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Girard-Newton identities for symmetric functions

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Albert Girard (1629), and, later, Isaac Newton (1666), expressed the elementary symmetric functions [1]

$$s_j \;=\; \sum_{i_1<i_2<\ldots<i_j} x_{i_1} x_{i_2} \ldots x_{i_j}$$

in terms of symmetric power sum functions

$$p_j \;=\; x_1^j + \ldots + x_n^j$$

where $x_1, \ldots, x_n$ are indeterminates.

Basic properties of *exp* and *log*, either as convergent or formal power series, produce the relation. Thus, consider

$$\prod_i (1 - z x_i) \;=\; \exp \sum_i \log(1 - z x_i)$$

with an indeterminate z, and evaluate this in the two obvious ways. First, of course, the left-hand side essentially defines the elementary symmetric functions:

$$\prod_i (1 - z x_i) \;=\; \sum_j (-1)^j \, s_j \, z^j$$

On the right-hand side, use the power series for *log*, interchange order of summation, use the fact that *exp* converts sums to products, and expand *exp*: this will inevitably produce the relation. Indeed, the point is not the specific formula, but the device by which to recover it. We have

$$\exp\Big(-\sum_i \sum_{n\ge 1} \frac{z^n\, x_i^n}{n}\Big) \;=\; \exp\Big(-\sum_{k\ge 1} \frac{z^k\, p_k}{k}\Big) \;=\; \prod_{k\ge 1} \exp\Big(-\frac{z^k\, p_k}{k}\Big) \;=\; \prod_{k\ge 1} \sum_{\ell\ge 0} \frac{(-z^k\, p_k/k)^\ell}{\ell!}$$

$$= \sum_{j\ge 0} z^j \sum_{\ell_1+2\ell_2+\ldots=j} \frac{(-p_1/1)^{\ell_1}}{\ell_1!} \frac{(-p_2/2)^{\ell_2}}{\ell_2!} \frac{(-p_3/3)^{\ell_3}}{\ell_3!} \cdots \frac{(-p_n/n)^{\ell_n}}{\ell_n!} \cdots$$

Equating the coefficients of z^j in the latter and in $\sum_j (-1)^j \, s_j \, z^j$ expresses the elementary symmetric function s_ℓ in terms of sums-of-powers p_j:

$$(-1)^j \, s_j \;=\; \sum_{\ell_1+2\ell_2+\ldots=j} \frac{(-p_1/1)^{\ell_1}}{\ell_1!} \frac{(-p_2/2)^{\ell_2}}{\ell_2!} \frac{(-p_3/3)^{\ell_3}}{\ell_3!} \cdots \frac{(-p_n/n)^{\ell_n}}{\ell_n!} \cdots$$

Since $\ell_i \ge 1$, the right-hand side of the latter is smaller than it might otherwise appear, namely, the formula for s_j it terminates at the j^{th} term:

$$(-1)^j \, s_j \;=\; \sum_{\ell_1+2\ell_2+\ldots+j\ell_j=j} \frac{(-p_1/1)^{\ell_1}}{\ell_1!} \frac{(-p_2/2)^{\ell_2}}{\ell_2!} \frac{(-p_3/3)^{\ell_3}}{\ell_3!} \cdots \frac{(-p_j/j)^{\ell_j}}{\ell_j!}$$

This expresses the elementary symmetric functions in terms of the symmetric power sums. Note that their is a clear limitation on the integers appearing in denominators.

[1] Girard's priority is mentioned in http://en.wikipedia.org/wiki/Newton%27s_identity

In the opposite direction, while we already know on general principles that the symmetric power sums are expressible in terms of the elementary symmetric functions, a variant of the above argument gives a formulaic expression, as follows. Again, the point is the device by which to recover the formula, not the formula itself.

From the intermediate result (above)

$$\sum_{0\le j\le n} (-1)^j\, s_j\, z^j \;=\; \prod_i (1-zx_i) \;=\; \exp\big(-\sum_{k\ge 1}\frac{z^k\,p_k}{k}\big)$$

move the *exp* to the left-hand side, as a logarithm:

$$\log\big(\sum_{0\le j\le n}(-1)^j\, s_j\, z^j\big) \;=\; -\sum_{k\ge 1}\frac{z^k\,p_k}{k}$$

Moving the sign to the other side,

$$-\log\big(1-\sum_{1\le j\le n}(-1)^{j-1}\, s_j\, z^j\big) \;=\; \sum_{k\ge 1}\frac{z^k\,p_k}{k}$$

Expand the logarithm on the left-hand side:

$$\sum_{\ell\ge 1}\big(s_1 z - s_2 z^2 + \ldots + (-1)^{n-1} s_n z^n\big)^\ell\,/\,\ell$$

$$= \sum_{\ell\ge 1}\;\sum_{k_1+k_2,+\ldots+k_n=\ell} z^{k_1+2k_2+\ldots+nk_n}\,\frac{1}{\ell}\binom{\ell}{k_1\;k_2\;\ldots\;k_n} s_1^{k_1}(-s_2)^{k_2}\ldots\big((-1)^{n-1}s_n\big)^{k_n}$$

$$= \sum_{k_1,k_2,\ldots,k_n} z^{k_1+2k_2+\ldots+nk_n}\,\frac{1}{k_1+k_2+\ldots+k_n}\binom{k_1+k_2+\ldots+k_n}{k_1\;k_2\;\ldots\;k_n} s_1^{k_1}(-s_2)^{k_2}\ldots\big((-1)^{n-1}s_n\big)^{k_n}$$

$$= \sum_{k\ge 1} z^k \sum_{k_1+2k_2+\ldots+nk_n=k}\frac{(k_1+k_2+\ldots+k_n-1)!}{k_1!\,k_2!\,\ldots\,k_n!}\, s_1^{k_1}(-s_2)^{k_2}\ldots\big((-1)^{n-1}s_n\big)^{k_n}$$

Equating coefficients of z^k,

$$\sum_{k_1+2k_2+\ldots+nk_n=k}\frac{(k_1+k_2+\ldots+k_n-1)!}{k_1!\,k_2!\,\ldots\,k_n!}\, s_1^{k_1}(-s_2)^{k_2}\ldots\big((-1)^{n-1}s_n\big)^{k_n} \;=\; \frac{p_k}{k}$$