Boundary Conditions and Null Lagrangians in the Calculus of Variations and Elasticity

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Abstract. We explicitly characterize boundary conditions that are compatible with low order variational principles. The freedom afforded by adding in a null Lagrangian without altering the Euler–Lagrange equation significantly expands the range of variationally admissible boundary conditions, although not all possibilities are permitted. Applications to several fundamental problems arising in elastostatics, including bars, beams, and plates, are presented.

June 15, 2022

1. Introduction.

The aim of this paper is to better understand the role of boundary conditions in the calculus of variations. Standard texts, including [**12**, **19**, **20**, **26**], treat fixed and natural boundary conditions, but have little to say about whether other types of boundary conditions, e.g., inhomogeneous Neumann boundary conditions, or Robin boundary conditions, can be incorporated into a variational framework. One reason is that, at least for first order variational problems, the inhomogeneous fixed boundary conditions and homogeneous natural boundary conditions are, in fact, the only possible uncoupled† boundary conditions that induce the vanishing of the boundary terms arising after integration by parts in the first variation. However, this is not the end of the story.

It is well known, [**20**, **32**, **33**, **35**], that one can modify a variational problem without altering the corresponding Euler–Lagrange equations by adding a null Lagrangian to the integrand. As noted by Ericksen, [**15**], in the context of liquid crystal theory, this modification (he calls null Lagrangians "nilpotent energies") does, however, change the associated natural boundary conditions, and hence enables one to enlarge the range of boundary value problems that can be handled by variational techniques. This observation was further developed, in the context of continuum mechanics, by Edelen and Lagoudas, [**13**, **14**], and by Lancia, Podio–Guidugli, and Vergara Caffarelli, [**27**, **37**]. However, this idea has yet to be fully fleshed out, even in the simple situations to be presented here. Boundary conditions that are compatible with a variational principle will be called *variationally admissible* here; they are also termed *self-adjoint* or *conservative*, [**3**, **19**]. I should also note that the classical text by Forsyth, [**18**], does treat boundary conditions for variational problems in much greater detail than elsewhere but, as we will see, falls short in some of the analysis, and, as best as I can tell, does not consider the effect of modifying the Lagrangian in this manner.

The first result is that essentially any scalar first order variational problem can be modified by a suitable null Lagrangian in order to admit any (reasonable) uncoupled boundary conditions. On the other hand, coupled boundary conditions — also known as mixed boundary conditions, [**8**] — are not always variationally admissible and we explicitly characterize those that are. Moreover, the uncoupled result is particular to this simplest case, and no longer holds for higher order scalar variational problems, variational problems involving several unknowns, or multivariate variational problems. We illustrate the various options in some of the more basic situations: first and second order variational problems involving one and two independent variables and a single dependent variable, first order problems involving one independent variable and two dependent variables, and first order problems involving two or three independent and dependent variables, the latter of importance in (hyper-)elastostatics, [**3**, **23**, **28**]. For second order variational problems in one independent and one dependent variable, the variationally admissible uncoupled boundary conditions are the fixed conditions and generalizations of the simply supported, sliding, and

† By "uncoupled" — "separated" in the terminology of Bliss, [**8**] — we mean that each boundary condition only involves values of the unknowns at a single point on the boundary. Periodic, quasi-periodic, and more general coupled boundary conditions are thus excluded when making the ensuing assertion.

free end conditions arising in Euler–Bernoulli beam theory, [**3**, **40**]. We further derive the natural boundary conditions for a general second order variational problem involving two independent variables and one dependent variable, and show how Timoshenko's boundary conditions for linear plate mechanics, [**42**], are a particular case, provided one uses the physically relevant quadratic variational principle, which depends upon a particular choice of null Lagrangian.

In a sense, these results can be viewed as a (partial) solution to an extended version of the inverse problem of the calculus of variations that includes boundary conditions. The basic inverse problem asks when is a given system of differential equations is the Euler–Lagrange equations of some variational principle, and is answered by the classical Helmholtz conditions, [**32**]. Our concern is to characterize which boundary value problems can be assigned a variational structure as posed. We do not attempt a more general form of the inverse problem, which asks when a system is equivalent to a system coming from a variational principle; see the surveys [**1**, **38**] for details (in the absence of boundary conditions).

2. Scalar First Order Variational Problems.

We begin with a variational problem involving a scalar-valued function of a single variable. The basic minimization problem is to determine a suitable function $u = f(x)$ that minimizes the *objective functional*

$$J[u] = \int_a^b L(x, u, u')\, dx \tag{2.1}$$

which we assume to be defined on a compact interval $[a, b] \subset \mathbb{R}$. The integrand $L(x, u, p)$, where p represents u', is known as the *Lagrangian* for the variational problem. To avoid technicalities, we will impose the nondegeneracy condition

$$\frac{\partial^2 L}{\partial p^2}\,(x, u, p) \neq 0 \tag{2.2}$$

although in this study, this will only be required at the endpoints a, b. All functions will be assumed to be sufficiently smooth in order that one can perform the necessary manipulations.

In most treatments of the subject, one imposes one of two possible uncoupled boundary conditions at each endpoint a, b of the interval. (The case of coupled boundary conditions will be discussed at the end of the section.) The first is a *fixed boundary condition* that specifies the value of the minimizing function at an endpoint, e.g., $u(a) = \alpha$. The second possibility is that of a *free boundary*, in which no conditions are imposed a priori on the minimizer at the endpoint in question. In the latter case, as we will see, the variational formulation leads to the imposition of a natural boundary condition there.

The basic idea of the calculus of variations is to compare the proposed minimizer $u(x)$ with a nearby function. We thus introduce a *variation* in the form $\Phi(\varepsilon, x)$ — classically written as δu — in which $\varepsilon \in \mathbb{R}$ is assumed small, $\Phi(0, x) \equiv 0$, and so that the function

$u(x) + \Phi(\varepsilon, x)$ satisfies the same boundary conditions as $u(x)$ for all (sufficiently small) ε. Thus, if u is to be a minimizer, the scalar function[†]

$$h(\varepsilon) = J[\,u + \Phi\,] = \int_a^b L(x, u + \Phi, u' + \Phi'\,)\,dx \tag{2.3}$$

must have a minimum at $\varepsilon = 0$, and hence $h'(0) = 0$. Assuming sufficient smoothness of the integrand allows us to bring the derivative with respect to ε inside the integral and so, by the chain rule,

$$\begin{aligned} h'(0) = \frac{d}{d\varepsilon}\, J[\,u + \Phi\,]\,\bigg|_{\varepsilon=0} &= \int_a^b \frac{d}{d\varepsilon}\, L(x, u + \Phi, u' + \Phi'\,)\,\bigg|_{\varepsilon=0} dx \\ &= \int_a^b \left[\, \varphi(x)\, \frac{\partial L}{\partial u}\,(x, u, u') + \varphi'(x)\, \frac{\partial L}{\partial p}\,(x, u, u')\,\right] dx, \end{aligned} \tag{2.4}$$

where

$$\varphi(x) = \frac{d}{d\varepsilon}\, \Phi(\varepsilon, x)\,\bigg|_{\varepsilon=0}. \tag{2.5}$$

Integrating the second term in the final integral by parts, we arrive at the basic variational formula

$$\begin{aligned} 0 = h'(0) = \varphi(b)\, \frac{\partial L}{\partial p}\,\big(\, b, u(b), u'(b)\,\big) - \varphi(a)\, \frac{\partial L}{\partial p}\,\big(\, a, u(a), u'(a)\,\big) \\ + \int_a^b \varphi(x) \left[\, \frac{\partial L}{\partial u}\,(x, u, u') - \mathrm{D}_x \left(\frac{\partial L}{\partial p}\,(x, u, u') \right) \right] dx, \end{aligned} \tag{2.6}$$

where the notation D_x refers to the *total derivative*, in which one differentiates with respect to x treating u as a function thereof, [**32**]. We will refer to the first two terms on the right hand side as the *variational boundary terms*.

