Nonlinear PDE continuum limits in data science and machine learning

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University of Minnesota

University of Wisconsin PDE & GA seminar
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This research was supported by NSF-DMS grants 0914567, 1500829 and 1713691.

Outline

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Motivating example: Google Goggles

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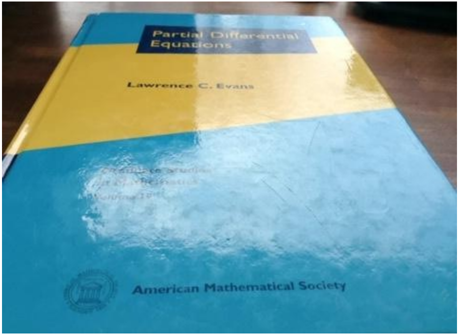

Figure: Query image

Motivating example: Google Goggles

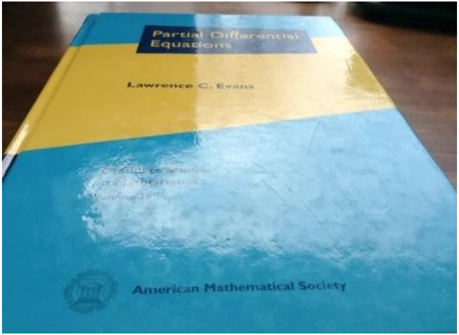

Figure: Query image

Figure: Retrieved images

Multi-query image retrieval

Problem: Find images in a dataset S that are similar to multiple query images.

Pareto method: "Solve" the multi-objective optimization problem

$$\arg\min_{I \in S}(\mathrm{dist}(I, Q_1), \ldots, \mathrm{dist}(I, Q_d)).$$

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Query 2

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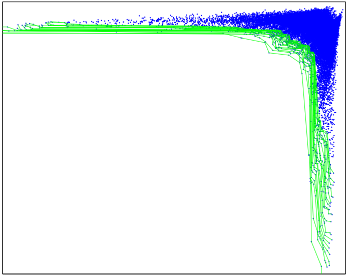
Pareto points:



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Problems:

1. Difficult to choose weights
2. Ignores relevant solutions

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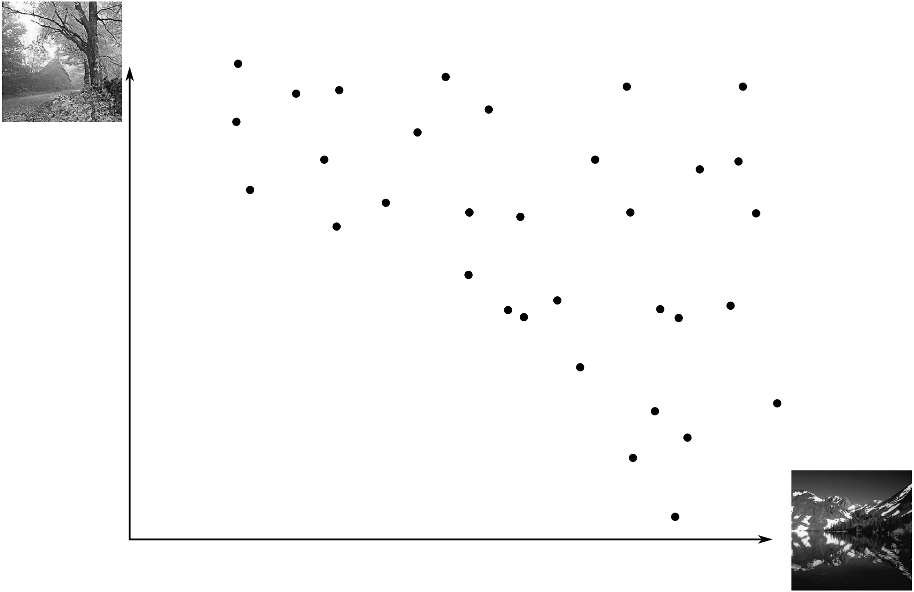

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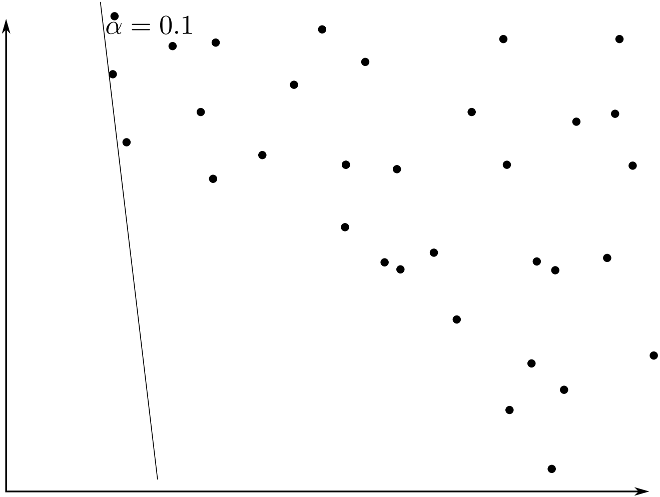

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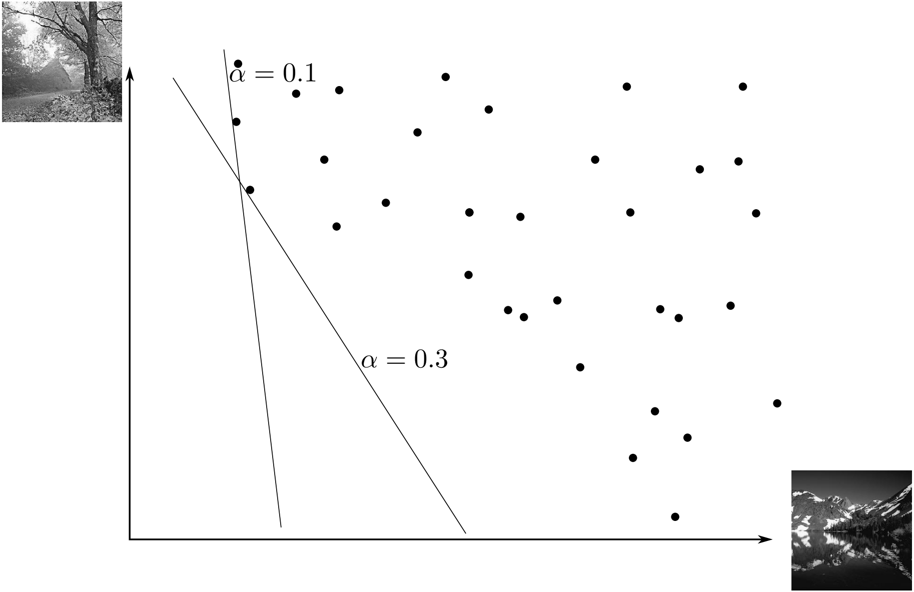

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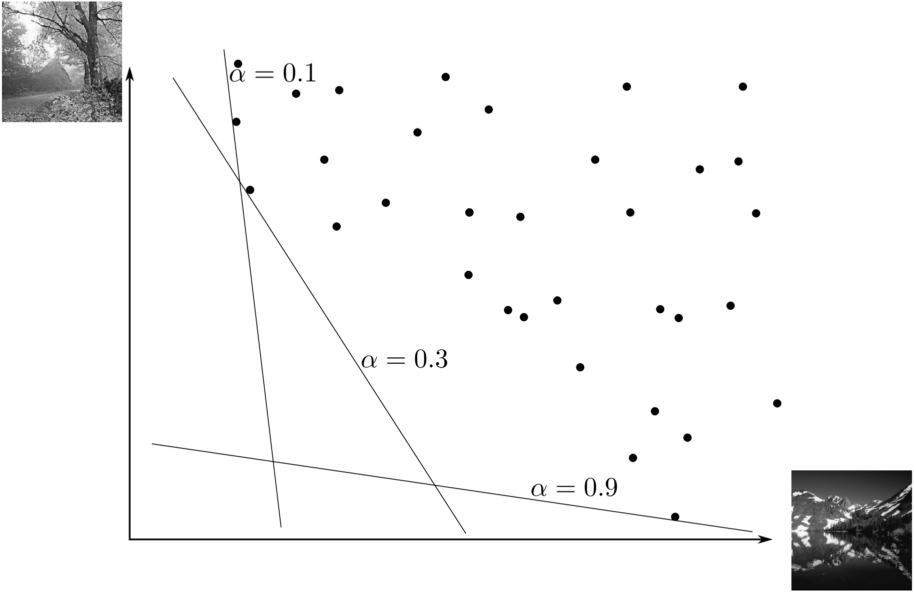
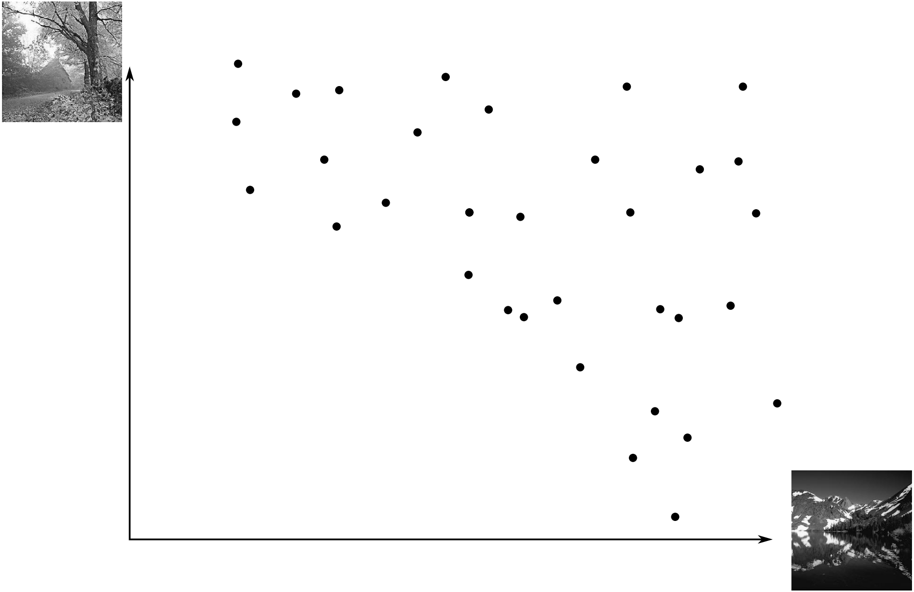


Nondominated solutions



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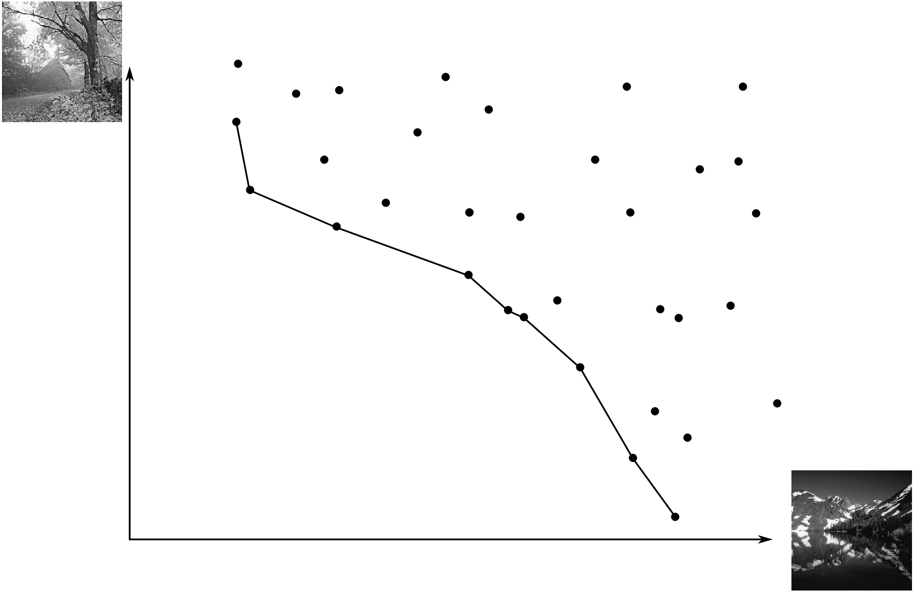

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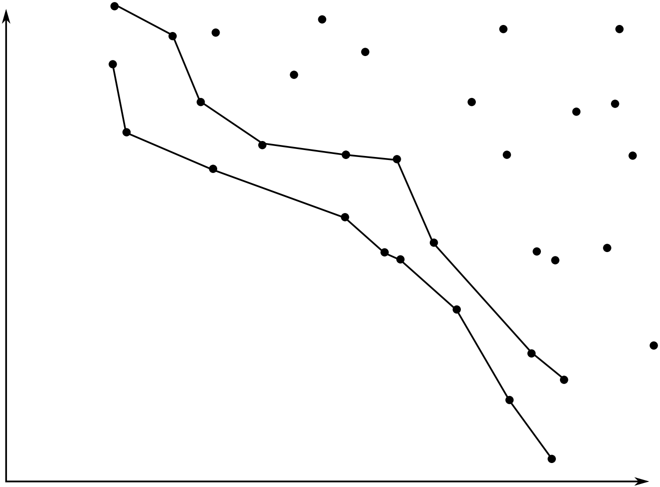

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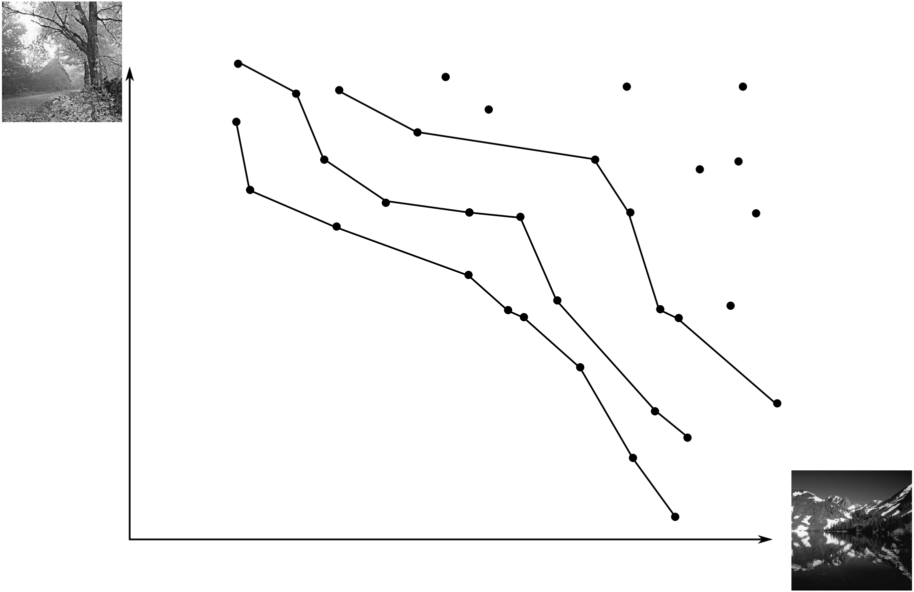

