Solutions for Homework Set #3

page 39, problem 1

The Fourier sine series for $f(x) = x$, $(0 < x < \pi)$

$$(*) \quad x \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

Since the function $f(x) = x$ on the interval $-\pi < x < \pi$ is odd and the Fourier sine series is odd, $(*)$ is in fact the ordinary Fourier series for $f(x) = x$ on $(-\pi < x < \pi)$. The theory in Section 13 shows that the series converges to x for $-\pi < x < \pi$, to 0 for $x = \pm\pi$ and is periodic with period 2π. This gives the graph of the function pictured in Fig 4, Section 5

page 39, problem 3

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2-1}, \quad 0 \le x \le \pi$$

Set $x = 0$. Then, since $\cos 0 = 1$,

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$$

Set $x = \frac{\pi}{4}$. Then $\cos 2n\left(\frac{\pi}{2}\right) = \cos n\pi = (-1)^n$,
so we get, $\sin\frac{\pi}{2} = 1$

$$1 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1} = \frac{1}{2} - \frac{\pi}{4}$$

page 39, problem 5

$$x \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}, \quad 0<x<\pi$$

This is the Fourier cosine series for the function $f(x) = x$ on the interval $0<x<\pi$. Since the cosine series represents an even function, it is just the ordinary Fourier series for the function $g(x) = |x|$ on the interval $-\pi<x<\pi$

$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}, \quad -\pi<x<\pi$$

Now $g(x)$ is continuous on $(-\pi,\pi)$ and $g(-\pi) = g(\pi)$. Also, $g'(x)$ is piecewise continuous on $(-\pi,\pi)$. Thus the Fourier convergence theorem says that the series converges to $g(x) = |x| = x$ ~~at $x=0$ and $g(\pi)=\pi$ at $x=\pi$~~. for all x in the interval $0 \le x \le \pi$.

For $x=0$ we have $\cos(2n-1)0=1$ so

$$0=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

or $$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$$

page 39, problem 6

a) $$x^2\sim\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx, \quad 0<x<\pi$$

This is the Fourier cosine series for the function x^2 on $0<x<\pi$, hence the ordinary Fourier series for the even function $f(x)=x^2$ on $-\pi<x<\pi$. Since $f(x)$ is continuous on $-\pi<x<\pi$, $f(\pi)=f(-\pi)$ and $f'(x)$ is piecewise continuous on $(-\pi,\pi)$, the convergence theorem states that

$$(*)\quad x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx, \quad -\pi\le x\le\pi$$

Set $x=0$ in $(*)$, so that $\cos 0=1$.
Then we have

$$0^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}$$

Set $x=\pi$ in $(*)$, so that $\cos n\pi=(-1)^n$.
Then we have

$$\pi^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{1}{n^2}\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

b) $x^4 \sim \frac{\pi^4}{5} + 8\sum_{n=1}^{\infty} (-1)^n \frac{(n\pi)^2-6}{n^4} \cos nx, \quad 0<x<\pi$

This is the Fourier cosine series for x^4 on $(0,\pi)$, hence the ordinary Fourier series for the even function $f(x)=x^4$ on $[-\pi,\pi]$. Since $f(x)$ is continuous on this interval, $f(-\pi)=f(\pi)$ and $f'(x)$ is piecewise continuous, the convergence theorem states that

$$(**) \quad x^4 = \frac{\pi^4}{5} + 8\sum_{n=1}^{\infty} (-1)^n \frac{(n\pi)^2-6}{n^4} \cos nx, \quad -\pi \le x \le \pi$$

Set $x=\pi$ in $(**)$. Then $\cos n\pi = (-1)^n$, so

$$\pi^4 = \frac{\pi^4}{5} + 8\left(\sum_{n=1}^{\infty} \frac{\pi^2}{n^2} - 6\sum_{n=1}^{\infty} \frac{1}{n^4}\right)$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ from part a), we have

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{48}\left[-\frac{4\pi^4}{5} + 8\pi^2\left(\frac{\pi^2}{6}\right)\right] = \frac{\pi^4(40-24)}{48(30)}$$

$$= \frac{\pi^4}{90}$$

page 4, problem 7

$$\cos ax \sim \frac{2a \sin a\pi}{\pi}\left[\frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2-a^2} \cos nx\right]$$

$a \neq 0, \pm 1, \pm 2, \ldots \qquad -\pi < x < \pi$

This is the Fourier series for the function $f(x) = \cos ax$ on the interval $[-\pi, \pi]$. Since $f(x)$ is continuous on this interval, $f(-\pi) = f(\pi)$, and $f'(x)$ is piecewise continuous, the convergence theorem states that

$$(*) \quad \cos ax = \frac{2a \sin a\pi}{\pi}\left[\frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2-a^2} \cos nx\right],$$

$$-\pi \leq x \leq \pi, \quad a \neq 0, \pm 1, \cdots$$

Set $x = 0$ in $(*)$ to get

$$1 = \frac{2a \sin a\pi}{\pi}\left[\frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2-a^2}\right]$$

$$\Longrightarrow 1 + 2a^2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2-a^2} = \frac{a\pi}{\sin a\pi}, \quad a \neq 0, \pm 1, \cdots$$