So far we have not explicitly specified the conditions to be imposed on our variation Φ other than setting $\Phi(0, x) \equiv 0$. Suppose first that, for each fixed ε, the variation $\Phi(\varepsilon, x)$ has compact support in the open interval (a, b), which implies that it vanishes at the boundary: $\Phi(\varepsilon, a) = \Phi(\varepsilon, b) = 0$, which, by (2.5), implies the same for $\varphi(a) = \varphi(b) = 0$. The latter implies that the variational boundary terms in (2.6) vanish. The vanishing of the final integral term is then governed by the *Fundamental Lemma*, whose proof can be found in any text on the subject.

Lemma 2.1. *If $f(x)$ is continuous on $[\,a, b\,]$, and $\displaystyle\int_a^b f(x)\,\varphi(x)\,dx = 0$ for every C^∞ function $\varphi(x)$ with compact support in $(\,a, b\,)$, then $f(x) \equiv 0$ for all $a \le x \le b$.*

This implies that the minimizer $u(x)$ must satisfy the usual *Euler–Lagrange equation*

$$\frac{\partial L}{\partial u}\,(x, u, u') - \mathrm{D}_x\, \frac{\partial L}{\partial p}\,(x, u, u') = 0, \tag{2.7}$$

[†] Primes on Φ mean derivatives with respect to x.

which, under the nondegeneracy condition (2.2), constitutes a second order ordinary differential equation that any sufficiently smooth minimizer $u(x)$ must satisfy. One word of caution: there do exist seemingly reasonable functionals whose minimizers are not, in fact, C^2, and hence do not solve the Euler–Lagrange equation in the classical sense; see [**7**, **26**, **35**] for examples. Such pathologies are not considered here.

Now let us look at the boundary terms in (2.6). If u satisfies a fixed boundary condition at an endpoint, say $u(a) = \alpha$, then in order that the variation continue to satisfy the same boundary condition, we must have $\varphi(a) = 0$, and hence the corresponding variational boundary term vanishes. On the other hand, if a is a free boundary, then there are no conditions initially imposed on either $u(a)$ or $\varphi(a)$. Thus, in order that the variational requirement (2.6) hold, the term multiplying $\varphi(a)$ must vanish instead:

$$\frac{\partial L}{\partial p}\,\big(\, a, u(a), u'(a)\,\big) = 0. \tag{2.8}$$

This is known as a *natural boundary condition*, and imposes a constraint that any minimizer must satisfy at the free boundary $x = a$. An identical argument proves that at the other endpoint either u satisfies a fixed boundary condition $u(b) = \beta$, or the corresponding natural boundary condition holds:

$$\frac{\partial L}{\partial p}\,\big(\, b, u(b), u'(b)\,\big) = 0. \tag{2.9}$$

We conclude that, in the case of uncoupled boundary conditions, any sufficiently smooth minimizer for the variational problem (2.1) must satisfy a two-point boundary value problem for the associated second order Euler–Lagrange equation (2.7), where, at each endpoint one imposes either an (inhomogeneous) fixed boundary condition or a (homogeneous) natural boundary condition. As always, the above are just necessary conditions for local minimizers. Maximizers, if such exist, must also satisfy the same conditions, which serve to characterize the *critical functions* of the variational principle subject to the given boundary constraints. Determining whether a critical function is a (local) minimizer, or maximizer, or neither, involves further criteria based on the second variation, [**19**, **20**, **26**, **35**], that will not concern us here. It is also worth noting that minimizers (and maximizers) may not exist, need not be unique, and can be local or global. In continuum mechanics, under suitable assumptions, the stable equilibrium configurations are the (local and global) minimizers of the variational problem representing the stored energy in the system.

Consequently, imposing any other type of boundary conditions at an endpoint, say $x = a$, will force the critical function to satisfy more than one boundary condition there — the natural boundary condition (2.8) plus the imposed boundary condition, thereby effectively prescribing both $u(a)$ and $u'(a)$. In this case, the basic existence and uniqueness theorem for the initial value problem for a second order ordinary differential equation, [**10**], would imply that they are satisfied by one and only one solution to the Euler–Lagrange equation. This solution will probably not satisfy the additional boundary condition(s) at the other endpoint, in which case there would be no critical functions associated with the variational principle, and hence no candidate minimizers.

Example 2.2. Let us apply this method to the following problem: find the shortest path between a point $\mathbf{a} = (a, \alpha)$ and a vertical straight line $\ell_b = \{ x = b \}$ in the xy plane. Assuming the solution is given by the graph of a function $y = u(x)$, the length is given by the arc length functional

$$J[u] = \int_a^b \sqrt{1 + u'(x)^2}\, dx, \tag{2.10}$$

subject to the single fixed boundary condition $u(a) = \alpha$. The Lagrangian is

$$L = \sqrt{1 + p^2}, \qquad \text{and hence} \qquad \frac{\partial L}{\partial u} = 0, \qquad \frac{\partial L}{\partial p} = \frac{p}{\sqrt{1 + p^2}}\,. \tag{2.11}$$

The Euler–Lagrange equation becomes

$$0 = -\,\mathrm{D}_x \frac{u'(x)}{\sqrt{1 + u'(x)^2}} = -\,\frac{u''(x)}{(1 + u'(x)^2)^{3/2}}\,,$$

which simplifies to $u'' = 0$, whose solutions are straight lines. To determine the second, natural boundary condition at $x = b$, we apply (2.9), whence

$$\frac{u'(b)}{\sqrt{1 + u'(b)^2}} = 0, \qquad \text{or, simply,} \qquad u'(b) = 0.$$

This means that any critical function $u(x)$ must have horizontal tangent at the point $x = b$, or, equivalently, it must be perpendicular to the vertical line ℓ_b.

