \otimes \mu)$ in the lower left corner. Let $x \in V$. Going up, then to the right, we obtain the endomorphism which maps x to

$$j(v \otimes \lambda) \circ j(w \otimes \mu)\ (x) = j(v \otimes \lambda)(j(w \otimes \mu)(x)) = j(v \otimes \lambda)(\mu(x)\, w) = \mu(x)\, j(v \otimes \lambda)(w) = \mu(x)\, \lambda(w)\, v$$

Going the other way around, to the right then up, we obtain the endomorphism which maps x to

$$j(\, c((v \otimes \lambda) \times (w \otimes \mu)))\, (x) = j(\, \lambda(w)(v \otimes \mu)\,)\, (x) = \lambda(w)\, \mu(x)\, v$$

These two outcomes are the same. ///

[20.6] Via the isomorphism $\mathrm{End}_k(V) \approx V \otimes_k V^*$, show that the linear map

$$\mathrm{tr}\, : \mathrm{End}_k(V) \to k$$

is the linear map

$$V \otimes V^* \to k$$

induced by the bilinear map $v \times \lambda \to \lambda(v)$.

Note that the induced map

$$V \otimes_k V^* \to k \quad \text{by} \quad v \otimes \lambda \to \lambda(v)$$

is another **contraction map** on tensors. Part of the issue is to compare the coordinate-bound trace with the induced (contraction) map $t(v \otimes \lambda) = \lambda(v)$ determined uniquely from the bilinear map $v \times \lambda \to \lambda(v)$. To this end, let $e_1, \ldots, e_n$ be a basis for V, with dual basis $\lambda_1, \ldots, \lambda_n$. The corresponding matrix coefficients $T_{ij} \in k$ of a k-linear endomorphism T of V are

$$T_{ij} = \lambda_i(Te_j)$$

(Always there is the worry about interchange of the indices.) Thus, in these coordinates,

$$\mathrm{tr}\, T = \sum_i \lambda_i(Te_i)$$

Let $T = j(e_s \otimes \lambda_t)$. Then, since $\lambda_t(e_i) = 0$ unless $i = t$,

$$\mathrm{tr}\, T = \sum_i \lambda_i(Te_i) = \sum_i \lambda_i(j(e_s \otimes \lambda_t)e_i) = \sum_i \lambda_i(\lambda_t(e_i) \cdot e_s) = \lambda_t(\lambda_t(e_t) \cdot e_s) = \begin{cases} 1 & (s = t) \\ 0 & (s \neq t) \end{cases}$$

On the other hand,

$$t(e_s \otimes \lambda_t) = \lambda_t(e_s) = \begin{cases} 1 & (s = t) \\ 0 & (s \neq t) \end{cases}$$

Thus, these two k-linear functionals agree on the monomials, which span, they are equal. ///

[20.7] Prove that $\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA)$ for two endomorphisms of a finite-dimensional vector space V over a field k, with trace defined as just above.

Since the maps

$$\mathrm{End}_k(V) \times \mathrm{End}_k(V) \to k$$

by

$$A \times B \to \mathrm{tr}\,(AB) \qquad \text{and/or} \qquad A \times B \to \mathrm{tr}\,(BA)$$

are bilinear, it suffices to prove the equality on (images of) monomials $v \otimes \lambda$, since these span the endomophisms over k. Previous examples have converted the issue to one concerning $V \overset{\otimes}{_k} V^*$. (We have already shown that the isomorphism $V \otimes_k V^* \approx \mathrm{End}_k(V)$ is converts a *contraction* map on tensors to composition of endomorphisms, and that the trace on tensors defined as another contraction corresponds to the trace of matrices.) Let tr now denote the contraction-map trace on tensors, and (temporarily) write

$$(v \otimes \lambda) \circ (w \otimes \mu) = \lambda(w)\, v \otimes \mu$$

for the contraction-map composition of endomorphisms. Thus, we must show that

$$\mathrm{tr}\ (v \otimes \lambda) \circ (w \otimes \mu) = \mathrm{tr}\ (w \otimes \mu) \circ (v \otimes \lambda)$$

The left-hand side is

$$\mathrm{tr}\ (v \otimes \lambda) \circ (w \otimes \mu) = \mathrm{tr}\,(\,\lambda(w)\, v \otimes \mu) = \lambda(w)\,\mathrm{tr}\,(v \otimes \mu) = \lambda(w)\,\mu(v)$$

The right-hand side is

$$\mathrm{tr}\ (w \otimes \mu) \circ (v \otimes \lambda) = \mathrm{tr}\,(\,\mu(v)\, w \otimes \lambda) = \mu(v)\,\mathrm{tr}\,(w \otimes \lambda) = \mu(v)\,\lambda(w)$$

