5. Linear algebra I: dimension

1. *Some simple results*

Several observations should be made. Once stated explicitly, the proofs are easy. [1]

• The intersection of a (non-empty) set of subspaces of a vector space V is a subspace.

Proof: Let $\{W_i : i \in I\}$ be a set of subspaces of V. For w in every W_i, the additive inverse $-w$ is in W_i. Thus, $-w$ lies in the intersection. The same argument proves the other properties of subspaces. ///

The **subspace spanned** by a set X of vectors in a vector space V is the intersection of all subspaces containing X. From above, this intersection is a subspace.

• The subspace spanned by a set X in a vector space V is the collection of all linear combinations of vectors from X.

Proof: Certainly every linear combination of vectors taken from X is in any subspace containing X. On the other hand, we must show that any vector in the intersection of subspaces containing X is a linear combination of vectors in X. Now it is not hard to check that the collection of such linear combinations is *itself* a subspace of V, and contains X. Therefore, the intersection is no larger than this set of linear combinations. ///

[1] At the beginning of the abstract form of this and other topics, there are several results which have little informational content, but, rather, only serve to assure us that the definitions/axioms have not included phenomena too violently in opposition to our expectations. This is not surprising, considering that the definitions have endured several decades of revision exactly to address foundational and other potential problems.

A *linearly independent* set of vectors *spanning* a subspace W of V is a **basis** for W.

[1.0.1] **Proposition:** Given a basis $e_1, \ldots, e_n$ for a vector space V, there is *exactly one* expression for an arbitrary vector $v \in V$ as a linear combination of $e_1, \ldots, e_n$.

Proof: That there is *at least* one expression follows from the spanning property. On the other hand, if

$$\sum_i a_i e_i = v = \sum_i b_i e_i$$

are two expressions for v, then subtract to obtain

$$\sum_i (a_i - b_i) e_i = 0$$

Since the e_i are linearly independent, $a_i = b_i$ for all indices i. ///

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2. *Bases and dimension*

The argument in the proof of the following fundamental theorem is the *Lagrange replacement principle*. This is the first non-trivial result in linear algebra.

[2.0.1] **Theorem:** Let $v_1, \ldots, v_m$ be a linearly independent set of vectors in a vector space V, and let $w_1, \ldots, w_n$ be a basis for V. Then $m \le n$, and (renumbering the vectors w_i if necessary) the vectors

$$v_1, \ldots, v_m, w_{m+1}, w_{m+2}, \ldots, w_n$$

are a basis for V.

Proof: Since the w_i's are a basis, we may express v_1 as a linear combination

$$v_1 = c_1 w_1 + \ldots + c_n w_n$$

Not all coefficients can be 0, since v_1 is not 0. Renumbering the w_i's if necessary, we can assume that $c_1 \neq 0$. Since the scalars k are a *field*, we can express w_1 in terms of v_1 and $w_2, \ldots, w_n$

$$w_1 = c_1^{-1} v_1 + (-c_1^{-1} c_2) w_2 + \ldots + (-c_1^{-1} c_2) w_n$$

Replacing w_1 by v_1, the vectors $v_1, w_2, w_3, \ldots, w_n$ span V. They are still linearly independent, since if v_1 were a linear combination of $w_2, \ldots, w_n$ then the expression for w_1 in terms of $v_1, w_2, \ldots, w_n$ would show that w_1 was a linear combination of $w_2, \ldots, w_n$, contradicting the linear independence of $w_1, \ldots, w_n$.

Suppose inductively that $v_1, \ldots, v_i, w_{i+1}, \ldots, w_n$ are a basis for V, with $i < n$. Express v_{i+1} as a linear combination

$$v_{i+1} = a_1 v_1 + \ldots + a_i v_i + b_{i+1} w_{i+1} + \ldots + b_n w_n$$

Some b_j is non-zero, or else v_i is a linear combination of $v_1, \ldots, v_i$, contradicting the linear independence of the v_j's. By renumbering the w_j's if necessary, assume that $b_{i+1} \neq 0$. Rewrite this to express w_{i+1} as a linear combination of $v_1, \ldots, v_i, w_{i+1}, \ldots, w_n$

$$w_{i+1} = (-b_{i+1}^{-1} a_1) v_1 + \ldots + (-b_{i+1}^{-1} a_i) v_i + (b_{i+1}^{-1}) v_{i+1}$$

$$+ (-b_{i+1}^{-1} b_{i+2}) w_{i+2} + \ldots + (-b_{i+1}^{-1} b_n) w_n$$

Thus, $v_1, \ldots, v_{i+1}, w_{i+2}, \ldots, w_n$ span V. Claim that these vectors are linearly independent: if for some coefficients a_j, b_j

$$a_1 v_1 + \ldots + a_{i+1} v_{i+1} + b_{i+2} w_{i+2} + \ldots + b_n w_n = 0$$

then some a_{i+1} is non-zero, because of the linear independence of $v_1, \ldots, v_i, w_{i+1}, \ldots, w_n$. Thus, rearrange to express v_{i+1} as a linear combination of $v_1, \ldots, v_i, w_{i+2}, \ldots, w_n$. The expression for w_{i+1} in terms of $v_1, \ldots, v_i, v_{i+1}, w_{i+2}, \ldots, w_n$ becomes an expression for w_{i+1} as a linear combination of $v_1, \ldots, v_i, w_{i+2}, \ldots, w_n$. But this would contradict the (inductively assumed) linear independence of $v_1, \ldots, v_i, w_{i+1}, w_{i+2}, \ldots, w_n$.