Suppose we try to impose a non-natural, non-fixed boundary condition at the endpoint — for example, the *Robin condition*, [**34**],

$$u'(b) = \beta\, u(b) + \gamma, \tag{2.12}$$

where $\beta \neq 0$ and γ are constants. The solution to this boundary value problem for the Euler–Lagrange equation is easily calculated:

$$u(x) = \frac{(\alpha\beta + \gamma)\, x + \alpha(1 - \beta\, b) - \gamma\, a}{1 - \beta\,(b - a)} \tag{2.13}$$

provided the denominator does not vanish. However, unless $\alpha\beta + \gamma = 0$, so the graph of u is a horizontal line, this function does not provide a minimum to arc length functional among functions subject to the prescribed boundary conditions. Indeed, one can construct functions that satisfy the Robin boundary condition (2.12) whose arc length is arbitrarily close to that of the horizontal line, which is $b - a$. For example, we can slightly perturb the line by, say, setting

$$u_\varepsilon(x) = \begin{cases} \alpha, & a \leq x \leq b - \varepsilon, \\ \alpha + (\alpha\beta + \gamma)\,(x - b + \varepsilon)/(1 - \varepsilon\,\beta), & b - \varepsilon \leq x \leq b, \end{cases}$$

where $0 < \varepsilon < |\,\beta\,|$, which does satisfy (2.12); see Figure 1. Its arc length

$$b - a + \varepsilon \left(\sqrt{1 + \left(\frac{\alpha\beta + \gamma}{1 - \varepsilon\,\beta} \right)^2} - 1 \right) \longrightarrow b - a \qquad \text{as} \qquad \varepsilon \to 0.$$

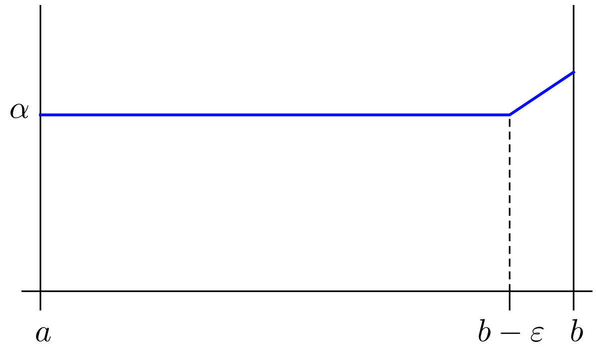

Figure 1. Perturbed Horizontal Line.

One can even smooth off its corner, which has the effect of slightly decreasing its total arc length and thus does not affect the convergence to $b - a$ as $\varepsilon \to 0$. Thus, while the Robin boundary value problem has no minimizing solution, it does admit smooth functions that come arbitrarily close to the minimum possible value, which is $b - a$. A similar behavior is expected for other types of non-variationally admissible boundary conditions.

Despite the perhaps disappointing conclusion of Example 2.2, it turns out that it is possible to impose other types of boundary conditions on a variational principle, by suitably modifying the functional. With this aim, we introduce an important notion of independent interest in the calculus of variations.

Definition 2.3. A function $N(x,u,p)$ is called a *null Lagrangian* if and only if its associated Euler–Lagrange expression (2.7) vanishes identically: $E(N) \equiv 0$.

The key result is a special case of a general theorem characterizing higher order and multidimensional null Lagrangians; see [**32**; Theorem 4.7].

Proposition 2.4. *A function $N(x,u,p)$ defined for all*[†] $(x,u,p) \in \mathbb{R}^3$ *is a null Lagrangian if and only if it is a total derivative, so*

$$N(x,u,u') = \mathrm{D}_x[\,A(x,u)\,] = \frac{\partial A}{\partial x} + u' \,\frac{\partial A}{\partial u} \tag{2.14}$$

for some function A that depends only on x, u.

With Proposition 2.4 in hand, we can apply the Fundamental Theorem of Calculus to write the functional associated with a null Lagrangian in the following form:

$$I[\,u\,] = \int_a^b N(x,u,u')\,dx = \int_a^b \mathrm{D}_x[\,A(x,u)\,]\,dx = A\big(b,u(b)\big) - A\big(a,u(a)\big). \tag{2.15}$$

In other words, the value of the functional associated with a null Lagrangian depends only on the values of the function at the endpoints of the interval, and hence it is *constant* for any function $u(x)$ that satisfies our usual fixed boundary conditions $u(a) = \alpha$, $u(b) = \beta$.

† More generally, N can be defined on a subset of $\mathbb{R}^3$ with trivial topology, [**2**, **32**].

The flexibility afforded by null Lagrangians allows us to expand the range of boundary conditions that can be handled by a variational analysis. Namely, we can modify the original variational problem by adding in a null Lagrangian, which does not alter the Euler–Lagrange equations but does change the associated natural boundary conditions. In other words, if $N = \mathrm{D}_x A$ is a null Lagrangian, then the *modified objective functional*

$$
\begin{aligned}
\widetilde{J}[u] &= \int_a^b \left[L(x,u,u') + N(x,u,u') \right] dx = \int_a^b \left[L(x,u,u') + \mathrm{D}_x A(x,u) \right] dx \\
&= A\big(b,u(b)\big) - A\big(a,u(a)\big) + \int_a^b L(x,u,u')\, dx
\end{aligned}
\tag{2.16}
$$

has the same Euler–Lagrange equations as $J[u] = \displaystyle\int_a^b L(x,u,u')\, dx$. Moreover, when subject to fixed boundary conditions, they have exactly the same critical functions, and hence the same minimizers, since their values differ only by the initial two boundary terms in the final expression, which depend only on the values of u at the endpoints a, b. On the other hand, as we will see, the two variational problems have different natural boundary conditions, and this flexibility allows us, at least in the present situation, to admit any uncoupled boundary conditions into the variational framework.

Remark: The second expression for the modified objective functional (2.16) is, in the terminology of Bliss, [**8**], a version of the *problem of Bolza*, with separated end-conditions.

To prove the preceding claim, and explain how to construct the required null Lagrangian, let us assume that both boundary conditions explicitly involve the derivative of the function $u(x)$ at the endpoint. We can already handle fixed boundary conditions, and the remaining "mixed" case, in which one end is fixed and the boundary condition at the other end involves the derivative, is left as an exercise for the reader. Under mild algebraic assumptions, we can solve the boundary conditions for the derivative:

$$
u'(a) = \beta_1\big(u(a)\big), \qquad u'(b) = \beta_2\big(u(b)\big),
\tag{2.17}
$$

for some functions β_1, β_2 depending only on the value of u at the endpoint in question. (They may, of course, depend on the respective endpoints a, b, but this is taken care of by allowing different functions at each end.)