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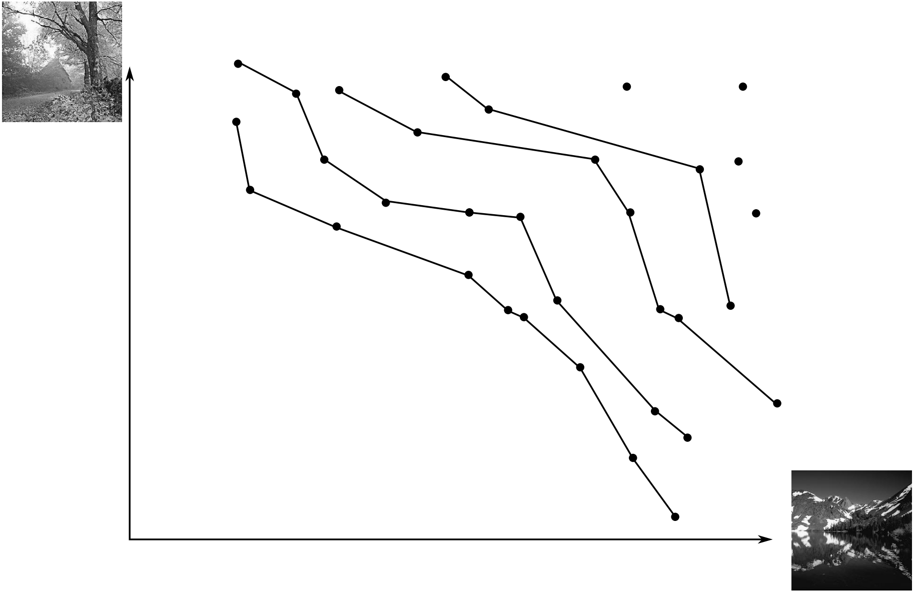

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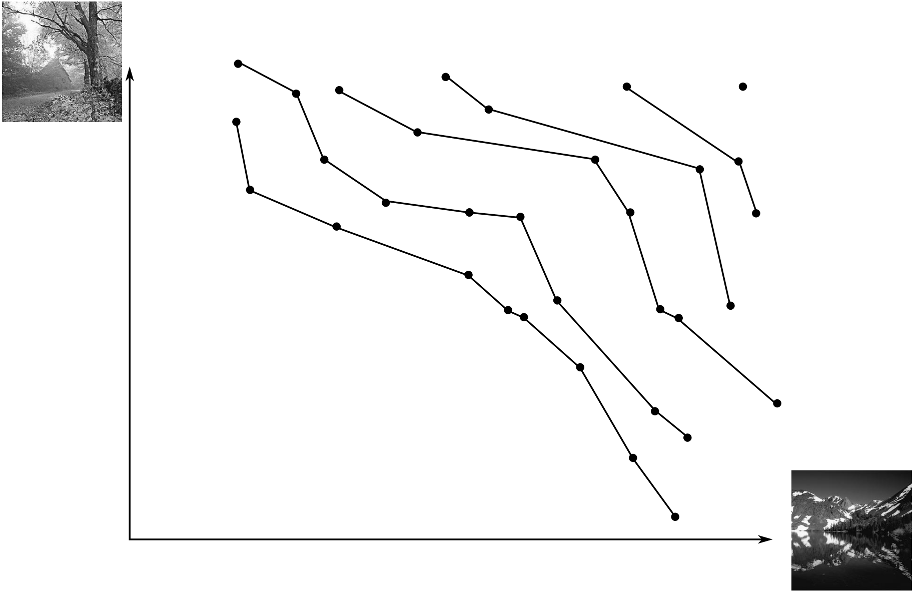

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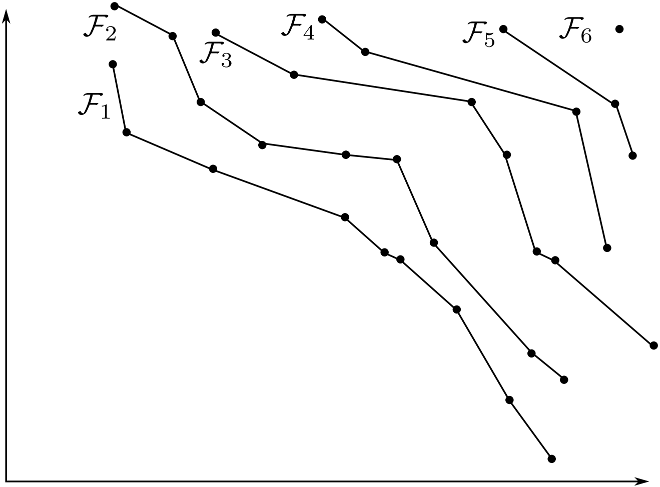

Multi-query image retrieval

First Pareto front:

Query 1

1

2

3

4

5

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7

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10

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12

13

14

15

Query 2

Hsiao, K.-J., Calder, J., and Hero III, A. O. (2015). Pareto-depth for multiple-query image retrieval. *IEEE Transactions on Image Processing*, 24(2):583–594.

Nondominated sorting

Let $X_1, \ldots, X_n$ be points in $\mathbb{R}^d$ and set $S = \{X_1, \ldots, X_n\}$.

Define the partial order

$$x \leqq y \iff x_i \leq y_i \text{ for all } i \in \{1, \ldots, d\}.$$

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Definition

Nondominated sorting is the process of arranging S into layers $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots,$ defined by

$$\mathcal{F}_1 = \text{Minimal elements of } S,$$

$$\mathcal{F}_k = \text{Minimal elements of } S \setminus (\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_{k-1}).$$

Applications

Multi-objective optimization

- Genetic algorithms [Deb et al., 2002]
- Gene selection and ranking [Hero, 2003]
- Database systems [Papadias et al., 2005]
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Combinatorics and probability

- Longest monotone subsequences [Ulam, 1961]
- Longest chain in Euclidean space [Hammersley, 1972]
- Patience sorting [Aldous and Diaconis, 1999]
- Young Tableaux [Viennot, 1984]
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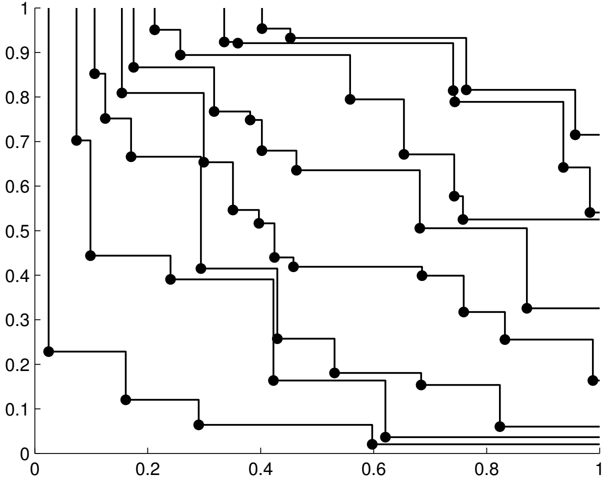
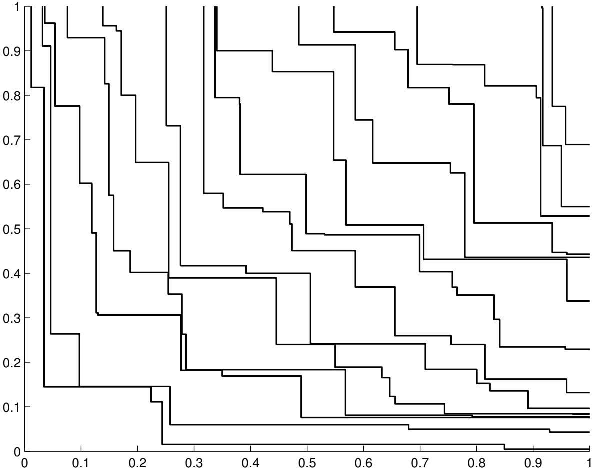
Other applications

- Molecular biology [Pevzner, 2000]
- Integrated circuit design [Adhar, 2007]

Demo: 50 Random samples

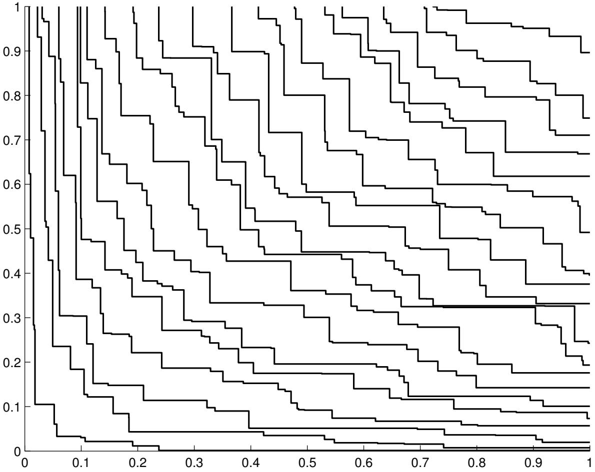


Demo: Uniform distribution



$n = 10^2$ points

Demo: Uniform distribution



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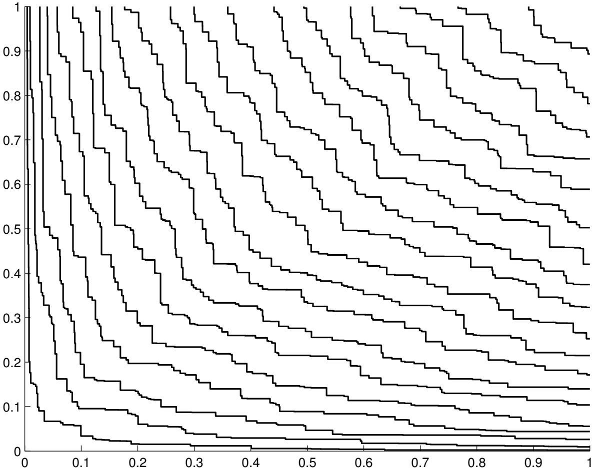

$n = 10^4$ points

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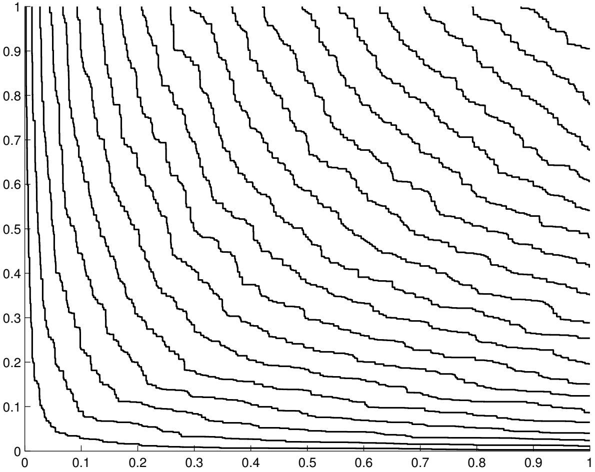
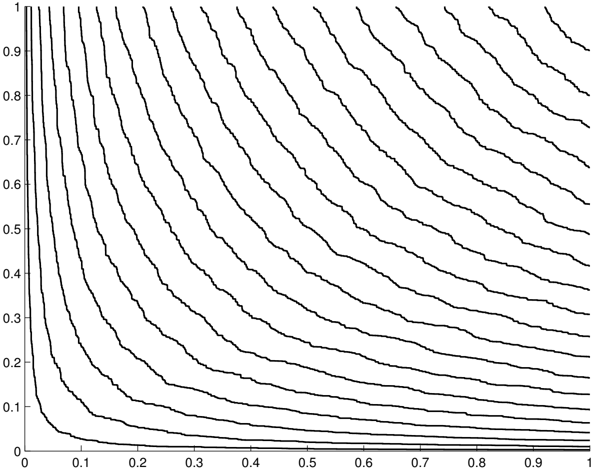


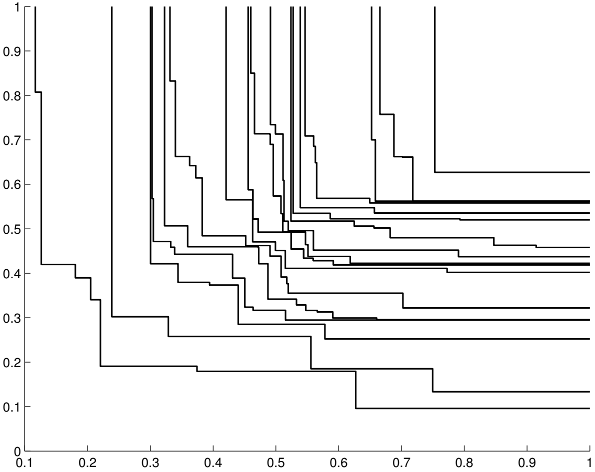
$n = 10^5$ points

Demo: Uniform distribution



$n = 10^6$ points

Demo: Gaussian distribution



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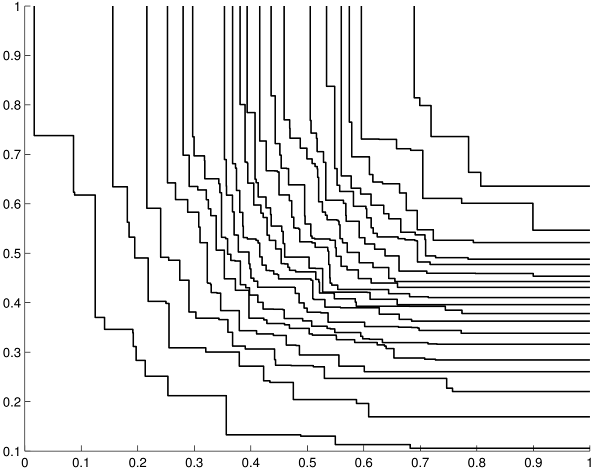