Consider the possibility that $m > n$. Then, by the previous argument, $v_1, \ldots, v_n$ is a basis for V. Thus, v_{n+1} is a linear combination of $v_1, \ldots, v_n$, contradicting their linear independence. Thus, $m \le n$, and $v_1, \ldots, v_m, w_{m+1}, \ldots, w_n$ is a basis for V, as claimed. ///

Now define the (k-)**dimension** [2] of a vector space (over field k) as the number of elements in a (k-)basis. The theorem says that this number is well-defined. Write

$$\dim V = \text{ dimension of } V$$

A vector space is **finite-dimensional** if it has a finite basis. [3]

[2.0.2] **Corollary:** A linearly independent set of vectors in a finite-dimensional vector space can be augmented to be a basis.

Proof: Let $v_1, \ldots, v_m$ be as linearly independent set of vectors, let $w_1, \ldots, w_n$ be a basis, and apply the theorem. ///

[2.0.3] **Corollary:** The dimension of a *proper* subspace of a finite-dimensional vector space is strictly less than the dimension of the whole space.

Proof: Let $w_1, \ldots, w_m$ be a basis for the subspace. By the theorem, it can be extended to a basis $w_1, \ldots, w_m, v_{m+1}, \ldots, v_n$ of the whole space. It must be that $n > m$, or else the subspace is the whole space. ///

[2.0.4] **Corollary:** The dimension of k^n is n. The vectors

$$\begin{array}{rcl} e_1 &=& (1,0,0,\ldots,0,0) \\ e_2 &=& (0,1,0,\ldots,0,0) \\ e_3 &=& (0,0,1,\ldots,0,0) \\ &\ldots& \\ e_n &=& (0,0,0,\ldots,0,1) \end{array}$$

are a basis (the **standard basis**).

Proof: Those vectors *span* k^n, since

$$(c_1, \ldots, c_n) = c_1 e_1 + \ldots + c_n e_n$$

[2] This is an instance of terminology that is nearly too suggestive. That is, a naive person might all too easily accidentally assume that there is a connection to the colloquial sense of the word *dimension*, or that there is an appeal to physical or visual intuition. Or one might assume that it is somehow *obvious* that *dimension* is a well-defined invariant.

[3] We proved only the finite-dimensional case of the well-definedness of dimension. The infinite-dimensional case needs *transfinite induction* or an equivalent.

On the other hand, a linear dependence relation

$$0 = c_1 e_1 + \ldots + c_n e_n$$

gives

$$(c_1, \ldots, c_n) = (0, \ldots, 0)$$

from which each c_i is 0. Thus, these vectors are a basis for k^n. ///

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3. *Homomorphisms and dimension*

Now we see how dimension behaves under homomorphisms.

Again, a vector space **homomorphism** [4] $f : V \longrightarrow W$ from a vector space V over a field k to a vector space W over the same field k is a function f such that

$$\begin{array}{rcll} f(v_1 + v_2) & = & f(v_1) + f(v_2) & (\text{for all } v_1, v_2 \in V) \\ f(\alpha \cdot v) & = & \alpha \cdot f(v) & (\text{for all } \alpha \in k,\ v \in V) \end{array}$$

The **kernel** of f is

$$\ker f = \{v \in V : f(v) = 0\}$$

and the **image** of f is

$$\mathrm{Im} f = \{f(v) : v \in V\}$$

A homomorphism is an **isomorphism** if it has a two-sided inverse homomorphism. For vector spaces, a homomorphism that is a bijection is an isomorphism. [5]

• A vector space homomorphism $f : V \longrightarrow W$ sends 0 (in V) to 0 (in W, and, for $v \in V$, $f(-v) = -f(v)$. [6]

[3.0.1] **Proposition:** The kernel and image of a vector space homomorphism $f : V \longrightarrow W$ are vector subspaces of V and W, respectively.

Proof: Regarding the kernel, the previous proposition shows that it contains 0. The last bulleted point was that additive inverses of elements in the kernel are again in the kernel. For $x, y \in \ker f$

$$f(x + y) = f(x) + f(y) = 0 + 0 = 0$$

so $\ker f$ is closed under addition. For $\alpha \in k$ and $v \in V$

$$f(\alpha \cdot v) = \alpha \cdot f(v) = \alpha \cdot 0 = 0$$

so $\ker f$ is closed under scalar multiplication. Thus, the kernel is a vector subspace.