Given a variational problem with Euler–Lagrange equation supplemented by the prescribed boundary conditions (2.17), let us add in a suitable null Lagrangian (2.14) in order that the natural boundary conditions associated with the modified Lagrangian

$$
\widetilde{L}(x,u,p) = L(x,u,p) + N(x,u,p) = L(x,u,p) + p\, \frac{\partial A}{\partial u}(x,u) + \frac{\partial A}{\partial x}(x,u)
\tag{2.18}
$$

are equivalent to the desired boundary conditions (2.17). Thus, at the right hand endpoint, in view of (2.9), this means

$$
0 = \frac{\partial \widetilde{L}}{\partial p}\big(\, b, u(b), u'(b) \,\big) = \frac{\partial L}{\partial p}\big(\, b, u(b), u'(b) \,\big) + \frac{\partial A}{\partial u}\big(b, u(b)\big)
$$

is equivalent to the boundary equation $u'(b) = \beta_2\bigl(u(b)\bigr)$. For this to occur, we must require

$$\frac{\partial A}{\partial u}\bigl(b, u(b)\bigr) = \frac{\partial L}{\partial p}\bigl(\, b, u(b), \beta_2\bigl(u(b)\bigr)\,\bigr). \tag{2.19}$$

Similarly, at the left hand endpoint, we require

$$\frac{\partial A}{\partial u}\bigl(a, u(a)\bigr) = \frac{\partial L}{\partial p}\bigl(\, a, u(a), \beta_1\bigl(u(a)\bigr)\,\bigr). \tag{2.20}$$

These two conditions suffice to prescribe (2.17) as the natural boundary conditions for the variational problem associated with the modified Lagrangian (2.18), thus justifying our claim that by a suitable choice of $A(x,u)$, or, equivalently, by adding in a suitable null Lagrangian $N = dA/dx$ we can arrange for any boundary conditions of the above form to be the natural boundary conditions associated with the variational problem.

We can combine the requirements (2.19–20) into a simpler form as follows. Choose an "interpolating function" $B(x,u)$ such that

$$B\bigl(a, u(a)\bigr) = \beta_1\bigl(u(a)\bigr), \qquad B\bigl(b, u(b)\bigr) = \beta_2\bigl(u(b)\bigr). \tag{2.21}$$

For example, we can use linear interpolation and set

$$B(x,u) = \frac{x-a}{b-a}\,\beta_2(u) - \frac{x-b}{b-a}\,\beta_1(u). \tag{2.22}$$

Then (2.19–20) are implied by the interpolated equation

$$\frac{\partial A}{\partial u}(x,u) = -\,\frac{\partial L}{\partial p}\bigl(\, x, u, B(x,u)\,\bigr), \quad \text{and thus} \quad A(x,u) = -\int \frac{\partial L}{\partial p}\bigl(\, x, u, B(x,u)\,\bigr)\, du \tag{2.23}$$

is any anti-derivative of the integrand. We have thus proved a general result about variational problems with specified boundary conditions.

Theorem 2.5. *Let $J[u] = \displaystyle\int_a^b L(x,u,u')\,dx$ be a variational problem whose minimizers are subject to the boundary conditions*

$$u'(a) = \beta\bigl(a, u(a)\bigr), \qquad u'(b) = \beta\bigl(b, u(b)\bigr), \tag{2.24}$$

for some function $\beta(x,u)$. Let $A(x,u)$ be defined by (2.23). *Then the modified variational problem*

$$\widetilde{J}[u] = \int_a^b [\, L(x,u,u') + \mathrm{D}_x A(x,u)\,]\,dx = A\bigl(b,u(b)\bigr) - A\bigl(a,u(a)\bigr) + \int_a^b L(x,u,u')\,dx \tag{2.25}$$

has the same Euler–Lagrange equations as $J[u]$, and natural boundary conditions (2.24).

Observe that the modified variational problem (2.25) differs from the original only through the addition of certain "corrections" that depend only on the boundary values of u. The point is that any solution to the resulting boundary value problem will be a candidate minimizer for the modified variational problem.

Example 2.6. Let us consider the problem of minimizing arc length (2.10) subject to the Robin boundary conditions

$$u'(a) = \beta_1\, u(a) + \gamma_1, \qquad u'(b) = \beta_2\, u(b) + \gamma_2, \tag{2.26}$$

in which $\beta_1, \beta_2, \gamma_1, \gamma_2$ are prescribed constants. We assume $\beta_1\, \beta_2 \neq 0$, leaving the cases of inhomogeneous Neumann conditions to the reader. In accordance with (2.22), set

$$B(x,u) = \beta(x)\, u + \gamma(x) \quad \text{where} \quad \beta(x) = \frac{x-a}{b-a}\,\beta_2 - \frac{x-b}{b-a}\,\beta_1, \quad \gamma(x) = \frac{x-a}{b-a}\,\gamma_2 - \frac{x-b}{b-a}\,\gamma_1.$$

Substituting the second formula in (2.11) into (2.23), the required function A is obtained by integration:

$$A(x,u) = -\int \frac{\beta(x)\, u + \gamma(x)}{\sqrt{1 + [\,\beta(x)\, u + \gamma(x)\,]^2}}\, du = -\,\frac{\sqrt{1 + [\,\beta(x)\, u + \gamma(x)\,]^2}}{\beta(x)}\,,$$

provided $\beta(x) \neq 0$. The modified variational problem (2.25) can thus be written in the form[†]

$$\begin{aligned} \widetilde{J}[\,u\,] &= -\,\frac{\sqrt{1 + [\,\beta_2\, u(b) + \gamma_2\,]^2}}{\beta_2} + \frac{\sqrt{1 + [\,\beta_1\, u(a) + \gamma_1\,]^2}}{\beta_1} + \int_a^b \sqrt{1 + u'(x)^2}\, dx \\ &= -\,\frac{\sqrt{1 + u'(b)^2}}{\beta_2} + \frac{\sqrt{1 + u'(a)^2}}{\beta_1} + \int_a^b \sqrt{1 + u'(x)^2}\, dx. \end{aligned} \tag{2.27}$$

Thus, while the basic arc length functional does not, in general, admit a minimizer that satisfies the Robin boundary conditions, the modified arc length (2.27), which has the same Euler–Lagrange equation, (usually) does.

Let us solve the Robin boundary value problem for the Euler–Lagrange equation, which, as noted above, is merely $u'' = 0$, the solutions of which are straight lines $u = cx + d$. Substituting into the Robin boundary conditions (2.26) produces

$$c = \beta_1\,(c\,a + d) + \gamma_1 = \beta_2\,(c\,b + d) + \gamma_2.$$

Thus, if

$$(b-a)\,\beta_1\beta_2 + \beta_2 - \beta_1 \neq 0,$$

the problem admits a unique solution, while if the left hand side is zero, then there is either a one-parameter family of solutions that all give the same value to the modified variational problem (even though they have differing arc lengths), or there is no solution, depending on the values of γ_1, γ_2.

[†] We avoid writing out the more complicated integral expression (2.16) involving the associated null Lagrangian.