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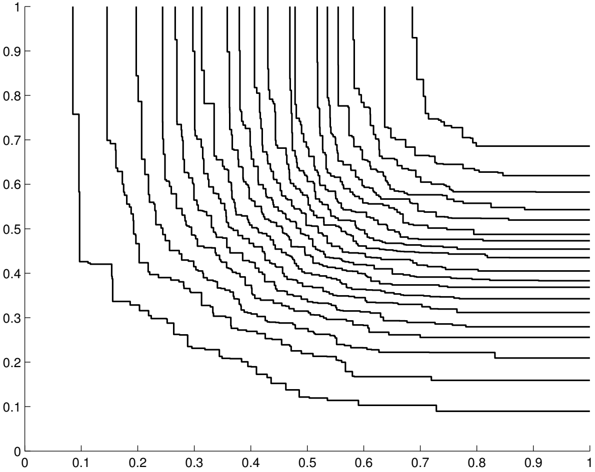

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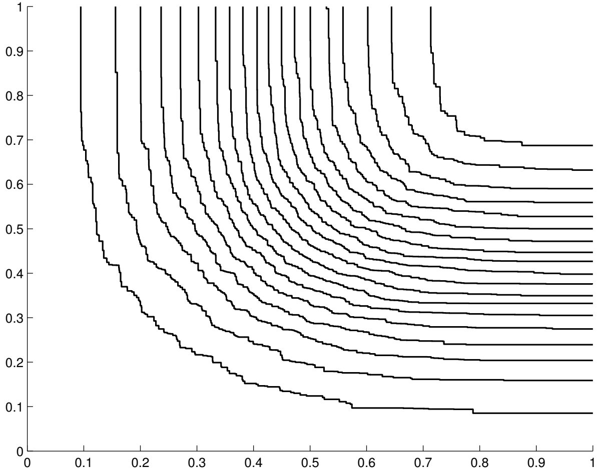

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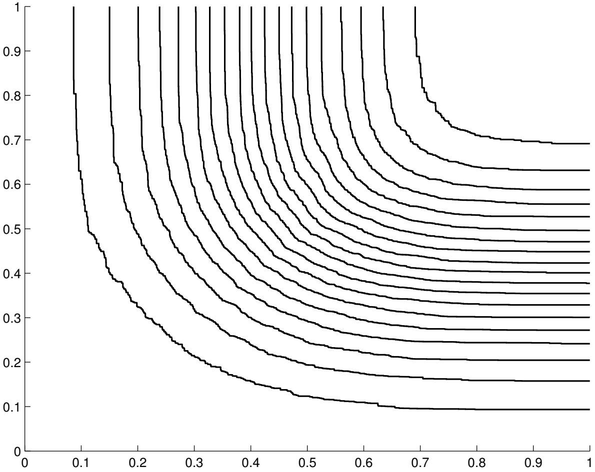

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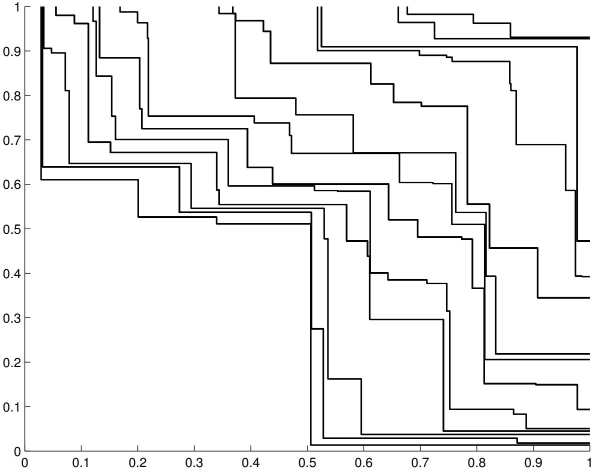
Demo: Uniform distribution on $[0,1]^2 \setminus [0,0.5]^2$



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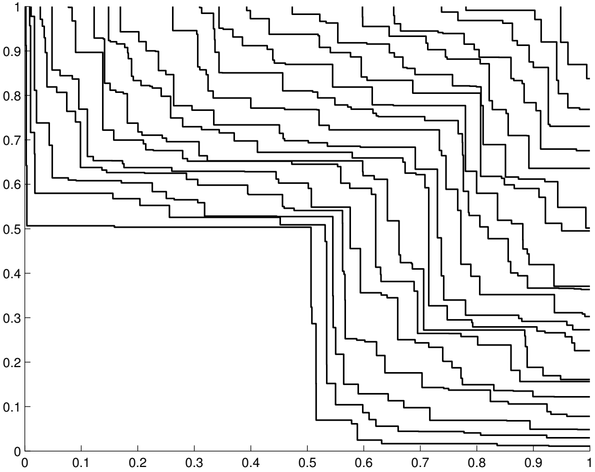

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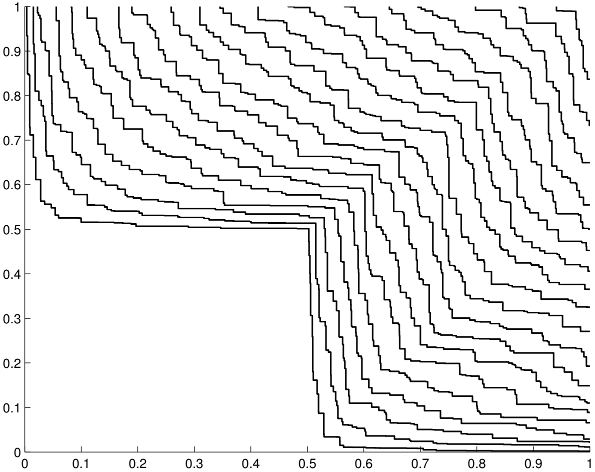

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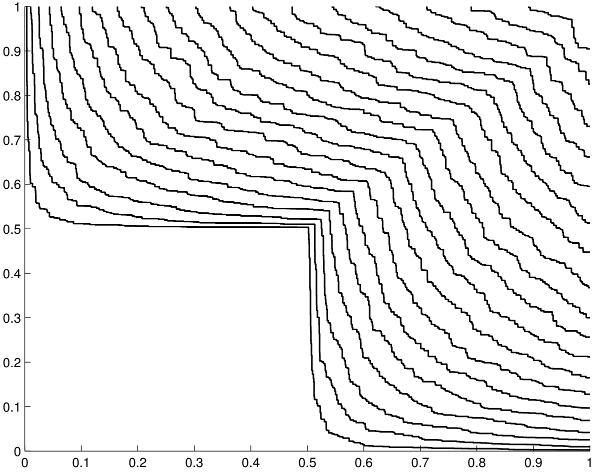

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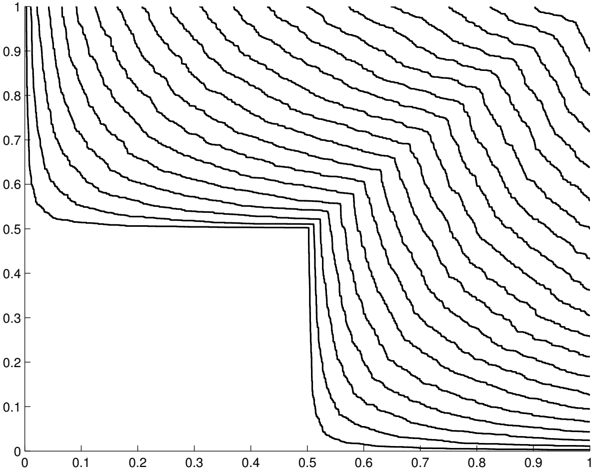

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A PDE continuum limit for nondominated sorting

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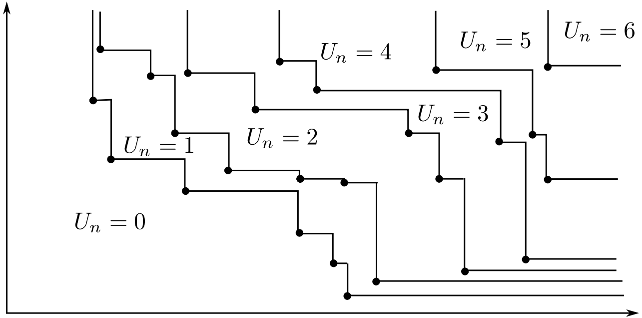

Theorem (Calder, Esedoḡlu, Hero, 2014)

There exists a universal constant $c_d > 0$ *such that with probability one*

$$n^{-\frac{1}{d}} U_n \longrightarrow c_d u \text{ locally uniformly as } n \to \infty$$

where $u \in C^{0,\frac{1}{d}}([0,\infty)^d)$ *is the unique nondecreasing* ($u_{x_i} \geq 0$) *viscosity solution of*

$$(\mathrm{P}) \begin{cases} u_{x_1} \cdots u_{x_d} = f & \text{in } \mathbb{R}^d_+ := (0,\infty)^d \\ u = 0 & \text{on } \partial\mathbb{R}^d_+. \end{cases}$$

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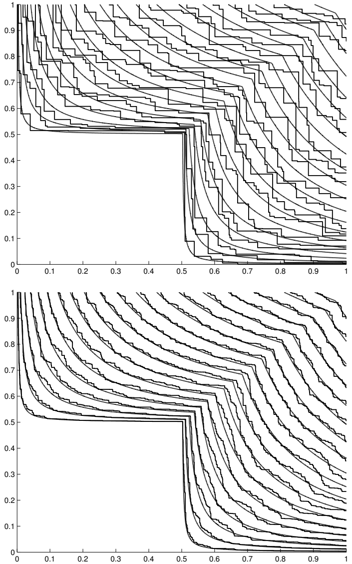
Calder, J., Esedoḡlu, S., and Hero, A. O. (2014). A Hamilton-Jacobi equation for the continuum limit of non-dominated sorting. *SIAM Journal on Mathematical Analysis*, 46(1):603–638.

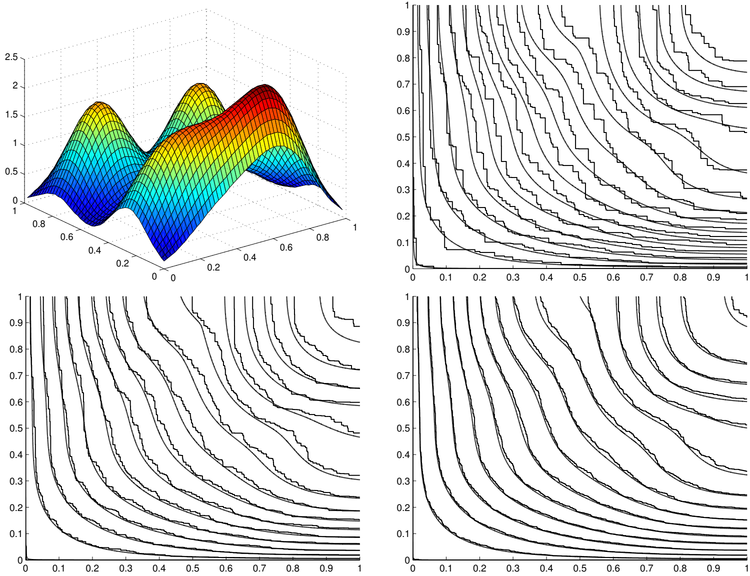
Calder, J. (2016). A direct verification argument for the Hamilton-Jacobi equation continuum limit of nondominated sorting. *Nonlinear Analysis Series A: Methods, Theory & Applications*, 141:88–108

Current work: Rate of convergence (Brendan Cook)

Demo: $f = 1 - \chi_{[0,0.5]^2}$



Demo: Multimodal f



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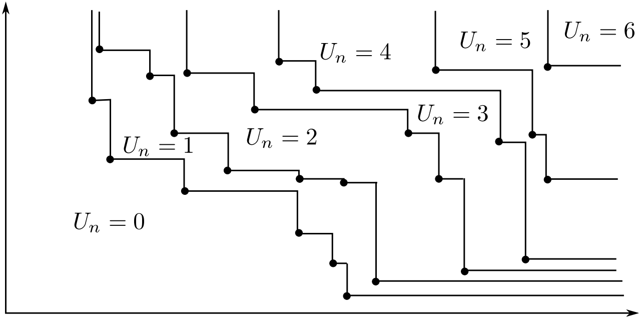

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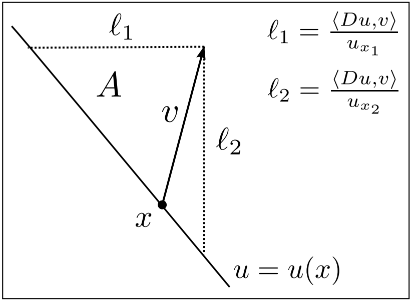

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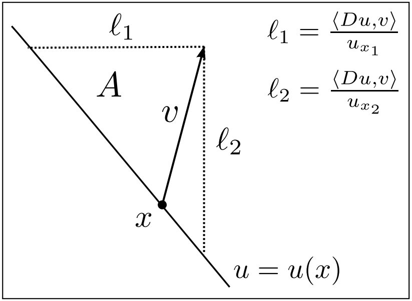

$$\langle Du, v \rangle \quad \approx \quad u(x + v) - u(x)$$

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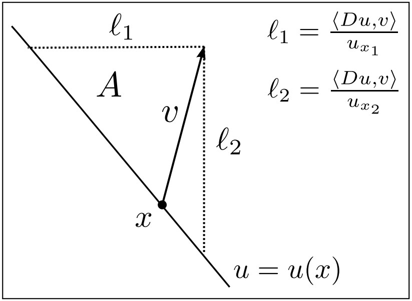

$$\begin{aligned} \langle Du, v\rangle &\approx u(x+v) - u(x) \\ &\approx (\#\text{ fronts in } A) n^{-\alpha} \end{aligned}$$

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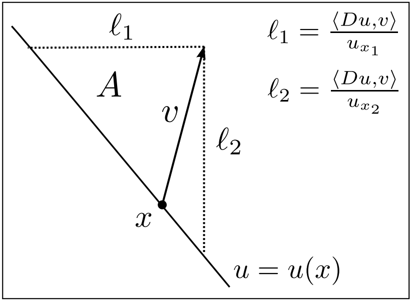

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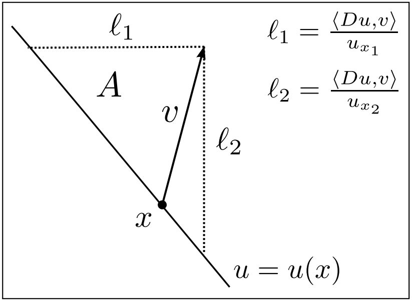