Similarly, $f(0) = 0$ shows that 0 is in the image of f. For $w = f(v)$ in the image of f and $\alpha \in k$

$$\alpha \cdot w = \alpha \cdot f(v) = f(\alpha v) \in \mathrm{Im} f$$

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[4] Or **linear map** or **linear operator**.

[5] In most of the situations we will encounter, bijectivity of various sorts of homomorphisms is sufficient (and certainly necessary) to assure that there is an inverse map of the same sort, justifying this description of *isomorphism*.

[6] This follows from the analogous result for groups, since V with its additive structure is an abelian group.

For $x = f(u)$ and $y = f(v)$ both in the image of f,

$$x + y = f(u) + f(v) = f(u + v) \in \operatorname{Im} f$$

And from above

$$f(-v) = -f(v)$$

so the image is a vector subspace. ///

[3.0.2] **Corollary:** A linear map $f : V \longrightarrow W$ is injective if and only if its kernel is the trivial subspace $\{0\}$.

Proof: This follows from the analogous assertion for groups. ///

[3.0.3] **Corollary:** Let $f : V \longrightarrow W$ be a vector space homomorphism, with V finite-dimensional. Then

$$\dim \ker f + \dim \operatorname{Im} f = \dim V$$

Proof: Let $v_1, \ldots, v_m$ be a basis for $\ker f$, and, invoking the theorem, let $w_{m+1}, \ldots, w_n$ be vectors in V such that $v_1, \ldots, v_m, w_{m+1}, \ldots, w_n$ form a basis for V. We claim that the images $f(w_{m+1}), \ldots, f(w_n)$ are a basis for $\operatorname{Im} f$. First, show that these vectors *span*. For $f(v) = w$, express v as a linear combination

$$v = a_1 v_1 + \ldots + a_m v_m + b_{m+1} w_{m+1} + \ldots + b_n w_n$$

and apply f

$$\begin{aligned} w &= a_1 f(v_1) + \ldots + a_m f(v_m) + b_{m+1} f(w_{m+1}) + \ldots + b_n f(w_n) \\ &= a_1 \cdot 0 + \ldots + a_m \cdot 0(v_m) + b_{m+1} f(w_{m+1}) + \ldots + b_n f(w_n) \\ &= b_{m+1} f(w_{m+1}) + \ldots + b_n f(w_n) \end{aligned}$$

since the v_is are in the kernel. Thus, the $f(w_j)$'s *span* the image. For linear independence, suppose

$$0 = b_{m+1} f(w_{m+1}) + \ldots + b_n f(w_n)$$

Then

$$0 = f(b_{m+1} w_{m+1} + \ldots + b_n w_n)$$

Then, $b_{m+1} w_{m+1} + \ldots + b_n w_n$ would be in the kernel of f, so would be a linear combination of the v_i's, contradicting the fact that $v_1, \ldots, v_m, w_{m+1}, \ldots, w_n$ is a basis, unless all the b_j's were 0. Thus, the $f(w_j)$ are linearly independent, so are a basis for $\operatorname{Im} f$. ///

Exercises

5.[3.0.1] For subspaces V, W of a vector space over a field k, show that

$$\dim_k V + \dim_k W \;=\; \dim_k(V + W) + \dim_k(V \cap W)$$

5.[3.0.2] Given two bases $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ for a vector space V over a field k, show that there is a unique k-linear map $T : V \longrightarrow V$ such that $T(e_i) = f_i$.

5.[3.0.3] Given a basis $e_1, \ldots, e_n$ of a k-vectorspace V, and given arbitrary vectors $w_1, \ldots, w_n$ in a k-vectorspace W, show that there is a unique k-linear map $T : V \longrightarrow W$ such that $Te_i = w_i$ for all indices i.

5.[3.0.4] The space $\mathrm{Hom}_k(V, W)$ of k-linear maps from one k-vectorspace V to another, W, is a k-vectorspace under the operation

$$(\alpha \dot{T})(v) = \alpha \cdot (T(v))$$

for $\alpha \in k$ and $T \in \mathrm{Hom}_k(V, W)$. Show that

$$\dim_k \mathrm{Hom}_k(V, W) = \dim_k V \cdot \dim_k W$$

5.[3.0.5] A **flag** $V_1 \subset \ldots \subset V_\ell$ of subspaces of a k-vectorspace V is simply a collection of subspaces satisfying the indicated inclusions. The **type** of the flag is the list of *dimensions* of the subspaces V_i. Let W be a k-vectorspace, with a flag $W_1 \subset \ldots \subset W_\ell$ of the same *type* as the flag in V. Show that there exists a k-linear map $T : V \longrightarrow W$ such that T restricted to V_i is an isomorphism $V_i \longrightarrow W_i$.

5.[3.0.6] Let $V_1 \subset V_\ell$ be a flag of subspace inside a finite-dimensional k-vectorspace V, and $W_1 \subset \ldots \subset W_\ell$ a flag inside another finite-dimensional k-vectorspace W. We do not suppose that the two flags are of the same type. Compute the dimension of the space of k-linear homomorphisms $T : V \longrightarrow W$ such that $TV_i \subset W_i$.