What about coupled boundary conditions, which relate the values of the minimizer and its derivatives at the endpoints? The simplest are the usual *periodic boundary conditions*

$$u(a) = u(b), \qquad u'(a) = u'(b). \tag{2.28}$$

Any variation must satisfy the same periodic conditions: $\varphi(a) = \varphi(b)$, $\varphi'(a) = \varphi'(b)$, and hence the difference of the two variational boundary terms in (2.6) will vanish provided $\partial L/\partial p$ is also periodic in x, meaning

$$\frac{\partial L}{\partial p}\,(a,u,p) = \frac{\partial L}{\partial p}\,(b,u,p). \tag{2.29}$$

More generally we could try to impose a pair of coupled boundary conditions of the form

$$F_1\big(\,u(a), u'(a), u(b), u'(b)\,\big) = F_2\big(\,u(a), u'(a), u(b), u'(b)\,\big) = 0,$$

where each $F_i(u,p,v,q)$ depends on 4 arguments. The variation $\varphi(x)$ will thus satisfy the linearization of each:

$$\frac{\partial F_i}{\partial u}\,\varphi(a) + \frac{\partial F_i}{\partial p}\,\varphi'(a) + \frac{\partial F_i}{\partial v}\,\varphi(b) + \frac{\partial F_i}{\partial q}\,\varphi'(b) = 0. \tag{2.30}$$

Now, if there are no algebraic constraints relating $\varphi(a)$ and $\varphi(b)$, meaning they can achieve independent values, then (2.6) will imply that both natural boundary conditions must be satisfied, and hence, unless the boundary value problem is overdetermined, variational admissibility implies that the boundary conditions decouple into the usual natural conditions at each end. For this not to be the case, the coupled boundary condition must relate the values of the critical function at the endpoints, say

$$F\big(\,u(a), u(b)\,\big) = 0, \tag{2.31}$$

and hence

$$\frac{\partial F}{\partial u}\big(\,u(a), u(b)\,\big)\,\varphi(a) + \frac{\partial F}{\partial v}\big(\,u(a), u(b)\,\big)\,\varphi(b) = 0.$$

The vanishing of the variational boundary terms in (2.6) thus requires

$$G\big(u(a), u'(a), u(b), u'(b)\big) = 0, \tag{2.32}$$

where

$$G(u,p,v,q) = \frac{\partial F}{\partial v}(u,v)\;\frac{\partial L}{\partial p}(a,u,p) + \frac{\partial F}{\partial u}(u,v)\;\frac{\partial L}{\partial p}(b,v,q).$$

This provides the second "natural coupled boundary condition" complementing the "fixed coupled condition" (2.31). Thus, variationally admissible coupled boundary conditions necessarily take the form (2.31, 32).

For example, the *quasiperiodic condition*

$$u(b) = \alpha\, u(a), \tag{2.33}$$

where α is a nonzero constant, requires

$$\frac{\partial L}{\partial p}\big(\, b, u(b), u'(b)\,\big) = \frac{1}{\alpha}\,\frac{\partial L}{\partial p}\big(\, a, u(a), u'(a)\,\big) \tag{2.34}$$

as its variationally admissible quasiperiodic counterpart. In the case of the arc length functional (2.10), the second quasiperiodic condition (2.34) becomes

$$\frac{u'(b)}{\sqrt{1+u'(b)^2}} = \frac{u'(a)}{\alpha\sqrt{1+u'(a)^2}}\,. \tag{2.35}$$

3. Second Order Scalar Variational Problems.

We next consider a second order scalar variational problem, with *objective functional*

$$J[\,u\,] = \int_a^b L(x,u,u',u'')\,dx, \tag{3.1}$$

prescribed by the *Lagrangian* $L(x,u,p,q)$, where now p,q represent the first and second order derivatives u',u''. In analogy with (2.2), we will impose the nondegeneracy condition

$$\frac{\partial^2 L}{\partial q^2}\big(\, x,u,p,q\,\big) \neq 0. \tag{3.2}$$

We introduce the variation

$$h(\varepsilon) = J[\,u+\Phi\,] = \int_a^b L(x,u+\Phi,u'+\Phi',u''+\Phi''\,)\,dx,$$

where $\Phi(0,x)\equiv 0$ and we set $\varphi(x) = \dfrac{\partial\Phi}{\partial\varepsilon}\,(0,x)$. As before, any sufficiently smooth critical function u satisfies

$$\begin{aligned}
0 = h'(0) &= \frac{d}{d\varepsilon}\,J[\,u+\Phi\,]\,\bigg|_{\varepsilon=0} = \int_a^b \frac{d}{d\varepsilon}\,L(x,u+\Phi,u'+\Phi',u''+\Phi''\,)\,\bigg|_{\varepsilon=0} dx \\
&= \int_a^b \left[\varphi(x)\,\frac{\partial L}{\partial u}\,(x,u,u',u'') + \varphi'(x)\,\frac{\partial L}{\partial p}\,(x,u,u',u'') + \varphi''(x)\,\frac{\partial L}{\partial q}\,(x,u,u',u'')\right] dx \\
&= B(b) - B(a) + \int_a^b \varphi(x)\,E(L)\,dx,
\end{aligned} \tag{3.3}$$

where the final formula results from integrating the second and third terms in the preceding integral by parts. Here

$$E(L) = \frac{\partial L}{\partial u}\,(x,u,u',u'') - \mathrm{D}_x\!\left(\frac{\partial L}{\partial p}\,(x,u,u',u'')\right) + \mathrm{D}_x^2\left(\frac{\partial L}{\partial q}\,(x,u,u',u'')\right), \tag{3.4}$$

is the associated Euler–Lagrange expression, while the variational boundary terms are obtained by evaluating

$$B(x) = g(x)\,\varphi(x) + h(x)\,\varphi'(x), \tag{3.5}$$

where

$$g(x) = \frac{\partial L}{\partial p}\,(x,u,u',u'') - \mathrm{D}_x \frac{\partial L}{\partial q}\,(x,u,u',u''), \qquad h(x) = \frac{\partial L}{\partial q}\,(x,u,u',u''),$$

at the endpoints $x = a, b$.

If $\Phi(\varepsilon, x)$ has compact support in (a, b) for each ε, then $\varphi(a) = \varphi'(a) = \varphi(b) = \varphi'(b) = 0$, and the boundary terms in (3.3) both vanish: $B(a) = B(b) = 0$. Thus, applying the Fundamental Lemma 2.1, we conclude that any critical function u must satisfy the fourth order Euler–Lagrange equation $E(L) = 0$. This differential equation will be supplemented by four boundary conditions — two at each endpoint in the uncoupled case. The key question is which pairs of boundary conditions are *compatible* with the variational structure afforded by (3.1).

The variation must maintain the boundary conditions, which serves to constrain $\Phi(\varepsilon, x)$ and/or some of its derivatives at each endpoint. We can classify the possible uncoupled cases as follows. Let us designate the *order* $1 \leq n \leq 3$ of a pair of boundary conditions at an endpoint, either a or b, to be the highest order derivative of u that explicitly occurs in them. We will allow nonlinear boundary conditions — indeed for non-quadratic variational problems the natural boundary conditions are inevitably nonlinear — but make a mild algebraic assumption that allows us to solve one of the two boundary conditions for the highest order derivative and then, substituting this expression into the second boundary condition, solve it for the highest order remaining derivative occurring therein. We will also substitute the resulting expression back into the first boundary condition so the derivative we subsequently solved for only occurs in the second condition. (This preliminary step will become clearer in the context of specific examples.) We will focus on the left hand endpoint $x = a$, noting that the same analysis holds, mutatis mutandis, at $x = b$.