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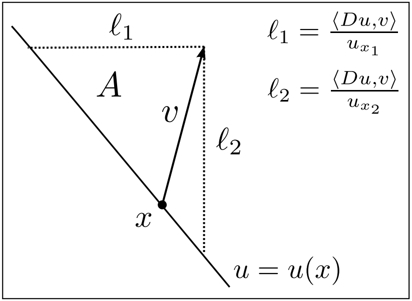

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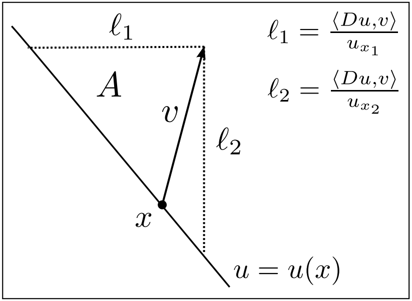

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Use $|A| \approx \frac{\langle Du, v\rangle^d}{u_{x_1} \cdots u_{x_d}}$

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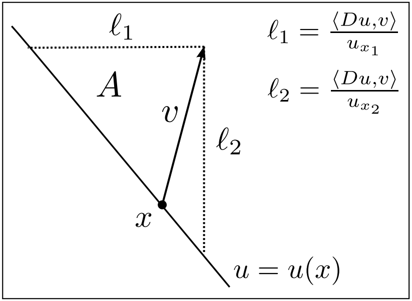

$$\begin{aligned}
\langle Du, v\rangle &\approx u(x+v) - u(x) \\
&\approx (\text{\# fronts in } A) n^{-\alpha} \\
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Use $|A| \approx \frac{\langle Du, v\rangle^d}{u_{x_1} \cdots u_{x_d}}$ to find

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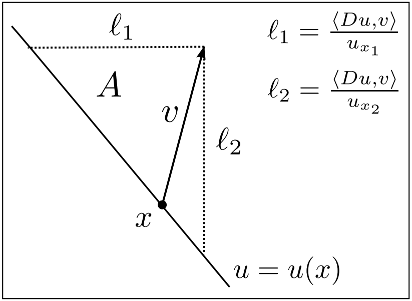

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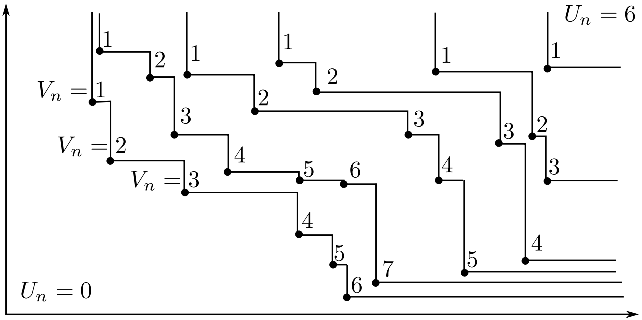
If $\alpha d = 1$, or $\alpha = 1/d$, then

$$\boxed{u_{x_1} \cdots u_{x_d} = f}$$

Ordering within each front

Let $X_1, \ldots, X_n$ be i.i.d. random variables with density f on $[0,1]^2$. Define

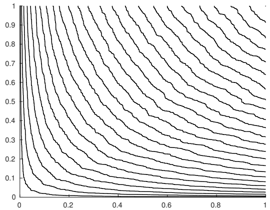
$$V_n(X_i) = \text{Index of } X_i \text{ within its Pareto front.}$$

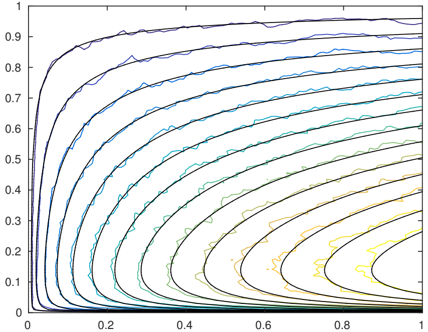


Demo: Uniform distribution on $[0,1]^2$

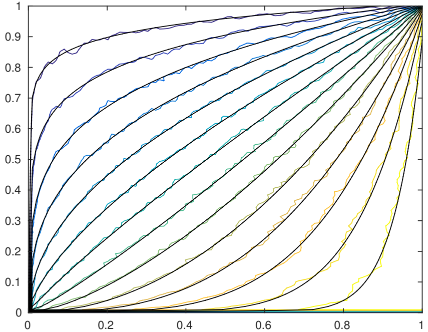
$$\text{(T)} \begin{cases} \langle Dv, D^\perp u\rangle = f & \text{in } (0,1)^2, \\ v = 0 & \text{on } (0,1)\times\{x_2 = 1\}. \end{cases}$$

$$\text{(T')} \begin{cases} \langle Dw, vD^\perp u\rangle = wf & \text{in } (0,1)^2, \\ w = 1 & \text{on } \{x_1 = 1\}\times(0,1). \end{cases}$$





(a) V_n vs. v



(b) W_n vs. w

Fast approximate sorting

Algorithm (PDE-based Ranking)

1. *Select k points from $X_1, \dots, X_n$ at random. Call them $Y_1, \dots, Y_k$. ($k \ll n$)*

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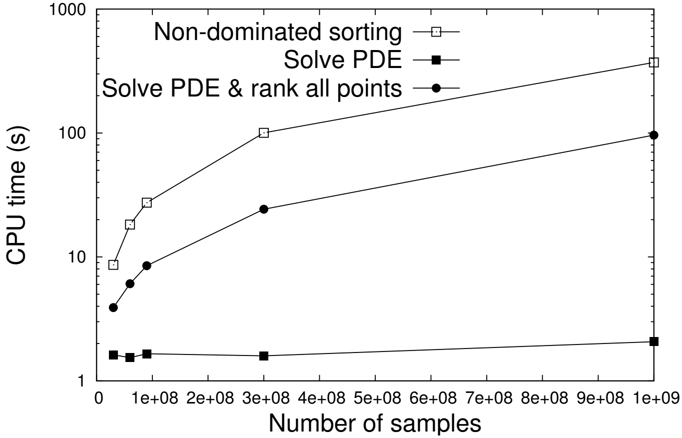
Notes:

- Total complexity is $O(k + h^{-d} + n)$.
- If we fix k and h, independent of n, then Steps 1-3 have $O(1)$ complexity.

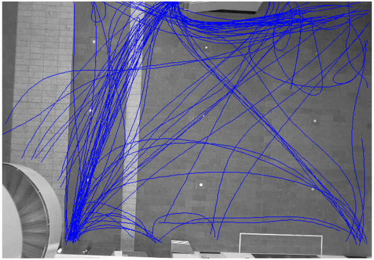
Calder, J., Esedoḡlu, S., and Hero, A. O. (2015). A PDE-based approach to nondominated sorting. *SIAM Journal on Numerical Analysis*, 53(1):82–104.

CPU Time (C/C++)

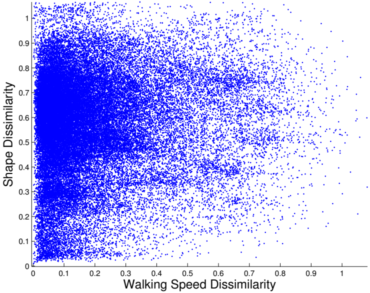


- # Subsamples $= k = 10^7$, Grid for solving PDE $= 250 \times 250$.
- $O(n \log n)$ non-dominated sorting of [Felsner and Wernisch, 1999].

Application in anomaly detection



(a) Example trajectories

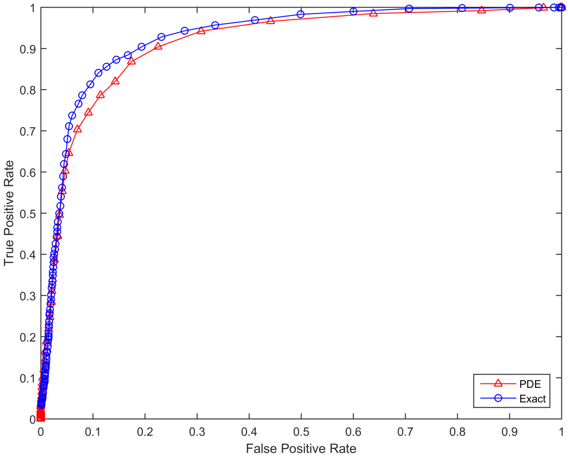


(b) 5×10^5 Pareto points

Abbasi, B., Calder, J., and Oberman, A.M. Anomaly detection and classification for streaming data using PDEs *SIAM Journal on Applied Mathematics*, 78(2), 921–941, 2018.

Results

Anomaly detection with PDE-based ranking: Reduces complexity from $O(n^2)$ to $O(n)$.



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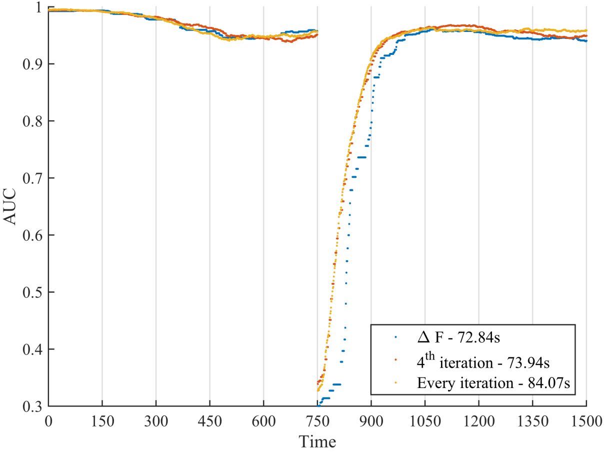
Anomaly detection for streaming data:



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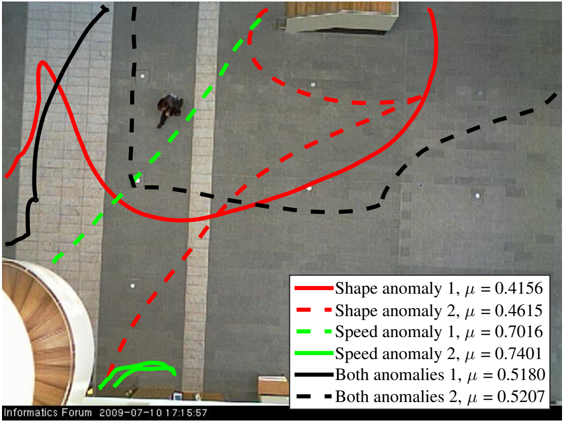
Examples of detected anomalies. . .

with classifications using the new transport equations.



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Outline

Convex hull peeling

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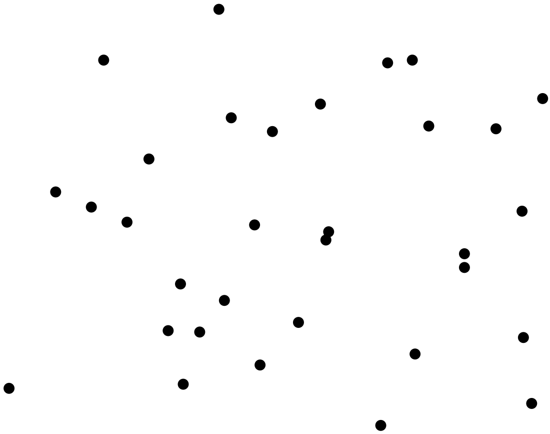

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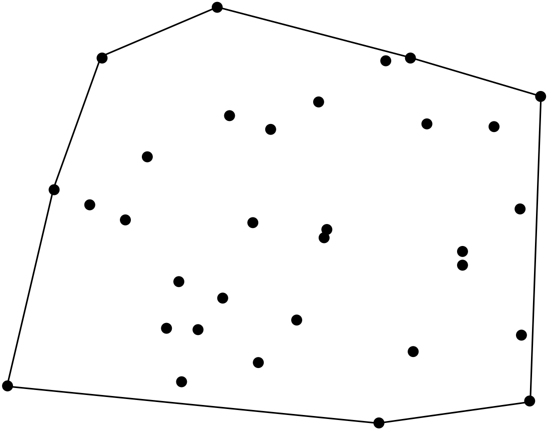

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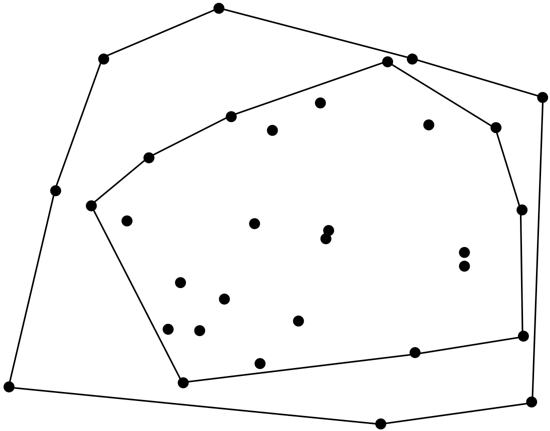

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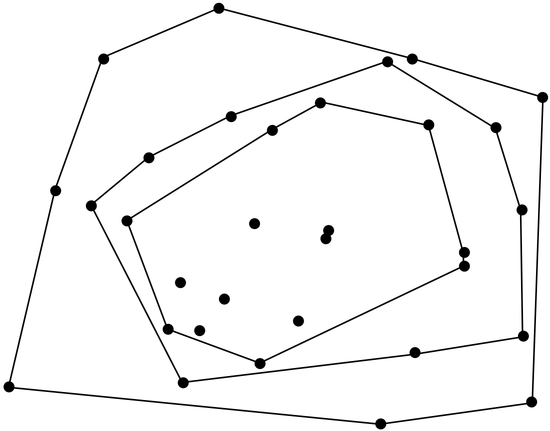

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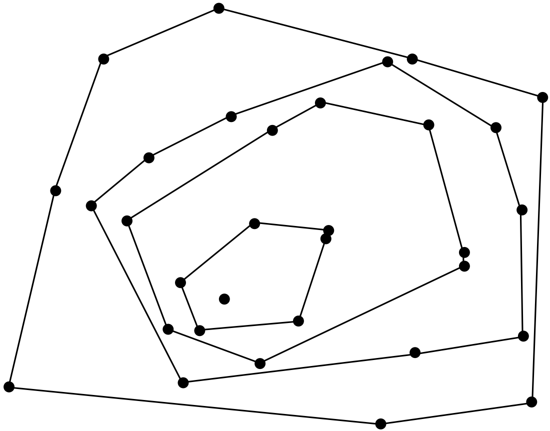

