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Poincaré-Birkhoff-Witt theorem

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We prove the Poincaré-Birkhoff-Witt Theorem on the structure of enveloping algebras of Lie algebras. The argument here is basically that given in Jacobson. The same argument is reproduced later in Varadarajan. By contrast, a somewhat different argument is given in Bourbaki, and by Humphreys.

N. Bourbaki, *Groupes et algèbres de Lie, Chap. 1*, Paris: Hermann, 1960.

N. Jacobson, *Lie Algebras*, Dover, 1962.

J. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, 1972.

V.S. Varadarajan, *Lie Groups, Lie Algebras, and their Representations*, Springer-Verlag, 1974, 1984.

It is not clear *a priori* that the *Jacobi identity*

$$[x,[y,z]] - [y,[x,z]] \;=\; [[x,y],z]$$

plays a role in the argument, but it does. At the same time, apart from Jacobson's device of use of the endomorphism L (see below), the argument is mostly very natural. And we note that the Jacobi identity is the assertion that the adjoint action of a Lie algebra on itself is a Lie algebra representation.

The following argument does not use any further properties of the Lie algebra $\mathfrak{g}$, so *must* be general. The result is constantly invoked, so frequently, in fact, that one might tire of citing it and declare that it is understood that everyone should keep this in mind. It is surprisingly difficult to prove.

Thinking of the universal property of the universal enveloping algebra, we might interpret the *free-ness* assertion of the theorem as an assertion that, in the range of possibilities for abundance or poverty of representations of the Lie algebra $\mathfrak{g}$, the actuality is *abundance* rather than *scarcity.*

[0.0.1] Theorem: For any basis $\{x_i : i \in I\}$ of a Lie algebra $\mathfrak{g}$ with *ordered* index set I, the monomials

$$x_{i_1}^{e_1} \ldots x_{i_n}^{e_n} \qquad (\text{with } i_1 < \ldots < i_n, \text{ and integers } e_i > 0)$$

form a *basis* for the enveloping algebra $U\mathfrak{g}$.

[0.0.2] Corollary: The natural map of a Lie algebra to its universal enveloping algebra is an *injection.* ///

Proof: Since we do not yet know that $\mathfrak{g}$ *injects* to $U\mathfrak{g}$, let $i : \mathfrak{g} \to U\mathfrak{g}$ be the natural Lie homomorphism. The easy part of the argument is to observe that these monomials *span.* Indeed, whatever *unobvious* relations may hold in $U\mathfrak{g}$,

$$U\mathfrak{g} \;=\; \mathbb{R} + \sum_{n=1}^{\infty} \underbrace{i(\mathfrak{g}) \ldots i(\mathfrak{g})}_{n}$$

though we are not claiming that the sum is direct (it is not). Let

$$U\mathfrak{g}^{\le N} \;=\; \mathbb{R} + \sum_{n=1}^{N} \underbrace{i(\mathfrak{g}) \ldots i(\mathfrak{g})}_{n}$$

Start from the fact that $i(x_k)$ and $i(x_\ell)$ commute modulo $i(\mathfrak{g})$, specifically,

$$i(x_k)\, i(x_\ell) - i(x_\ell)\, i(x_k) \;=\; i[x_k, x_\ell]$$

This reasonably suggests an induction proving that for α, β in $U\mathfrak{g}^{\leq n}$

$$\alpha\beta - \beta\alpha \in U\mathfrak{g}^{\leq n-1}$$

This much does not require much insight. We amplify upon this below.

The hard part of the argument is basically from Jacobson, and applies to not-necessarily finite-dimensional Lie algebras over arbitrary fields k of characteristic 0, using no special properties of $\mathbb{R}$. The assumption of characteristic 0 is not directly used; rather, the proper definition of *Lie algebra* in positive characteristic includes further conditions, which we ignore. Thus, our definition of *Lie algebra* in positive characteristic is incomplete.

Let T_n be

$$T_n = \underbrace{\mathfrak{g} \otimes \ldots \otimes \mathfrak{g}}_{n}$$

the space of *homogeneous tensors* of degree n, and T the **tensor algebra**

$$T = k \oplus T_1 \oplus T_2 \oplus \ldots$$

of $\mathfrak{g}$. For $x, y \in \mathfrak{g}$ let

$$u_{x,y} = (x \otimes y - y \otimes x) - [x, y] \in T_2 + T_1 \subset T$$

Let J be the two-sided ideal in T generated by the set of all elements $u_{x,y}$. Since $u_{x,y} \in T_1 + T_2$, the ideal J contains no non-zero elements of $T_0 \approx k$, so J is a *proper* ideal in T.

Let $U = T/J$ be the quotient, the **universal enveloping algebra** of $\mathfrak{g}$. Let

$$q : T \longrightarrow U$$

be the quotient map.