These elements of k are the same. ///

[20.8] Prove that tensor products are *associative*, in the sense that, for R-modules A, B, C, we have a *natural isomorphism*

$$A \otimes_R (B \otimes_R C) \approx (A \otimes_R B) \otimes_R C$$

In particular, *do* prove the *naturality*, at least the one-third part of it which asserts that, for every R-module homomorphism $f : A \to A'$, the diagram

$$\begin{array}{ccc} A \otimes_R (B \otimes_R C) & \xrightarrow{\approx} & (A \otimes_R B) \otimes_R C \\ \downarrow{\scriptstyle f\otimes(1_B\otimes 1_C)} & & \downarrow{\scriptstyle (f\otimes 1_B)\otimes 1_C} \\ A' \otimes_R (B \otimes_R C) & \xrightarrow{\approx} & (A' \otimes_R B) \otimes_R C \end{array}$$

commutes, where the two horizontal isomorphisms are those determined in the first part of the problem. (One might also consider maps $g : B \to B'$ and $h : C \to C'$, but these behave similarly, so there's no real compulsion to worry about them, apart from awareness of the issue.)

Since all tensor products are over R, we drop the subscript, to lighten the notation. As usual, to make a (linear) map *from* a tensor product $M \otimes N$, we induce uniquely from a bilinear map on $M \times N$. We have done this enough times that we will suppress this part now.

The thing that is slightly less trivial is construction of maps *to* tensor products $M \otimes N$. These are always obtained by composition with the canonical bilinear map

$$M \times N \to M \otimes N$$

Important at present is that we can create n-fold tensor products, as well. Thus, we prove the indicated isomorphism by proving that both the indicated iterated tensor products are (naturally) isomorphic to the un-parenthesis'd tensor product $A \otimes B \otimes C$, with canonical map $\tau : A \times B \times C \to A \otimes B \otimes C$, such that for every trilinear map $\varphi : A \times B \times C \to X$ there is a unique linear $\Phi : A \otimes B \otimes C \to X$ such that

$$\begin{array}{ccc} A \otimes B \otimes C & & \\ \uparrow{\scriptstyle \tau} & \searrow{\scriptstyle \Phi} & \\ A \times B \times C & \xrightarrow{\varphi} & X \end{array}$$

The set map

$$A \times B \times C \approx (A \times B) \times C \to (A \otimes B) \otimes C$$

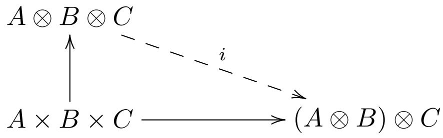
by

$$a \times b \times c \to (a \times b) \times c \to (a \otimes b) \otimes c$$

is linear in each single argument (for fixed values of the others). Thus, we are assured that there is a unique induced linear map

$$A \otimes B \otimes C \to (A \otimes B) \otimes C$$

such that



commutes.

Similarly, from the set map

$$(A \times B) \times C \approx A \times B \times C \to A \otimes B \otimes C$$

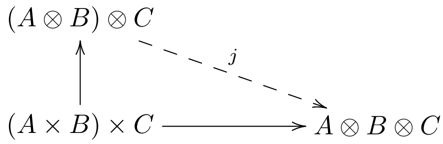
by

$$(a \times b) \times c \to a \times b \times c \to a \otimes b \otimes c$$

is linear in each single argument (for fixed values of the others). Thus, we are assured that there is a unique induced linear map

$$(A \otimes B) \otimes C \to A \otimes B \otimes C$$

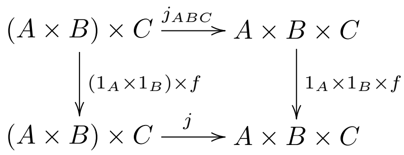
such that



commutes.

Then $j \circ i$ is a map of $A \otimes B \otimes C$ to itself compatible with the canonical map $A \times B \times C \to A \otimes B \otimes C$. By uniqueness, $j \circ i$ is the identity on $A \otimes B \otimes C$. Similarly (just very slightly more complicatedly), $i \circ j$ must be the identity on the iterated tensor product. Thus, these two maps are mutual inverses.

To prove naturality in one of the arguments A, B, C, consider $f : C \to C'$. Let j_{ABC} be the isomorphism for a fixed triple A, B, C, as above. The diagram of maps of cartesian products (of sets, at least)



does commute: going down, then right, is

$$j_{ABC'}\left((1_A \times 1_B) \times f\right)((a \times b) \times c)) = j_{ABC'}\left((a \times b) \times f(c)\right) = a \times b \times f(c)$$

Going right, then down, gives

$$(1_A \times 1_B \times f)\left(j_{ABC}((a \times b) \times c)\right) = (1_A \times 1_B \times f)\left(a \times b \times c)\right) = a \times b \times f(c)$$

These are the same. ///