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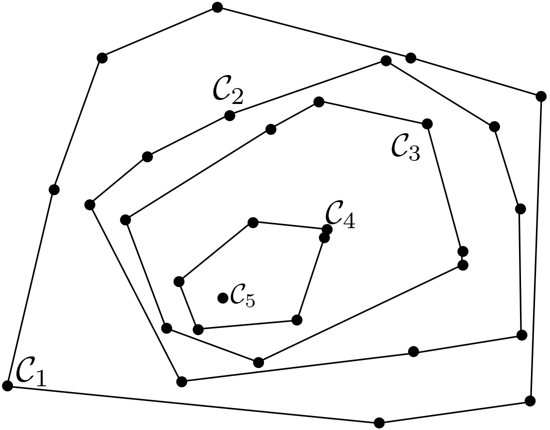

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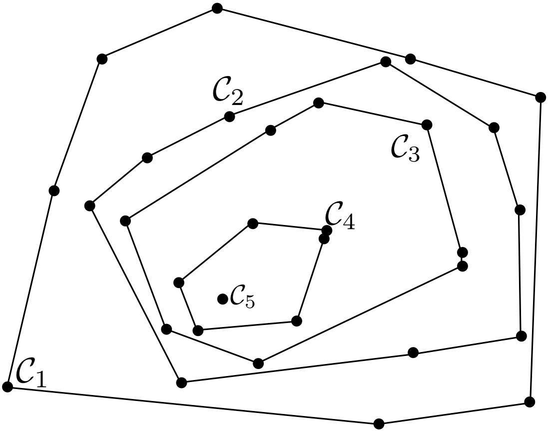
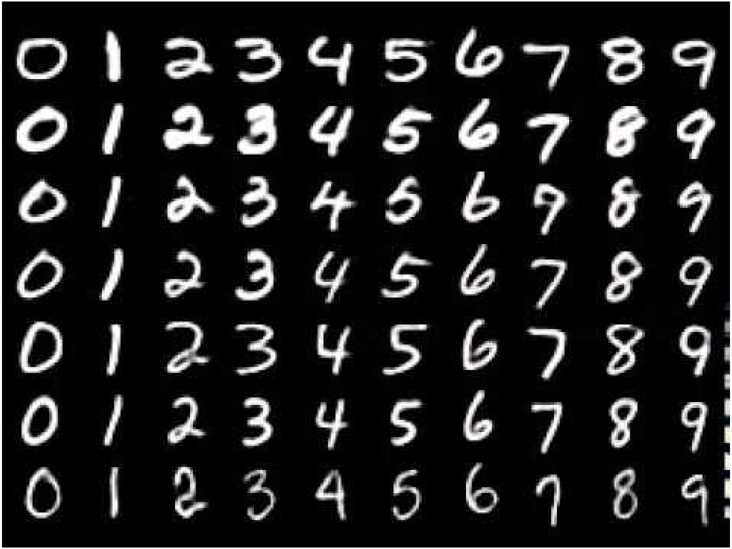


Convex hull peeling median := Centroid of final layer

MNIST handwritten digit dataset



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Definition

Convex hull peeling is the process of arranging S into convex layers $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \ldots,$ defined by

$$\mathcal{C}_1 = \text{Vertices of convex hull of } S,$$

$$\mathcal{C}_k = \text{Vertices of convex hull of } S \setminus (\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_{k-1}).$$

Convex hull peeling

Applications:

- Robust statistics, machine learning, etc.
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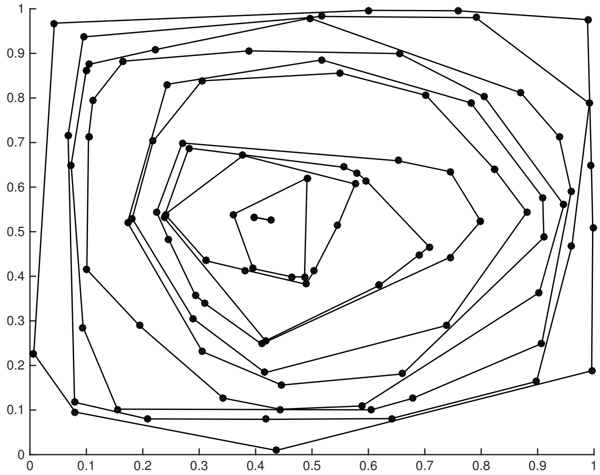
Convex hull peeling: Demo - Uniform distribution



$n = 10^2$ points

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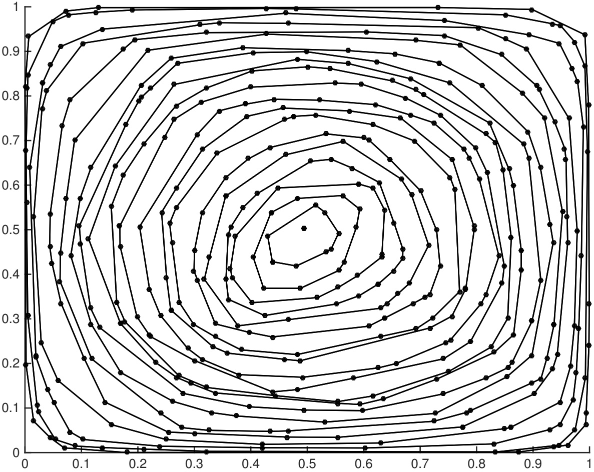

$n = 10^3$ points

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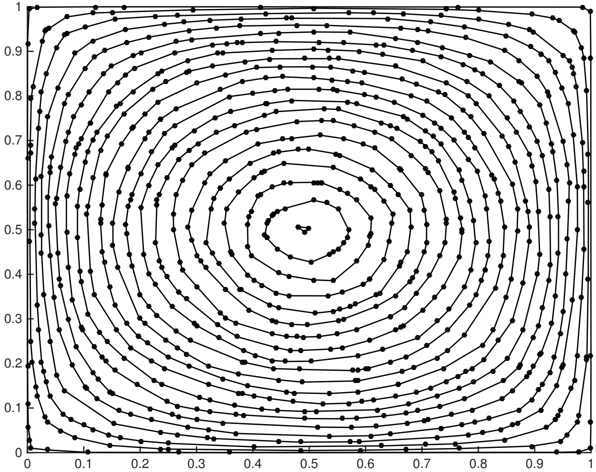
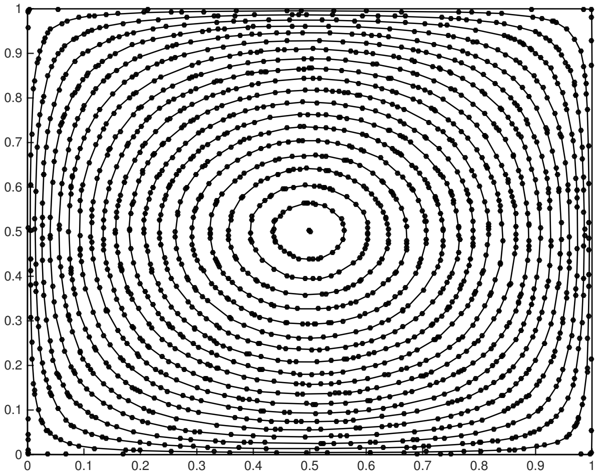


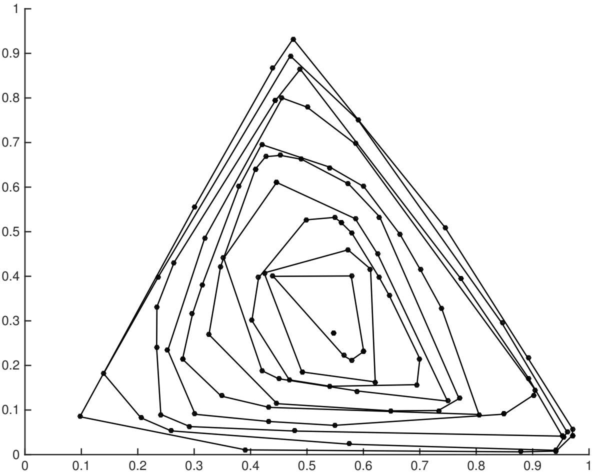
$n = 10^4$ points

Convex hull peeling: Demo - Uniform distribution



$n = 10^5$ points

Convex hull peeling: Demo - Triangle distribution



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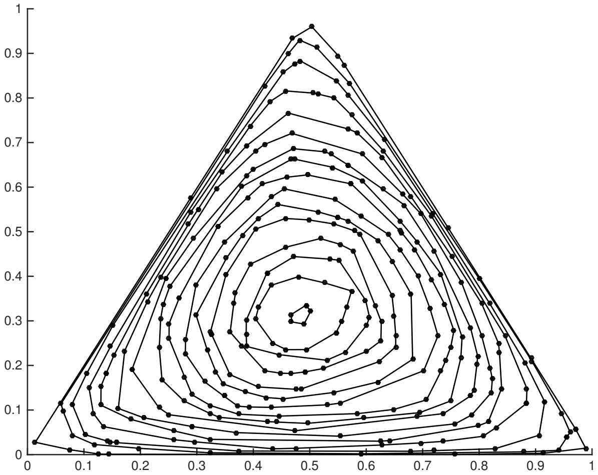

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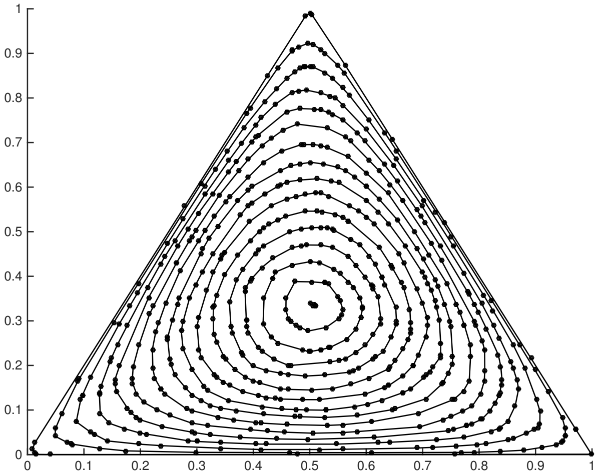

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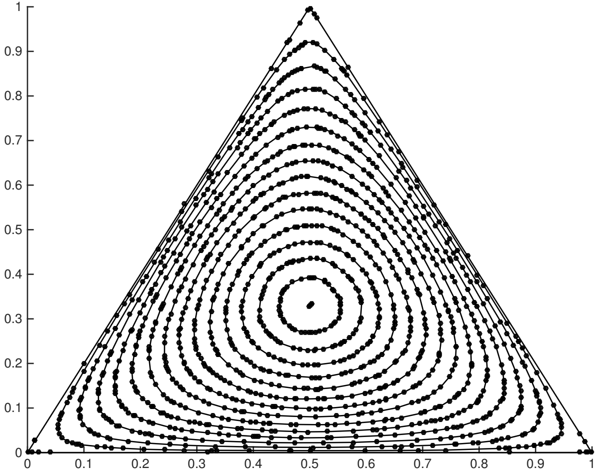

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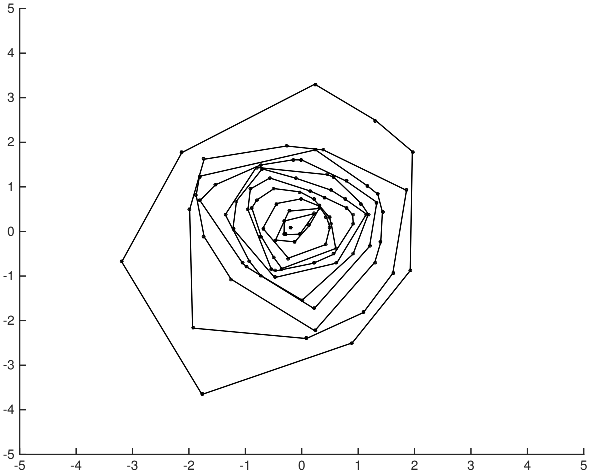
Convex hull peeling: Demo - Gaussian distribution



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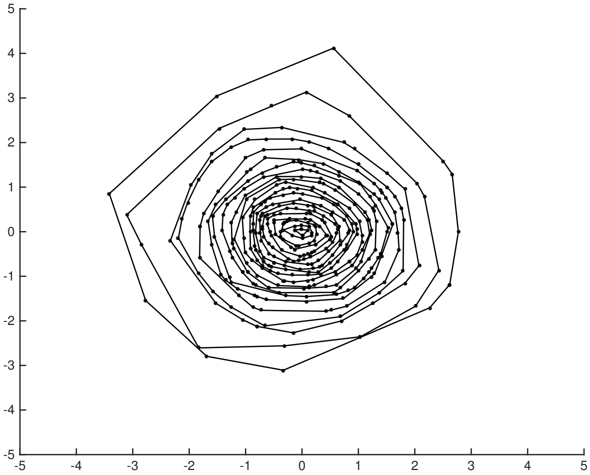

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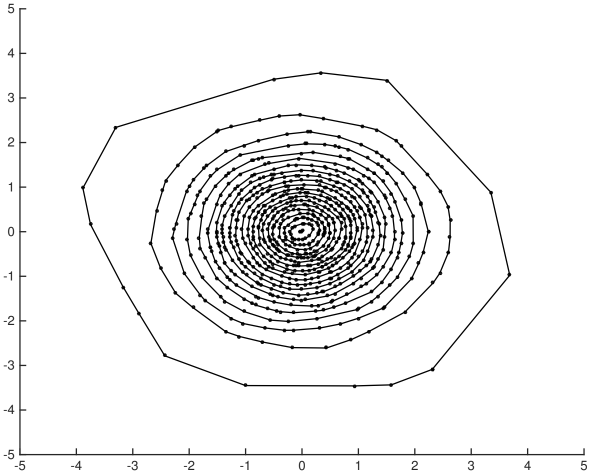

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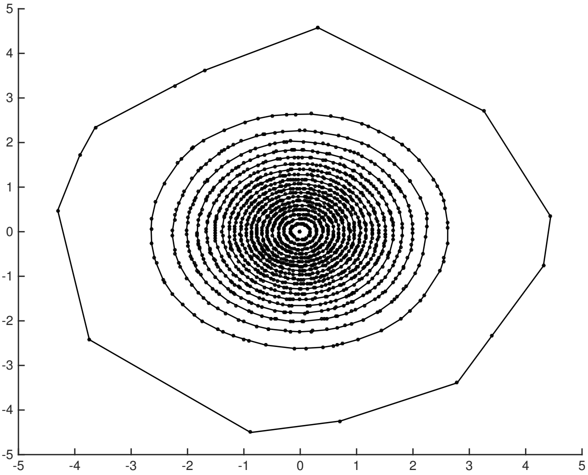