For any basis $\{x_i : i \in I\}$ of $\mathfrak{g}$ the images $q(x_{i_1} \otimes \ldots \otimes x_{i_n})$ in U of tensor monomials $x_{i_1} \otimes \ldots \otimes x_{i_n}$ *span* the enveloping algebra over k, since they span the tensor algebra.

With an *ordered* index set I for the basis of $\mathfrak{g}$, using the Lie bracket $[,]$, we can rearrange the x_{i_j}'s in a monomial. We *anticipate* that everything in U can be rewritten to be a sum of monomials $x_{i_1} \ldots x_{i_n}$ where

$$i_1 \leq i_2 \leq \ldots i_n$$

A monomial in with indices so ordered is a **standard monomial**.

To form the induction that proves that the (images of) standard monomials *span* U, consider a monomial $x_{i_1} \ldots x_{i_n}$ with indices not correctly ordered. There must be at least one index j such that

$$i_j > i_{j+1}$$

Since

$$x_{i_j} x_{i_{j+1}} - x_{i_{j+1}} x_{i_j} - [x_{i_j}, x_{i_{j+1}}] \in J$$

we have

$$x_{i_1} \ldots x_{i_n} = x_{i_1} \ldots x_{i_{j-1}} \cdot (x_{i_j} x_{i_{j+1}} - x_{i_{j+1}} x_{i_j} - [x_{i_j}, x_{i_{j+1}}]) \cdot x_{i_{j+2}} \ldots x_{i_n}$$
$$+ x_{i_1} \ldots x_{i_{j-1}} x_{i_{j+1}} x_{i_j} x_{i_{j+2}} \ldots x_{i_n} + x_{i_1} \ldots x_{i_{j-1}} [x_{i_j}, x_{i_{j+1}}] x_{i_{j+2}} \ldots x_{i_n}$$

The first summand lies inside the ideal J, while the third is a tensor of smaller degree. Thus, do induction on degree of tensors, and for each fixed degree do induction on the number of pairs of indices out of order.

The serious assertion is *linear independence*. Given a tensor monomial $x_{i_1} \otimes \ldots \otimes x_{i_n}$, say that the **defect** of this monomial is the number of pairs of indices j, j' such that $j < j'$ but $i_j > i_{j'}$. *Suppose* that we can define a linear map

$$L : T \to T$$

such that L is the identity map on standard monomials, and whenever $i_j > i_{j+1}$

$$L(x_{i_1} \otimes \ldots \otimes x_{i_n}) \;=\; L(x_{i_1} \otimes \ldots \otimes x_{i_{j+1}} \otimes x_{i_j} \otimes \ldots \otimes x_{i_n})$$

$$+L(x_{i_1} \otimes \ldots \otimes [x_{i_j}, x_{i_{j+1}}] \otimes \ldots \otimes x_{i_n})$$

If there is such L, then $L(J) = 0$, while L acts as the identity on any linear combination of standard monomials. This would prove that the subspace of T consisting of linear combinations of standard monomials meets the ideal J just at 0, so maps injectively to the enveloping algebra.

Incidentally, L would have the property that

$$L(y_{i_1} \otimes \ldots \otimes y_{i_n}) \;=\; L(y_{i_1} \otimes \ldots \otimes y_{i_{j+1}} \otimes y_{i_j} \otimes \ldots \otimes y_{i_n})$$

$$+L(y_{i_1} \otimes \ldots \otimes [y_{i_j}, y_{i_{j+1}}] \otimes \ldots \otimes y_{i_n})$$

for *any* vectors y_{i_j} in $\mathfrak{g}$.

Thus, the problem reduces to defining L. Do an induction to define L. First, define L to be the identity on $T_0 + T_1$. Note that the first condition on L is vacuous here, and the second condition follows since every monomial tensor of degree 1 or 0 is standard.

Now fix $n \geq 2$, and attempt to define L on monomials in $T_{\leq n}$ inductively by using the second required property: *define* $L(x_{i_1} \otimes \ldots \otimes x_{i_n})$ by

$$L(x_{i_1} \otimes \ldots \otimes x_{i_n}) \;=\; L(x_{i_1} \otimes \ldots \otimes x_{i_{j+1}} \otimes x_{i_j} \otimes \ldots \otimes x_{i_n})$$

$$+L(x_{i_1} \otimes \ldots \otimes [x_{i_j}, x_{i_{j+1}}] \otimes \ldots \otimes x_{i_n})$$

where $i_j > i_{j+1}$. One term on the right-hand side is of lower degree, and the other is of smaller defect. Thus, we do induction on degree of tensor monomials, and for each fixed degree do induction on defect.

