Name: ______________________________

Math 4567. Midterm Exam II (solutions)

March 10, 2010

There are a total of 100 points on this 55 minute exam. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. A standard calculator and TWO 8.5×11 inch sheets of notes are allowed, but no books, other notes, cell phones or other elecronic devices are allowed. Do all of your calculations on this test paper.

Problem	Score
1.	________
2.	________
3.	________
Total:	________

Problem 1 a) (20 points) *Use the method of separation of variables to solve the boundary value problem*

$$u_t = ku_{xx}, \quad 0 < x < c, \ t > 0.$$

with boundary conditions

$$u(0,t) = 0, \ u(c,t) = 0, \quad t > 0,$$

and initial condition

$$u(x,0) = f(x), \quad 0 < x < c,$$

where f is piecewise continuous with piecewise continuous derivative.

Solution*:* Look for solution $u = X(x)T(t)$ of the differential equation and boundary conditions. Separating variables we have

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda.$$

Thus we have the equation $T' + \lambda kT = 0$ and the Sturm-Liouville eigenvalue problem

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(c) = 0.$$

The solution of the T equation is $T(t) = T_0 \exp(-\lambda kt)$. For the S-L problem we have

Case 1: $\lambda = \alpha^2$, $\alpha > 0$: Thus $X = A \sin \alpha x + B \cos \alpha x$. The conditions are $X(0) = 0B$, $X(c) = 0 = A \sin \alpha c$, so $\alpha c = n\pi$, $\lambda = n^2\pi^2/c^2$, $n = 1, 2, \cdots$, and $X_n(x) = \sin \frac{n\pi x}{c}$. Further, $T_n(t) = \exp(-\frac{n^2\pi^2 kt}{c^2})$.

Case 2: $\lambda = 0$. Thus $X(x) = Ax + B$, $X(0) = 0 = B$ and $X(c) = 0 = Ac$, so $A = 0$. We conclude that $\lambda = 0$ is not an eigenvalue.

Case 3: $\lambda = -\alpha^2$, $\alpha > 0$. Thus $X(x) = A \exp(\alpha x) + B \exp(-\alpha x)$. $X(0) = 0 = A + B$, so $B = -A$. Then $X(c) = 0 = A(\exp(\alpha c) - \exp(-\alpha c)) = 2A \sinh(\alpha c)$, so $A = 0$. We conclude that there are no negative eigenvalues.

We look for a solution of our problem in the form

$$u(x,t) = \sum_{n=1}^{\infty} B_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi x}{c}) \exp(-\frac{n^2\pi^2 t}{c^2}).$$

The initial condition gives

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{c}),\ 0 \le x \le c,$$

which is the Fourier sine series for $f(x)$ on the interval $[0,c]$. Thus,

$$B_n = \frac{2}{c}\int_0^c f(\tau)\sin(\frac{n\pi\tau}{c})\ d\tau, \quad n = 1, 2, \cdots.$$

b) (10 points) *If*

$$f(x) = \begin{cases} 1 & 0 < x < \frac{c}{2} \\ 0 & \frac{c}{2} < x < c. \end{cases}$$

Verify that the formal solution is

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin^2(\frac{n\pi}{4})}{n} \exp(-\frac{n^2\pi^2 k}{c^2}t) \sin\frac{n\pi x}{c}.$$

Solution*:*

$$B_n = \frac{2}{c}\int_0^c f(\tau)\sin(\frac{n\pi\tau}{c})\ d\tau = \frac{2}{c}\int_0^{c/2} \sin(\frac{n\pi\tau}{c})\ d\tau = \frac{2}{c}(\frac{-c}{n\pi})\cos(\frac{n\pi\tau}{c})\Big|_0^{c/2}$$

$$= -\frac{2}{n\pi}\left(\cos(\frac{n\pi}{2}) - 1\right) = \frac{4}{n\pi}\sin^2(\frac{n\pi}{4}).$$

In the last step we used the double angle formula $\sin^2\beta = \frac{\cos(2\beta)-1}{2}$. Thus,

$$u(x,t) = \sum_n B_n X_n(x) T_n(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin^2(\frac{n\pi}{4})}{n}\exp(-\frac{n^2\pi^2 k}{c^2}t)\sin\frac{n\pi x}{c}.$$

c) (5 points) *Give an argument that your formal solution of part b) is an actual solution of the heat equation.*

Solution: Term-by-term differentiation of the series is valid for $t > 0$ because $n^\ell \exp(-\frac{n^2\pi^2 t}{c^2}) \to 0$ as $n \to \infty$ for any $\ell = 1, 2, \cdots$. The differentiated series converges uniformly in x on $[0, c]$ for any fixed $t > 0$.