Furthermore, we will modify the variational problem by adding in a null Lagrangian so as to preserve the underlying Euler–Lagrange equation. The analogue of Proposition 2.4 characterizes every second order null Lagrangians as a total derivative, so

$$N(x,u,p,q) = \mathrm{D}_x A(x,u,p) = \frac{\partial A}{\partial x} + p\,\frac{\partial A}{\partial u} + q\,\frac{\partial A}{\partial p} \tag{3.6}$$

for some function A that depends only on x, u, p. Replacing L by $L + N$ in the preceding computation, we arrive at the modified boundary function (3.5) with

$$\begin{aligned} g(x) &= \frac{\partial L}{\partial p} - \mathrm{D}_x \frac{\partial L}{\partial q} + \frac{\partial A}{\partial u} \\ &= -\,\frac{\partial^2 L}{\partial q^2}\,u'''(x) - \frac{\partial^2 L}{\partial p\,\partial q}\,u''(x) - \frac{\partial^2 L}{\partial u\,\partial q}\,u'(x) - \frac{\partial^2 L}{\partial x\,\partial q} + \frac{\partial L}{\partial p} + \frac{\partial A}{\partial u}\,, \\ h(x) &= \frac{\partial L}{\partial q} + \frac{\partial A}{\partial p}\,, \end{aligned} \tag{3.7}$$

where the partial derivatives of L and A are evaluated at $x, u(x), u'(x), u''(x)$.

First order boundary conditions: If the two boundary conditions at $x = a$ only involve $u(a)$ and $u'(a)$, then, under our algebraic assumption, we can solve them for

$$u(a) = \alpha, \qquad u'(a) = \beta, \tag{3.8}$$

where $\alpha, \beta \in \mathbb{R}$, and so we are dealing with the *fixed boundary conditions* at $x = a$. The corresponding variation must therefore satisfy the homogeneous fixed boundary conditions

$$\varphi(a) = 0, \qquad \varphi'(a) = 0. \tag{3.9}$$

In view of (3.5), this immediately implies that the boundary term $B(a) = 0$. Hence, as in the first order case, fixed boundary conditions are variationally admissible and no null Lagrangian modification is required.

Second order boundary conditions: These come in two subclasses. First, one boundary condition could only involve $u(a)$, and so be of fixed type. Thus, under our algebraic assumptions, we can solve for

$$u(a) = \alpha, \qquad u''(a) = \beta\big(u'(a)\big), \tag{3.10}$$

where α is a constant and $\beta(p)$ a scalar function. Alternatively, both boundary conditions involve derivatives of u and can be solved for

$$u'(a) = \alpha\big(u(a)\big), \qquad u''(a) = \beta\big(u(a)\big), \tag{3.11}$$

where now both $\alpha(u), \beta(u)$ are scalar functions.

In the first subcase (3.10), the first condition again implies $\varphi(a) = 0$. As for the second boundary condition, the value of $\varphi'(a)$ is no longer constrained, and so, in order that $B(a) = 0$, we must impose the corresponding natural boundary condition

$$h(a) = \frac{\partial L}{\partial q}\left(a, u(a), u'(a), u''(a) \right) + \frac{\partial A}{\partial p}\left(a, u(a), u'(a) \right) = 0. \tag{3.12}$$

Arguing as we did for first order boundary conditions, this will be equivalent to the second boundary condition in (3.10) provided we choose $A(x, u, p)$ such that

$$\frac{\partial A}{\partial p}(a, \alpha, p) = \frac{\partial L}{\partial q}\left(a, \alpha, p, \beta(p) \right). \tag{3.13}$$

Thus, we can always find a null Lagrangian that makes the boundary conditions (3.10) variationally admissible. We will refer to (3.10) as *generalized simply supported* boundary conditions, the justification of this choice of terminology appearing below.

On the other hand, the second subcase (3.11) is not variationally admissible. Substituting $u + \Phi$ into the boundary conditions, differentiating with respect to ε and setting $\varepsilon = 0$, we find that φ satisfies the corresponding *linearized boundary conditions*

$$\varphi'(a) = \alpha'\big(u(a)\big)\,\varphi(a), \qquad \varphi''(a) = \beta'\big(u(a)\big)\,\varphi(a).$$

Using the first of these, the corresponding boundary term (3.5) can thus be written in the form

$$B(a) = \left[\, g(a) + h(a)\,\alpha'\big(u(a)\big) \,\right] \varphi(a), \tag{3.14}$$

whose vanishing requires the vanishing of the expression in brackets. However, referring back to (3.7) and the nondegeneracy condition (3.2), there is a term in $g(a)$ that depends on $u'''(a)$ with non-vanishing coefficient. Thus, the vanishing of the bracket expression will impose a third boundary condition involving $u'''(a)$ and hence, except in exceptional circumstances, we do not have a variational formulation of the boundary conditions (3.11).

Third order boundary conditions come in three subclasses.

Generalized free boundary conditions:

$$u''(a) = \alpha\big(\,u(a), u'(a)\,\big), \qquad u'''(a) = \beta\big(\,u(a), u'(a)\,\big); \tag{3.15}$$

Generalized sliding boundary conditions:

$$u'(a) = \alpha\big(\,u(a)\,\big), \qquad u'''(a) = \beta\big(\,u(a), u''(a)\,\big); \tag{3.16}$$

and a final case:

$$u(a) = \alpha, \qquad u'''(a) = \beta\big(\,u'(a), u''(a)\,\big). \tag{3.17}$$

Here α, β are smooth functions of the indicated variables except in the last case when α is constant. As we will show, in the first two cases, variational admissibility imposes certain compatibility conditions between the functions α, β, while the third case is never variationally admissible. In all cases, the variation $\varphi(x)$ must satisfy the corresponding linearized boundary conditions, derived in the same manner as above.

Starting with the free conditions (3.15), both $\varphi(a)$ and $\varphi'(a)$ are unconstrained, and hence both boundary terms (3.7) must vanish when the boundary conditions are satisfied. Since both $u(a)$ and $u'(a)$ are similarly unconstrained, this requires

$$\frac{\partial A}{\partial u}\,(a,u,p) = P(u,p), \qquad \frac{\partial A}{\partial p}\,(a,u,p) = Q(u,p), \tag{3.18}$$

for $u, p \in \mathbb{R}$, where

$$P(u,p) = \beta(u,p)\,\frac{\partial^2 L}{\partial q^2} + \alpha(u,p)\,\frac{\partial^2 L}{\partial p\,\partial q} + p\,\frac{\partial^2 L}{\partial u\,\partial q} + \frac{\partial^2 L}{\partial x\,\partial q} - \frac{\partial L}{\partial p}\,, \quad Q(u,p) = \frac{\partial L}{\partial q}\,, \tag{3.19}$$

and the derivatives of L are all evaluated at $(x,u,p,q) = \big(a,u,p,\alpha(u,p)\big)$. We can solve (3.18), and hence construct a null Lagrangian such that the modified variational principle is compatible with the given boundary conditions, provided the functions (3.19) satisfy the integrability constraint

$$\frac{\partial P}{\partial p} = \frac{\partial Q}{\partial u}\,, \tag{3.20}$$

which effectively imposes a compatibility condition on the boundary functions α, β in order that the corresponding boundary conditions be variationally admissible.