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A two player game for convex hull peeling

Players: Paul and Carol
State space: $\mathcal{X} := \{X_1, \dots, X_n\}$

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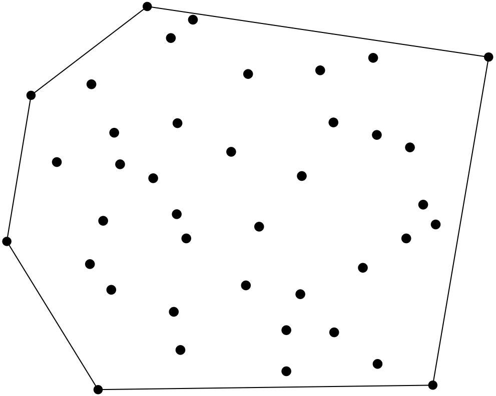

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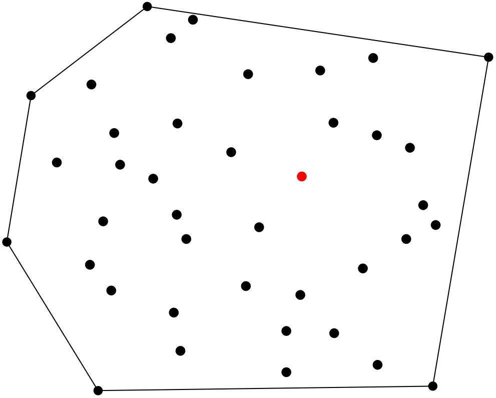

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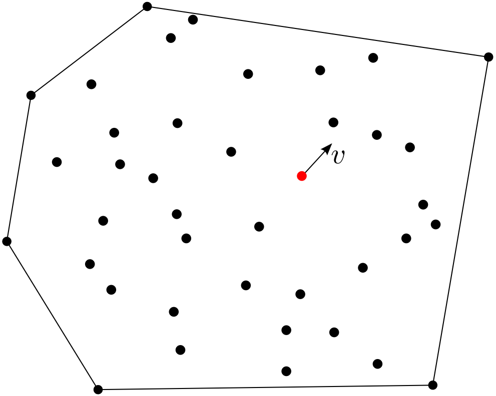

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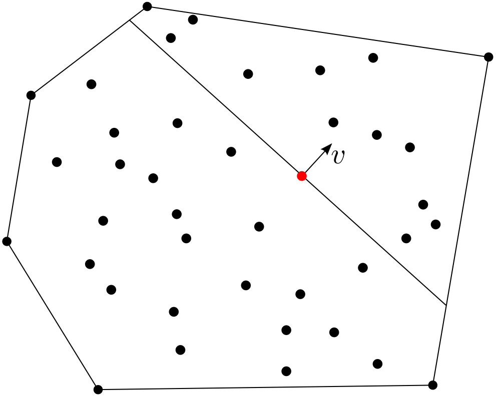

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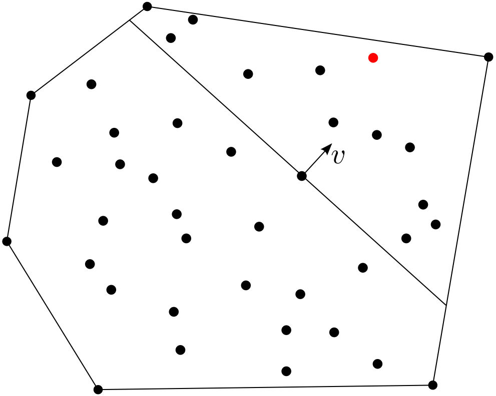

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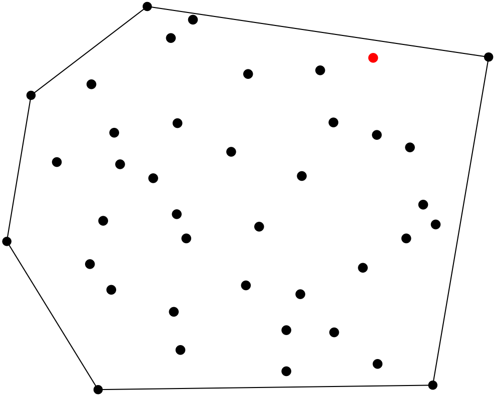

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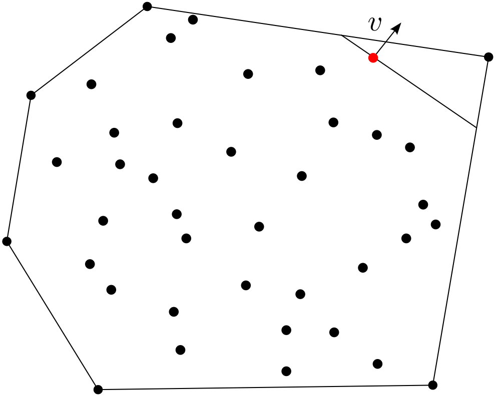


A two player game for convex hull peeling

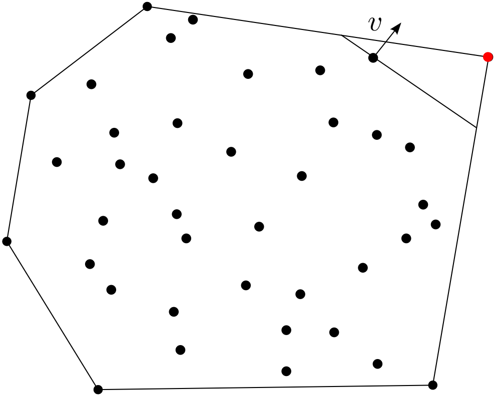
Players: Paul and Carol
State space: $\mathcal{X} := \{X_1, \ldots, X_n\}$

Paul's goal: Reach vertex of convex hull
Carol's goal: Obstruct Paul

Rules of the game: Token starts at $x^0 \in \mathcal{X}$ and is moved according to:

1. Paul picks $v \in \mathbb{S}^{d-1}$
2. Carol moves token to any $x^{k+1} \in \mathcal{X}$ satisfying

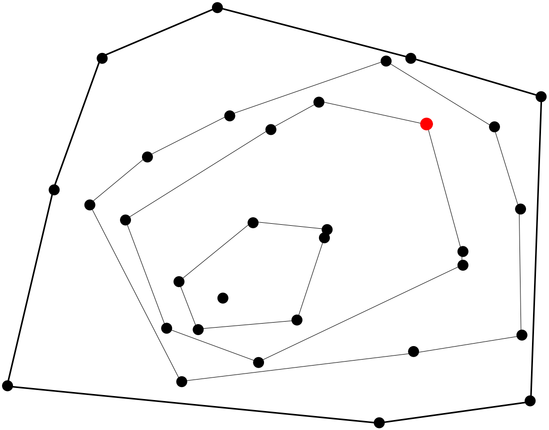
$$(x^{k+1} - x^k) \cdot v > 0.$$



A two player game for convex hull peeling

Paul's optimal choice: Any halfspace supporting current convex layer

Carol's optimal choice: Any point on the previous convex layer



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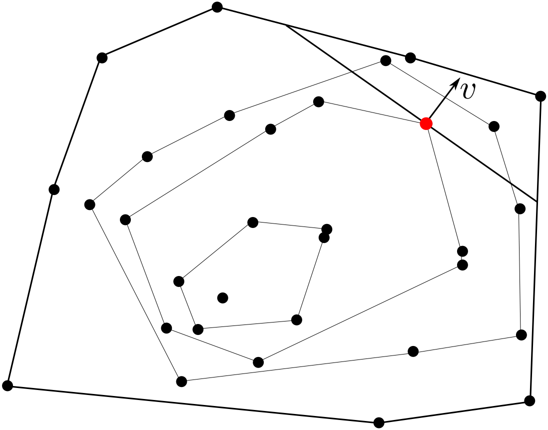

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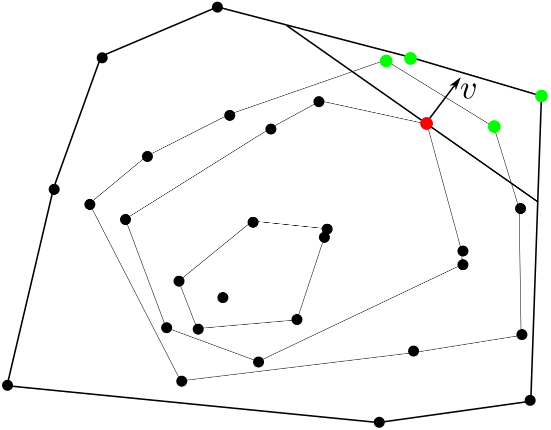

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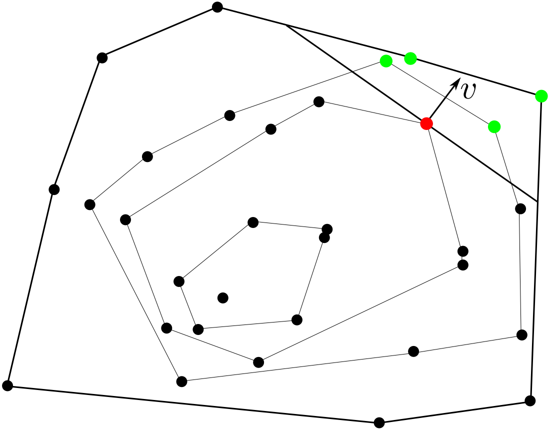

Value function $= U_n(x^0) =$ Convex depth function.

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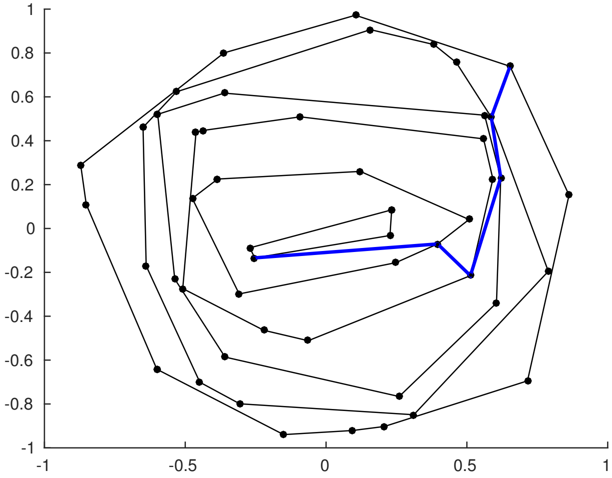
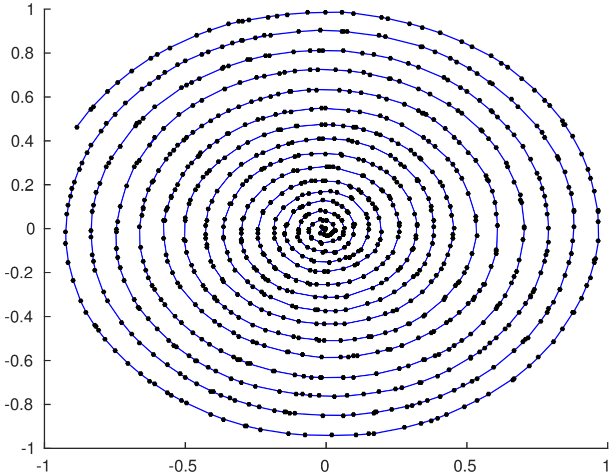


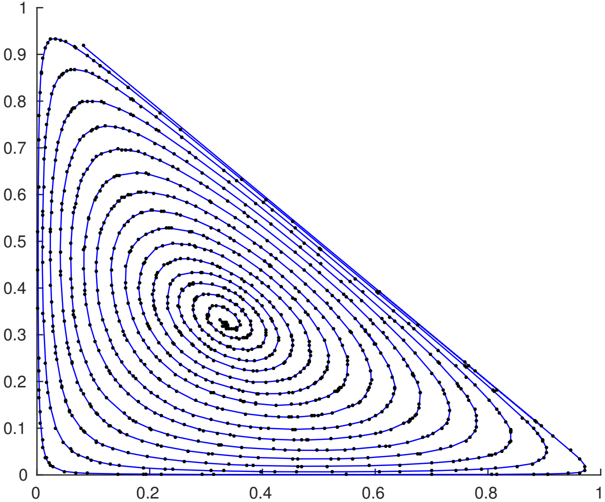
$n = 50$ points

A two player game for convex hull peeling



$n = 10^5$ points

A two player game for convex hull peeling

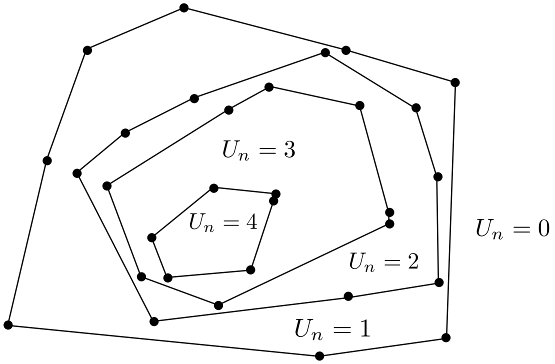


$n = 10^5$ points

A PDE continuum limit for convex hull peeling

Let $X_1, \ldots, X_n$ be i.i.d. with a continuous density f on a convex set $\Omega \subset \mathbb{R}^d$.