d) (10 points) *Show how the solution of part a) changes if the initial condition and right hand boundary condition remain the same, but the left hand boundary condition is replaced by*

$$u(0,t) = 1, \quad t > 0.$$

Solution*:* We look for the simplest solution $u = \Phi(x,t)$ of $u_t = ku_{xx}$ satisfying $u(0,t) = 1$, $u(c,t) = 0$. We try $u = \Phi(x)$, (independent of t). Then $\Phi'' = 0$, so $\Phi(x) = Ax + B$, and $\Phi(0) = 1 = B$, and $\Phi(c) = 0 = Ac + B$, so $a = -1/c$. Thus

$$\Phi(x) = 1 - \frac{x}{c}.$$

We will have a solution u of our problem provided $u = \tilde{u}(x.t) + \Phi(x)$ where $\tilde{u}$ satisfies part a), except that now the intitial condition must be $\tilde{u}(x,0) = f(x) - \Phi(x)$, $0 \leq x \leq c$. Thus

$$u(x,t) = (1 - \frac{x}{t}) + \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{c}) \exp(-\frac{n^2\pi^2 t}{c^2}),$$

$$B_n = \frac{2}{c}\int_0^c \Big(f(\tau) - (1 - \frac{\tau}{c})\Big) \sin(\frac{n\pi\tau}{c})\, d\tau,$$

Problem 2 *Let*

$$f(x) = \begin{cases} x & \text{if } -\pi \le x \le \frac{\pi}{2}, \\ \pi + 2 - 2\left(\frac{1+\pi}{\pi}\right)x & \text{if } \frac{\pi}{2} < x < \pi, \end{cases}$$

and let

$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

be the Fourier series for $f(x)$ in the interval $[-\pi, \pi]$.

a) (10 points) *For each x in the interval $-\pi \le x \le \pi$, state the exact numerical value of $S(x)$.*

Solution*:* Extend f to be 2π-periodic on the real line. Then

$$\lim_{x\to-\pi-} f(x) = \lim_{x\to\pi-} f(x) = \pi + 2 - 2(1+\pi) = -\pi,$$

$$\lim_{x\to-\pi+} f(x) = -\pi,$$

$$\lim_{x\to\frac{\pi}{2}-} f(x) = \frac{\pi}{2},$$

$$\lim_{x\to\frac{\pi}{2}+} f(x) = \pi + 2 - 2(\frac{1+\pi}{\pi})\frac{\pi}{2} = \pi + 2 - 1 - \pi = 1.$$

Therefore, using the pointwise convergence theorem for piecewise continuous functions with piecewise continuous derivative on the interval $[-\pi, \pi]$, we have

$$S(x) = \begin{cases} x & \text{if } -\pi < x < \frac{\pi}{2}, \\ \pi + 2 - 2(\frac{1+\pi}{\pi})x & \text{if } \frac{\pi}{2} < x < \pi, \\ -\pi & \text{if } x = \pm\pi, \\ \frac{1+\frac{\pi}{2}}{2} & \text{if } x = \frac{\pi}{2}. \end{cases}$$

b) (15 points) *Describe the behavior of the partial sums $S_N(x)$ as $N \to \infty$ near each of the points $x = -\pi$, $x = \frac{\pi}{2}$, $x = \pi$, including possible overshoots and undershoots.*

Solution*:*

1. Near $x = -\pi$ we have $S_N(x) \to f(x)$ uniformly in a neighborhood.
2. Near $x = +\pi$ we again have $S_N(x) \to f(x)$ uniformly in a neighborhood.
3. Near $x = \pi/2$, $S_N(x)$ overshoots by about 18% as $x \to \pi/2$ from below and undershoots by about 18% as $x \to \pi/2$ from above. The width of the overshoot/ undershoot region shrinks to 0 in the limit as $n \to \infty$.

c) (5 points) *Does the Parseval identity hold for this function and series? Why?*

Solution*:* The Parseval identity holds for any piecewise continuous function with piecewise continuous derivative, hence for this function.

Problem 3 (25 points) *Derive the eigenvalues and corresponding eigenfunctions for the Sturm-Liouville problem*

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < \pi,$$

$$X(0) = 0, \quad X'(\pi) = 0.$$

Consider all possibilities for λ.

Solution:

Case 1: $\lambda = \alpha^2$, $\alpha > 0$. The general solution is $X(x) = A\sin\alpha x + B\cos\alpha x$. The condition $X(0) = 0$ implies $B = 0$ and since $X'(x) = \alpha A\cos\alpha x - \alpha B\sin x$, the condition $X'(\pi) = 0$ implies $\alpha A\cos\alpha\pi = 0$. Thus $\alpha\pi = (2k-1)\pi/2$, $k = 1, 2, \cdots$. Therefore

$$\lambda_k = \frac{(2k-1)^2}{4}, \ X_k(x) = \sin(\frac{(2k-1)x}{2}) \quad k = 1, 2, \cdots.$$

Case 2: $\lambda = 0$. Then $X(x) = Ax + B$. The condition $X(0) = 0$ implies $B = 0$ and since $X'(x) = A$, the condition $X'(\pi) = 0$ implies $A = 0$. Thus 0 is not an eigenvalue.

Case 3: $\lambda = -\alpha^2$, $\alpha > 0$. Then $X(x) = A\exp(\alpha x) + B\exp(-\alpha x)$. The condition $X(0) = 0$ implies $B = -A$ and since $X'(x) = \alpha(A\exp(\alpha x) - B\exp(-\alpha x))$, the condition $X'(\pi) = 0$ implies $\alpha A[\exp(\alpha\pi) + \exp(-\alpha\pi)] = 0$, so $A = 0$. Thus there are no negative eigenvalues.