Absolute convergence implies convergence

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Definition 1 *The series* $\sum_{k=1}^{\infty} A_k$ *is* **absolutely convergent** *if the series*

$$\sum_{k=1}^{\infty} |A_k|$$

converges.

Thus the p-series for $p = 2$

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

is absolutely convergent, as is the alternating series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2}.$$

However, the alternating series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$$

is not absolutely convergent, even though it converges.

Theorem 1 *If* $\sum_{k=1}^{\infty} A_k$ *is absolutely convergent, then it is convergent. In other words, if the series of all nonnegative terms* $\sum_{k=1}^{\infty} |A_k|$ *converges, then so does the series* $\sum_{k=1}^{\infty} A_k$.

PROOF: Note that for any k we always have $A_k \leq |A_k|$, and $0 \leq |A_k|$, so that

$$0 \leq A_k + |A_k| \leq 2|A_k|.$$

Now assume that $\sum_{k=1}^{\infty} A_k$ is absolutely convergent. This implies that the series $2\sum_{k=1}^{\infty} |A_k|$ is convergent. Now let $a_k = A_k + |A_k|$ and $b_k = 2|A_k|$. Then part I of the comparison theorem applies to show that the nonnegative series

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (A_k + |A_k|)$$

converges. Then

$$\sum_{k=1}^{\infty} A_k = \sum_{k=1}^{\infty} (A_k + |A_k|) - \sum_{k=1}^{\infty} |A_k|$$

converges, since it is the difference of two convergent series. Q.E.D.

Example 1 *Consider the series*

$$\sum_{k=1}^{\infty} A_k = \sum_{k=1}^{\infty} \frac{\sin k}{k^2}.$$

Now $|A_k| \leq 1/k^2$, *and by part I of the comparison theorem with* $b_k = 1/k^2$ *we can show that the* A_k *series is absolutely convergent. This proves that the series* $\sum_{k=1}^{\infty} \frac{\sin k}{k^2}$ *must be convergent.*