Let U_n be the function that 'counts' the associated convex layers $\mathcal{C}_1, \mathcal{C}_2, \ldots$



Partial differential equation (PDE) continuum limit

Theorem (Joint with C. Smart)

There exists a universal constant α_d *such that with probability one*

$$n^{-\frac{2}{d+1}} U_n \longrightarrow \alpha_d u \quad \textit{uniformly on } \Omega,$$

where $u \in C(\overline{\Omega})$ *is the unique viscosity solution of*

$$\begin{cases} \nabla u \cdot \text{cof}(-\nabla^2 u)\nabla u = f^2 & \textit{in } \Omega \\ u = 0 & \textit{on } \partial\Omega. \end{cases} \tag{1}$$

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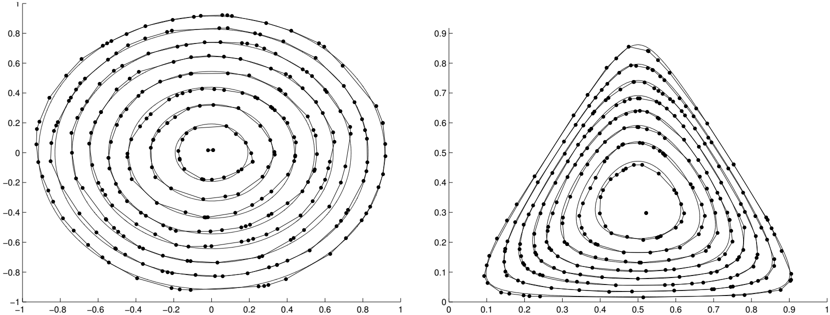
This is just motion by a power of Gauss curvature

$$\frac{dS}{dt} = f^{-2/(d+1)} \kappa_G^{1/(d+1)} \mathbf{n}.$$

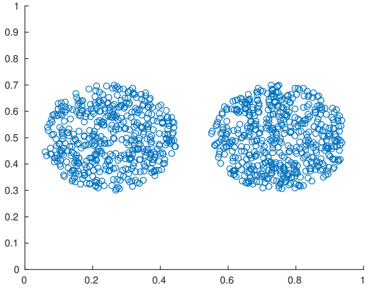
A PDE continuum limit for convex hull peeling



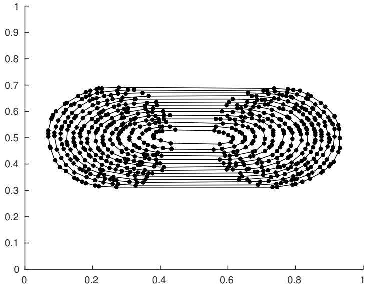
(c) (d)

Figure: Convex layers vs continuum limit for $n = 5 \times 10^3$.

A nonconvex example



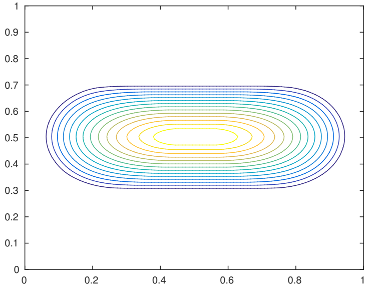
(a) Samples



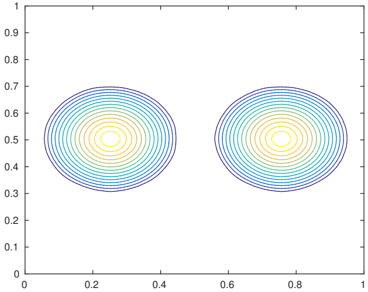
(b) Convex layers

Figure: Convex layers corresponding to disjoint clusters.

A nonconvex example



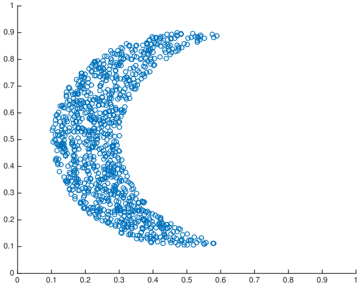
(a) One solution



(b) Another solution

Figure: Two different solutions continuum PDE.

The halfmoon



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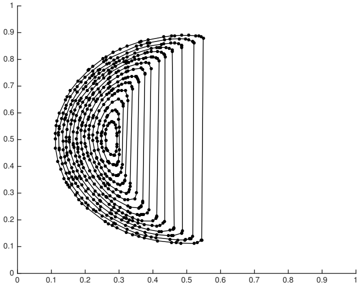
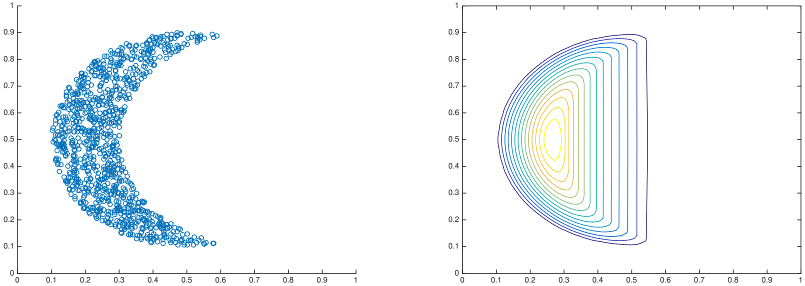


(b) Convex layers

Figure: Convex layers corresponding to the halfmoon distribution.

The halfmoon



(a) Samples (b) PDE

Figure: Solution of PDE for the halfmoon example.

Outline

Quick intro to learning

Fully supervised: In fully supervised learning, we are given training data (x_i, y_i) for $i = 1, \ldots, n$, where $x_i \in \mathcal{X}$ are the data points and $y_i \in \mathcal{Y}$ are the known labels.

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Classification when $\mathcal{Y}$ finite – Regression when $\mathcal{Y} = \mathbb{R}^d$.

Example: Automated image captioning

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A woman is throwing a **frisbee** in a park.

A **dog** is standing on a hardwood floor.

A **stop** sign is on a road with a mountain in the background

A little **girl** sitting on a bed with a teddy bear.

A group of **people** sitting on a boat in the water.

A giraffe standing in a forest with **trees** in the background.

[Yann LeCun, Yoshua Bengio, Geoffrey Hinton. Deep learning. Nature, 2015.]

Example: Automated image captioning fail

[Andrej Karpathy's NeuralTalk]

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A great introductory book [Chapelle et al., 2006].

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Semi-supervised smoothness assumption

Similar points $x, y \in \mathcal{X}$ in high density regions of the graph should have similar labels.

Laplacian regularization

$$\min_{u:\mathcal{X}\to\mathbb{R}} \sum_{x,y\in\mathcal{X}} w_{xy}^2 (u(x)-u(y))^2 \quad \text{subject to } u(x)=g(x) \text{ for all } x\in\mathcal{O}.$$

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The minimizer $u : \mathcal{X} \to \mathbb{R}$ satisfies the linear system

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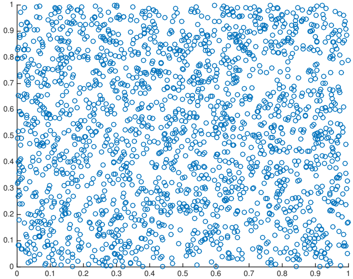
References:

- Original work [Zhu et al., 2003]
- Learning [Zhou et al., 2005][Ando and Zhang, 2007]
- Manifold ranking [He et al., 2006] [Wang et al., 2013] [Yang et al., 2013] [Zhou et al., 2011] [Xu et al., 2011]

Ill-posed with small amount of labeled data



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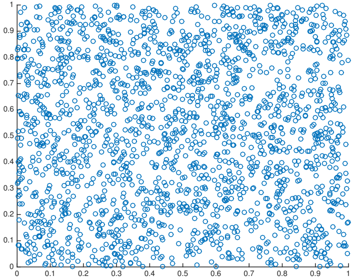

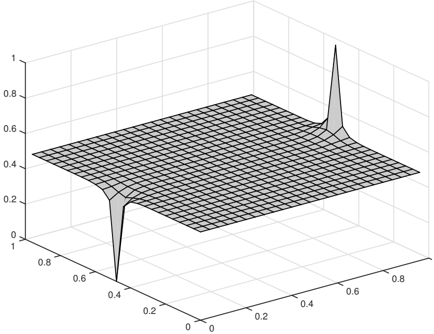

- Graph is $n = 10^5$ i.i.d. random variables uniformly drawn from $[0, 1]^2$.
- $w_{xy} = 1$ if $|x - y| < 0.01$ and $w_{xy} = 0$ otherwise.
- Over 95% of labels in $[0.4975, 0.5025]$.

[Nadler et al., 2009][El Alaoui et al., 2016]

ℓ_p-based Laplacian regularization

For any $p < \infty$:

$$\min_{u:\mathcal{X}\to\mathbb{R}} \sum_{x,y\in\mathcal{X}} w_{xy}^p |u(x) - u(y)|^p \quad \text{subject to } u(x) = g(x) \text{ for all } x \in \mathcal{O}. \tag{3}$$

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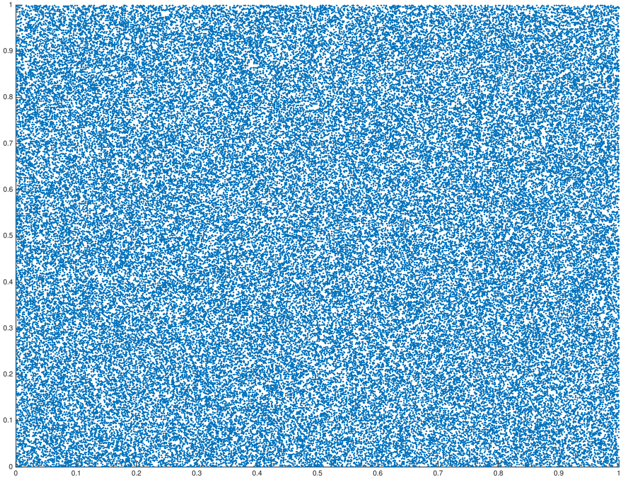
References:

- Finite p: [Bridle and Zhu, 2013][Alamgir and Luxburg, 2011]
- $p = \infty$: [Kyng et al., 2015] [Luxburg and Bousquet, 2004]
- Absolutely minimal Lipschitz extensions: [Aronsson et al., 2004]

p-Laplacian learning: $n = 10^5$ points, $h = 10^{-2}$



$p = 2$

Simulations are the work of Mauricio Flores (co-supervised by Gilad Lerman).

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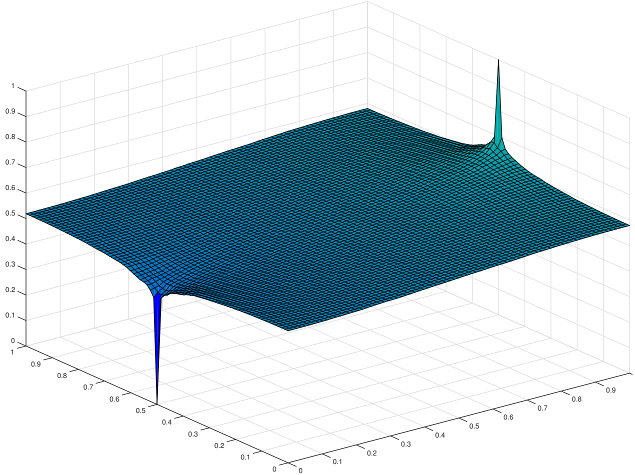

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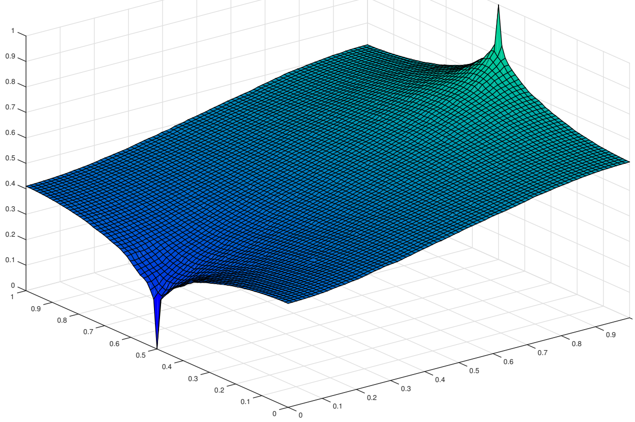

$p = 2.5$

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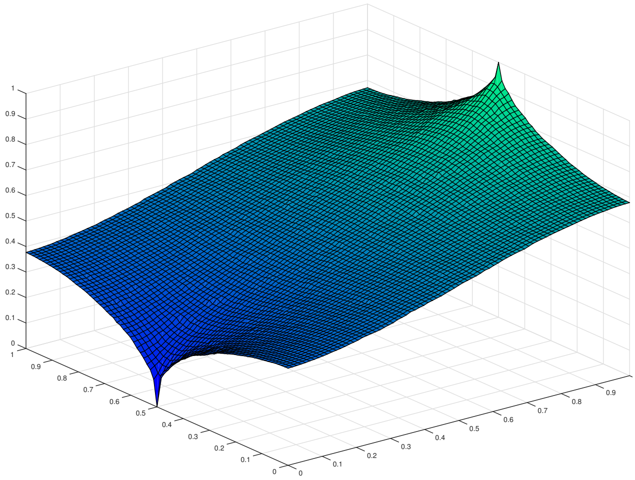

$p = 3$

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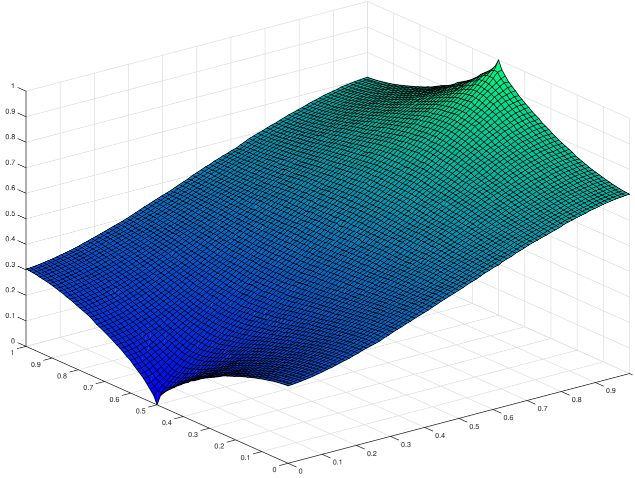
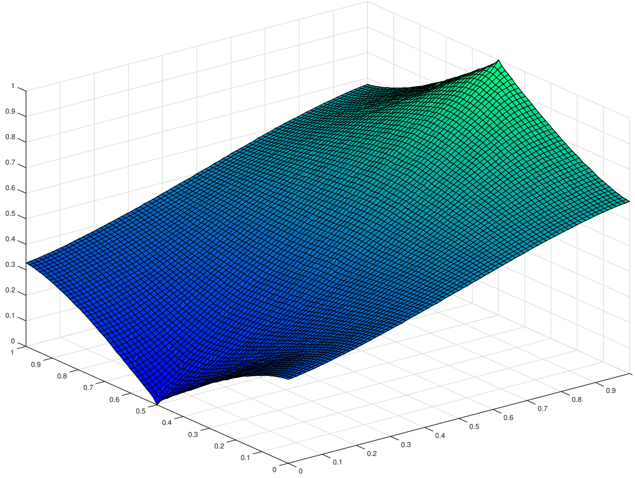


$p = 5$

Simulations are the work of Mauricio Flores (co-supervised by Gilad Lerman).