The potential problem is the well-definedness of this definition. Monomials of degree n and of defect 0 are already standard. For monomials of degree n and of defect 1 the definition is unambiguous, since there is just one pair of indices that are out of order.

So suppose that the defect is at least two. Let $j < j'$ be two indices so that both $i_j > i_{j+1}$ and $i_{j'} > i_{j'+1}$. To prove well-definedness it suffices to show that the two right-hand sides of the defining relation for $L(x_{i_1} \otimes \ldots \otimes x_{i_n})$ are the same element of T.

Consider the case that $j + 1 < j'$. Necessarily $n \geq 4$. (In this case the two rearrangements do not interact with each other.) Doing the rearrangement specified by the index j,

$$L(x_{i_1} \otimes \ldots \otimes x_{i_n}) \;=\; L(x_{i_1} \otimes \ldots \otimes x_{i_{j+1}} \otimes x_{i_j} \otimes \ldots \otimes x_{i_n})$$

$$+L(x_{i_1} \otimes \ldots \otimes [x_{i_j}, x_{i_{j+1}}] \otimes \ldots \otimes x_{i_n})$$

The first summand on the right-hand side has smaller defect, and the second has smaller degree, so we can use the inductive definition to evaluate them both. And still has $i_{j'} > i_{j'+1}$. Nothing is lost if we simplify notation by taking $j = 1, j' = 3$, and $n = 4$, since all the other factors in the monomials are inert. Further, to lighten the notation write x for x_{i_1}, y for x_{i_2}, z for x_{i_3}, and w for x_{i_4}. We use the inductive definition to obtain

$$\begin{aligned} L(x\otimes y\otimes z\otimes w) \;&=\; L(y\otimes x\otimes z\otimes w) + L([x,y]\otimes z\otimes w) \\ &=\; L(y\otimes x\otimes w\otimes z) + L(y\otimes x\otimes [z,w]) \\ &\quad + L([x,y]\otimes w\otimes z) + L([x,y]\otimes [z,w]) \end{aligned}$$

But then it is clear (or can be computed analogously) that the same expression is obtained when the roles of j and j' are reversed. Thus, the induction step is completed in case $j+1<j'$.

Now consider the case that $j+1=j'$, that is, the case in which the interchanges *do* interact. Here nothing is lost if we just take $j=1$, $j'=2$, and $n=3$. And write x for x_{i_1}, y for x_{i_2}, z for x_{i_3}. Thus,

$$i_1 > i_2 > i_3$$

Then, on one hand, applying the inductive definition by first interchanging x and y, and then further reshuffling,

$$L(x\otimes y\otimes z) \;=\; L(y\otimes x\otimes z) + L([x,y]\otimes z) \;=\; L(y\otimes z\otimes x) + L(y\otimes [x,z]) + L([x,y]\otimes z)$$

$$=\; L(z\otimes y\otimes x) + L([y,z]\otimes x) + L(y\otimes [x,z]) + L([x,y]\otimes z)$$

On the other hand, starting by doing the interchange of y and z gives

$$L(x\otimes y\otimes z) \;=\; L(x\otimes z\otimes y) + L(x\otimes [y,z]) \;=\; L(z\otimes x\otimes y) + L([x,z]\otimes y) + L(x\otimes [y,z])$$

$$=\; L(z\otimes y\otimes x) + L(z\otimes [x,y]) + L([x,z]\otimes y) + L(x\otimes [y,z])$$

It remains to see that the two right-hand sides are the same.

Since L is already well-defined, by induction, for tensors of degree $n-1$ (here in effect $n-1=2$), we can invoke the property

$$L(v\otimes w) \;=\; L(w\otimes v) + L([v,w])$$

for all $v,w\in\mathfrak{g}$. Apply this to the second, third, and fourth terms in the first of the two previous computations, to obtain

$$L(x\otimes y\otimes z)$$

$$=\; L(z\otimes y\otimes x) + \Big(L(x\otimes [y,z]) + L([[y,z],x])\Big) + \Big(L([x,z]\otimes y) + L([y,[x,z]])\Big) + \Big(L(z\otimes [x,y]) + L([[x,y],z])\Big)$$

The latter differs from the right-hand side of the *second* computation just by the expressions involving doubled brackets, namely

$$L([[y,z],x]) + L([y,[x,z]]) + L([[x,y],z])$$

Thus, we wish to prove that the latter is 0. Having the Jacobi identity in mind motivates some rearrangement: move $L([[x,y],z])$ to the right-hand side of the equation, multiply through by -1, and reverse the outer bracket in the first summand, to give the equivalent requirement

$$L([x,[y,z]]) - L([y,[x,z]]) \;=\; L([[x,y],z])$$

This equality follows from application of L to the Jacobi identity. ///