Turning to the sliding conditions (3.16), linearizing the first boundary condition as above, we deduce that (3.14) must vanish at $x = a$, which requires that

$$\begin{aligned}\frac{\partial A}{\partial u}\big(\,a,u,\alpha(u)\,\big) &+ \alpha'(u)\,\frac{\partial A}{\partial p}\big(\,a,u,\alpha(u)\,\big)\\ &= \beta(u,q)\,\frac{\partial^2 L}{\partial q^2} + q\,\frac{\partial^2 L}{\partial p\,\partial q} + \alpha(u)\,\frac{\partial^2 L}{\partial u\,\partial q} + \frac{\partial^2 L}{\partial x\,\partial q} - \frac{\partial L}{\partial p} - \alpha'(u)\,\frac{\partial L}{\partial q}\,,\end{aligned}$$

where the derivatives of L are evaluated at $(x, u, p, q) = \bigl(a, u, \alpha(u), q\bigr)$. However, here the dependence of A on p is not helpful, and we can assume the null Lagrangian (3.6) is prescribed by $A(x, u)$, reducing the previous equation to

$$\frac{\partial A}{\partial u} = \beta(u, q)\,\frac{\partial^2 L}{\partial q^2} + q\,\frac{\partial^2 L}{\partial p\,\partial q} + \alpha(u)\,\frac{\partial^2 L}{\partial u\,\partial q} + \frac{\partial^2 L}{\partial x\,\partial q} - \frac{\partial L}{\partial p} - \alpha'(u)\,\frac{\partial L}{\partial q}\,. \tag{3.21}$$

Since the left hand side does not depend on q, given the nondegeneracy condition (3.2), the right hand side serves to prescribe the dependence of β on q; once this is of the proper form, the null Lagrangian can be obtained by integrating the remaining q-independent terms on the right hand side of (3.21) with respect to u to construct $A(x, u)$.

As for the third possibility, (3.17), arguing as in the case (3.10), we must impose the corresponding natural boundary condition (3.12). But this would impose an additional boundary condition constraining $u''(a)$. We conclude that we are unable to find a null Lagrangian that makes (3.12) a consequence of the given boundary conditions.

Example 3.1. The linearized equilibrium equations for a one-dimensional elastic beam, [**3**, **40**], first developed by Jacob and Daniel Bernoulli and Leonhard Euler, are obtained by minimizing the stored energy functional

$$J[u] = \int_a^b \left[\, \tfrac{1}{2}\, c(x)\, u''^2 - f(x)\, u \,\right] dx, \tag{3.22}$$

prescribed by the *Lagrangian* $L(x, u, p, q) = \frac{1}{2}\, c(x)\, q^2 - f(x)\, u$. The function $c(x) > 0$ is assumed positive and measures the *stiffness* of the beam at the point x, while $f(x)$ represent an external forcing. The equilibrium configuration of the beam is characterized as a solution to the fourth order Euler–Lagrange equation

$$\frac{d^2}{dx^2}\left(c(x)\, \frac{d^2 u}{dx^2} \right) = f(x). \tag{3.23}$$

There are four physically important pairs of homogeneous boundary conditions that can be imposed at each endpoint. The associated names are physically motivated, and were adapted to our preceding general formulation.

a) *Fixed (clamped) end*: $u(a) = u'(a) = 0,$
b) *Simply supported end*: $u(a) = u''(a) = 0,$
c) *Sliding end*: $u'(a) = u'''(a) = 0,$
d) *Free end*: $u''(a) = u'''(a) = 0.$

A second pair of boundary conditions must be imposed at the other end $x = b$, and can be mixed or matched in any combination. Inhomogeneous boundary conditions, both linear and nonlinear, are also allowed and used to model applied displacements or applied forces at the ends, although not all of these are variational. On the other hand, we will see that any *constant* inhomogeneity for the above four types of boundary conditions, i.e., setting one or both of the indicated derivatives of u at the endpoint to be a fixed constant, is always admissible.

Let us investigate the full range of possibilities in detail based on the preceding analysis. First, inhomogeneous fixed boundary conditions (3.8) are always admissible. Next suppose we have inhomogeneous simply supported conditions of the form (3.11). In view of (3.13), using $A(x,p) = c(x)\,B(p)$ to form our null Lagrangian, where B is a scalar function of the derivative variable with $\beta(p) = B'(p)$ allows us to impose any inhomogeneous simply supported conditions. For sliding boundary conditions (3.16), requiring the right hand side of (3.21) to be independent of q implies that $\beta(u,q) = \alpha'(u)\,q + \gamma(u)$ for some scalar function γ. The remaining terms give the admissibility constraints

$$\alpha(u) = F'(u), \qquad \beta(u,q) = F''(u)\,q + G'(u), \qquad A(x,u) = c'(x)\,F(u) + c(x)\,G(u), \quad (3.24)$$

where F, G are arbitrary scalar functions. Finally consider the generalized free conditions (3.10). In this case, the functions (3.19) are

$$P(u,p) = c(a)\,\beta(u,p) + c'(a)\,\alpha(u,p), \qquad Q(u,p) = c'(a)\,\alpha(u,p), \qquad (3.25)$$

and the integrability constraint (3.20) required for variational admissibility reduces to

$$c(a)\,\frac{\partial \beta}{\partial p} + c'(a)\,\frac{\partial \alpha}{\partial p} = c'(a)\,\frac{\partial \alpha}{\partial u}\,. \qquad (3.26)$$

The potential physical relevance of these constraints is not immediately clear to the author.

Example 3.2. *Euler's elastica*, [**3**, **26**], governs the equilibrium configurations of a thin planar elastic rod, i.e., a nonlinear elastic beam. In the unforced case, we assume that the elastica is given by the graph† of a function $y = u(x)$, with stored energy functional is

$$\mathcal{I}[u] = \int \tfrac{1}{2}\,\kappa^2\,ds = \int \frac{(u'')^2\,dx}{2\big(1+(u')^2\big)^{5/2}}, \qquad \text{with Lagrangian} \qquad L = \frac{q^2}{2\,(1+p^2)^{5/2}}\,, \qquad (3.27)$$

where

$$\kappa = \frac{u''}{\big(1+(u')^2\big)^{3/2}}\,, \qquad ds = \sqrt{1+(u')^2}\,dx, \qquad (3.28)$$

are, respectively, the curvature and arc length of the elastica at the point $\big(x, u(x)\big)$. As Euler discovered, the resulting complicated, highly nonlinear fourth order Euler-Lagrange equation can be compactly written as

$$E(L) = \frac{d^2\kappa}{ds^2} + \tfrac{1}{2}\,\kappa^3 = 0, \qquad (3.29)$$

whose solutions $\kappa(s)$ can thus be expressed in terms of elliptic functions, [**26**, **30**], whose historical developments in fact commence with Euler's solution to the elastica. A direct derivation of (3.29) from the curvature form of the Lagrangian can be found in [**2**, **22**, **25**].