p-Laplacian learning: $n = 10^5$ points, $h = 10^{-2}$



$$p = \infty$$

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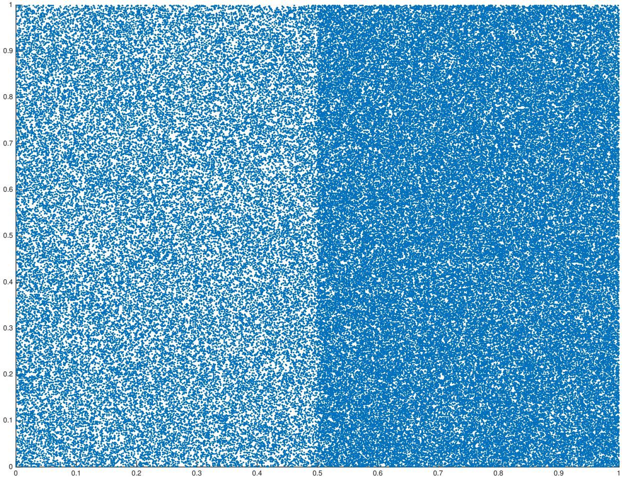
p-Laplacian learning: Varying density



$p = 2$

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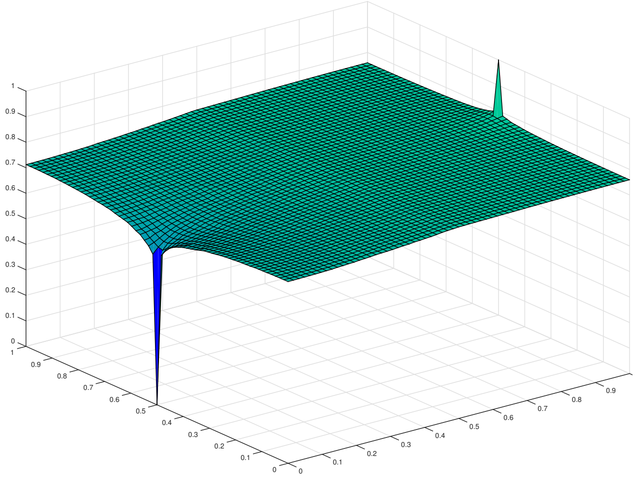

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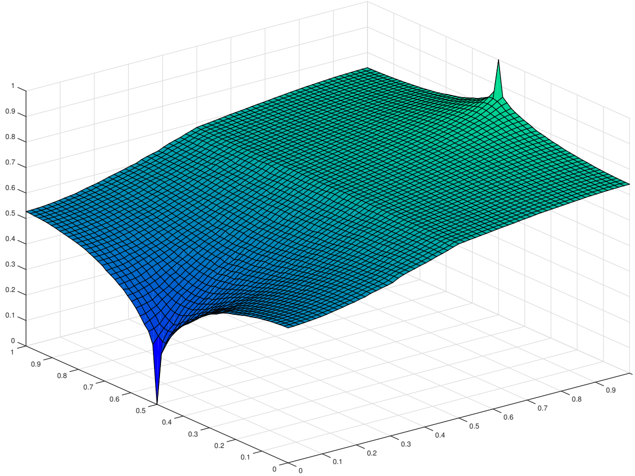

$p = 2.5$

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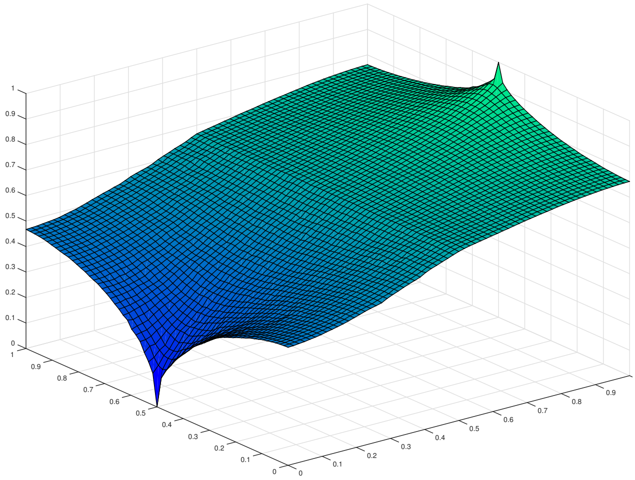
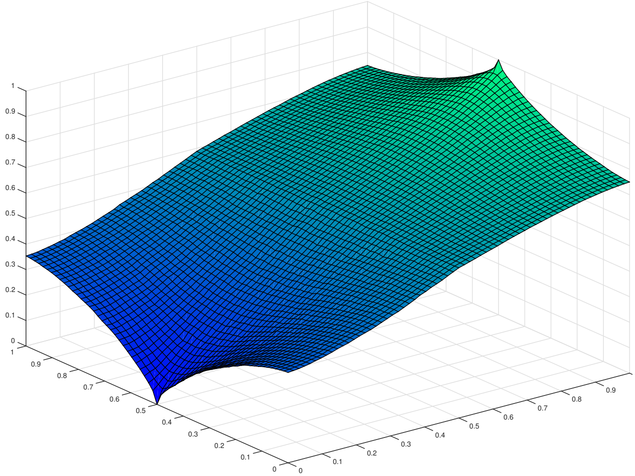


$p = 3$

Simulations are the work of Mauricio Flores (co-supervised by Gilad Lerman).

p-Laplacian learning: Varying density



$$p = 5$$

Simulations are the work of Mauricio Flores (co-supervised by Gilad Lerman).

Random model

- **Labeled data:** The labeled data is a fixed finite collection of N points

$$\mathcal{O} = \{y_1, \ldots, y_N\} \subset U \subset \mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d.$$

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- **Edge weights:** The edge weights are

$$w_{xy} = \Phi\left(\frac{|x-y|}{h}\right),$$

where $h > 0$, and $\Phi : [0,\infty) \to [0,\infty)$.

Random model

For $p < \infty$ we write

$$J_p(u) := \sum_{x,y \in \mathcal{X}_n} w_{xy}^p |u(x) - u(y)|^p,$$

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For $n \geq 1$, let $u_n : \mathcal{X}_n \to \mathbb{R}$ be the solution of

$$\boxed{\min_{u:\mathcal{X}_n \to \mathbb{R}} J_p(u) \quad \text{subject to } u(x) = g(x) \text{ for all } x \in \mathcal{O}.}$$

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Question: What can we say about u_n as $n \to \infty$?

Let

$$r_n = \sup \{ s > 0 \,|\, B(x,s) \cap \mathcal{X}_n = \varnothing \text{ for some } x \in U \}. \tag{5}$$

Theorem ($p = \infty$ [Calder, 2017a])

Suppose that $h_n \to 0$ *such that*

$$\lim_{n\to\infty} \frac{r_n^2}{h_n^3} = 0. \tag{6}$$

$$\textit{Then } u_n \longrightarrow u \quad \textit{uniformly as} \quad n \to \infty, \tag{7}$$

where $u \in C(\mathbb{T}^d)$ *is the unique viscosity solution of the* ∞*-Laplace equation*

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \mathbb{T}^d \setminus \mathcal{O} \\ u = g & \text{on } \mathcal{O} \end{cases} \tag{8}$$

Note that (6) holds almost surely when

$$\lim_{n\to\infty} \frac{n h_n^{3d/2}}{\log(n)} = \infty. \tag{9}$$

Theorem (Finite p [Calder, 2017b])

Let $d < p < \infty$, and suppose that $h_n \to 0$ such that

$$\lim_{n\to\infty} nh_n^p = 0 \quad \textit{and} \quad \lim_{n\to\infty} \frac{nh_n^{d+4}}{\log(n)} = \infty. \tag{10}$$

Then with probability one

$$u_n \longrightarrow u \quad \textit{uniformly as} \quad n \to \infty, \tag{11}$$

where $u \in C(\mathbb{T}^d)$ is the unique viscosity solution of the weighted p-Laplace equation

$$\begin{cases} \textit{div}\left(f^2|\nabla u|^{p-2}\nabla u\right) = 0 & \textit{in } \mathbb{T}^d \setminus \mathcal{O} \\ u = g & \textit{on } \mathcal{O} \end{cases} \tag{12}$$

A very similar result appeared recently in [Slepčev and Thorpe, 2017].

Regularity in semi-supervised learning

The PDE-limit can be used to prove Hölder regularity.

Theorem

Assume $p > d$. For every $\alpha < \frac{p-d}{p-1}$ there exists C, δ such that

$$\mathbb{P}\left[\forall x, y \in \mathcal{X}_n,\ |u_n(x) - u_n(y)| \le C(|x-y|^\alpha + n^{\frac{1}{p}} h)\right] \ge 1 - \exp\left(-\delta n h^{d+4} + C\log(n)\right).$$

Graph Laplacians

$$\min_{u:\mathcal{X}_n\to\mathbb{R}} J_p(u) = \sum_{x,y\in\mathcal{X}_n} w_{xy}^p |u(x)-u(y)|^p \quad \text{subject to } u(x)=g(x) \text{ for } x\in\mathcal{O}\subset\mathcal{X}_n$$

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References on graph p-Laplacian:

- [Manfredi et al., 2015] [Zhou and Schölkopf, 2005] [Amghibech, 2003] [Bühler and Hein, 2009] [Luo et al., 2010]

Graph Laplacian as $p \to \infty$

Note that solutions of

$$\Delta_p^{\mathcal{X}_n} u(x) = \sum_{y \in \mathcal{X}_n} w_{xy}^p |u(y) - u(x)|^{p-2}(u(y) - u(x)) = 0$$

satisfy

$$\left(\sum_{\substack{y \in \mathcal{X}_n \\ u(y) \geq u(x)}} w_{xy}^p |u(y) - u(x)|^{p-1} \right)^{1/p} = \left(\sum_{\substack{y \in \mathcal{X}_n \\ u(y) < u(x)}} w_{xy}^p |u(y) - u(x)|^{p-1} \right)^{1/p}.$$

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Send $p \to \infty$ to get

$$\max_{y \in \mathcal{X}_n} w_{xy}(u(y) - u(x)) = \max_{y \in \mathcal{X}_n} w_{xy}(u(x) - u(y)).$$

or

$$\Delta_\infty^{\mathcal{X}_n} u(x) := \max_{y \in \mathcal{X}_n} w_{xy}(u(y) - u(x)) + \min_{y \in \mathcal{X}_n} w_{xy}(u(y) - u(x)) = 0.$$

Graph Laplacians

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Reference:

1. [Kyng et al., 2015]

Game theoretic p-Lapacian

We can also consider the game theoretic p-Laplacian for semi-supervised learning:

$$\begin{cases} \dfrac{1}{d_n}\Delta_2^{\mathcal{X}_n} u_n + \lambda(p-2)\Delta_\infty^{\mathcal{X}_n} u_n = 0 & \text{in } \mathcal{X}_n \setminus \mathcal{O} \\ u = g & \text{in } \mathcal{O}, \end{cases}$$

where $d_n(x) = \sum_{y\in\mathcal{X}_n} w_{xy}^2$ and $\lambda = \lambda(\Phi)$.

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This is likely better conditioned numerically when p is large.

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Theorem (Finite p [Calder, 2017b])

Let $d < p < \infty$, and suppose that $h \to 0$ such that

$$\lim_{n\to\infty} \frac{nh^q}{\log(n)} = \infty, \tag{13}$$

where $q = \max\{d+4, 3d/2\}$. Then with probability one

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Notice no upper bound on h (i.e., we don't require $nh^p \to 0$).

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we have

$$\mathbb{E}[\Delta_p^{\mathcal{X}_n} \varphi(x)] = nh^d \int_{\mathbb{R}^d} \Phi(|z|)|\varphi(x + zh) - \varphi(x)|^{p-2}(\varphi(x + zh) - \varphi(x))f(x + zh)\, dz.$$

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Plug in Taylor expansions and plug away. . .

$$\mathbb{E}[\Delta_p^{\mathcal{X}_n} \varphi(x)] = \frac{1}{2} C_p f^{-1} \mathrm{div}(f^2 |\nabla \varphi|^{p-2} \nabla \varphi) nh^{d+p} + R(x) nh^{d+p+1},$$

where

$$|R(x)| \leq C \|\varphi\|_{C^3(\mathbb{R}^d)}^{p-1}.$$

Hölder continuity for p-Laplace equation

The maximum principle can be used to prove Hölder continuity when $p > d$:

$$\begin{cases} \Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0 & \text{in } U \\ u = g & \text{on } \partial U, \end{cases}$$

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It follows that

$$\boxed{u(x) - u(x_0) \leq C|x - x_0|^\alpha.}$$

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   1. For the variational graph p-Laplacian

      $$|u_n(x) - u_n(y)| \leq C n^{1/p} h \text{ for } |x-y| \leq h.$$

   2. For the game theoretic p-Laplacian, we use a different local barrier

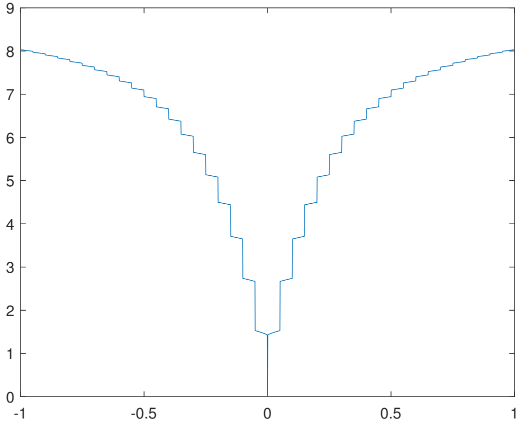
      $$v(x) = |x-y|^\alpha + M h_n^\alpha \sum_{k=1}^{\infty} \beta^k 1_{\{2|x-y| > (k-1)h_n\}}, \text{ where } \beta < 1.$$

The local barrier

$$v(x) = |x-y|^\alpha + Mh_n^\alpha \sum_{k=1}^{\infty} \beta^k 1_{\{2|x-y|>(k-1)h_n\}}$$

exploits the form of the graph ∞-Laplacian

$$\Delta_\infty^{\mathcal{X}_n} u(x) = \max_{y\in\mathcal{X}_n} w_{xy}(u(y)-u(x)) + \min_{y\in\mathcal{X}_n} w_{xy}(u(y)-u(x)).$$



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4. **Soft constraint:** Extend the results to the soft constraint

$$\min_{u:\mathcal{X}_n \to \mathbb{R}} J_p(u) + \lambda \sum_{y \in \mathcal{O}} |u(x) - g(x)|^q.$$

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