† Not all equilibrium configurations are of this form. Extending the analysis to general parametrized curves is not difficult.

Let us apply the preceding developments to determine which boundary conditions are variationally admissible. We will just summarize the results leaving the details of the calculation for the reader to fill in. As always, inhomogeneous fixed boundary conditions (3.8) are variationally admissible, as are inhomogeneous simply supported conditions (3.10), the latter requiring inclusion of the null Lagrangian based on

$$A(p) = \int \frac{\beta(p)\,dp}{(1+p^2)^{5/2}}\,. \tag{3.30}$$

In the case of sliding boundary conditions (3.16), using (3.21), we find

$$\beta(u,q) = \frac{5\,\alpha(u)^2 q^2}{2\,\bigl(1+\alpha(u)^2\bigr)} + \alpha'(u)\,q + \bigl(1+\alpha(u)^2\bigr)^{5/2} A'(u), \qquad A = A(u). \tag{3.31}$$

As for the free boundary conditions (3.15), setting

$$P(u,p) = \frac{(1+p^2)\,\beta(u,p) - \frac{5}{2}\,p\,\alpha(u,p)^2}{(1+p^2)^{7/2}}\,, \qquad Q(u,p) = \frac{\alpha(u,p)}{(1+p^2)^{5/2}}\,, \tag{3.32}$$

the integrability constraint (3.20) is required for admissibility.

4. Variational Problems Involving Several Unknowns.

Next we investigate the case of first order variational problems involving one independent and two dependent variables. We find, unlike the scalar first order case, compatibility constraints are required even for uncoupled boundary conditions in order that they be variationally admissible. The extension of this analysis to first order variational problems involving more dependent variables is straightforward.

We thus consider a functional

$$J[u,v] = \int_a^b L(x,u,v,u',v')\,dx, \tag{4.1}$$

prescribed by the *Lagrangian* $L(x,u,v,p,q)$ involving two unknown functions $u(x), v(x)$; here p, q represent u', v', respectively. We introduce variations $u(x)+\Phi(\varepsilon,x),\ v(x)+\Psi(\varepsilon,x)$, so that $\Phi(0,x) = \Psi(0,x) \equiv 0$, and set $\varphi(x) = \dfrac{\partial\Phi}{\partial\varepsilon}\,(0,x),\ \psi(x) = \dfrac{\partial\Psi}{\partial\varepsilon}\,(0,x)$. Arguing as before, we compute the derivative of the scalar function

$$h(\varepsilon) = J[\,u+\Phi, v+\Psi\,] = \int_a^b L(x, u+\Phi, v+\Psi, u'+\Phi', v'+\Psi'\,)\,dx$$

and require

$$\begin{aligned} 0 = h'(0) &= \int_a^b \frac{d}{d\varepsilon}\,L(x, u+\Phi, v+\Psi, u'+\Phi', v'+\Psi'\,)\,\Big|_{\varepsilon=0} dx \\ &= \int_a^b \left[\,\varphi(x)\,\frac{\partial L}{\partial u} + \varphi'(x)\,\frac{\partial L}{\partial p} + \psi(x)\,\frac{\partial L}{\partial v} + \psi'(x)\,\frac{\partial L}{\partial q}\,\right] dx \\ &= B(b) - B(a) + \int_a^b \bigl[\,\varphi(x)\,E_u(L) + \psi(x)\,E_v(L)\,\bigr]\,dx. \end{aligned} \tag{4.2}$$

Here

$$
\begin{aligned}
E_u(L) &= \frac{\partial L}{\partial u}\,(x,u,v,u',v') - \mathrm{D}_x\left(\frac{\partial L}{\partial p}\,(x,u,v,u',v') \right), \\
E_v(L) &= \frac{\partial L}{\partial v}\,(x,u,v,u',v') - \mathrm{D}_x\left(\frac{\partial L}{\partial q}\,(x,u,v,u',v') \right),
\end{aligned}
\tag{4.3}
$$

are the Euler–Lagrange expressions associated with each of the dependent variables, while the variational boundary terms are obtained by evaluating the following function at the endpoints $x = a, b$:

$$
B(x) = \varphi(x)\,\frac{\partial L}{\partial p}\,(x,u,v,u',v') + \psi(x)\,\frac{\partial L}{\partial q}\,(x,u,v,u',v'). \tag{4.4}
$$

Taking $\varphi(x), \psi(x)$ to have compact support in $(\,a, b\,)$ annihilates the boundary terms in (4.2). Thus, applying the Fundamental Lemma 2.1, we conclude that any critical functions u, v must satisfy the system of second order Euler–Lagrange equations

$$
E_u(L) = 0, \qquad E_v(L) = 0. \tag{4.5}
$$

In order that the boundary terms vanish at an endpoint, say $x = a$, we can either impose an inhomogeneous fixed condition $u(a) = \alpha$ or the corresponding homogeneous natural boundary condition $\dfrac{\partial L}{\partial p}\,\big(\,a, u(a), v(a), u'(a), v'(a)\,\big) = 0$; the same goes for the other variable: either $v(a) = \beta$ or $\dfrac{\partial L}{\partial q}\,\big(\,a, u(a), v(a), u'(a), v'(a)\,\big) = 0$.

Another option is a mixture, imposing one fixed boundary condition and letting the second be determined by naturality. In geometric terms, the problem is to find a solution curve $\big(u(x), v(x)\big)$ whose endpoint, at $x = a$, lies on a prescribed plane curve, say

$$
H\big(\,a, u(a), v(a)\,\big) = 0. \tag{4.6}
$$

We assume the curve is nonsingular, meaning that $H_a(u,v) = H(a,u,v)$ has non-vanishing gradient, $\nabla H_a = \left(\dfrac{\partial H_a}{\partial u}, \dfrac{\partial H_a}{\partial v} \right) \neq 0$, on the locus (4.6), which, by the Implicit Function Theorem, implies that one can simplify by solving for one of the boundary values in terms of the other; for example $v(a) = h\big(a, u(a)\big)$. The variations that infinitesimally preserve (4.6) satisfy the linearized boundary condition:

$$
\frac{\partial H}{\partial u}\big(\,a, u(a), v(a)\,\big)\,\varphi(a) + \frac{\partial H}{\partial v}\big(\,a, u(a), v(a)\,\big)\,\psi(a) = 0,
$$

which implies that they be tangent to the surface. Thus, the boundary variation (4.4) will vanish provided the natural boundary condition

$$
\begin{aligned}
&\frac{\partial H}{\partial v}\,\big(\,a, u(a), v(a)\,\big)\,\frac{\partial L}{\partial p}\,\big(\,a, u(a), v(a), u'(a), v'(a)\,\big) \\
&\qquad\qquad - \frac{\partial H}{\partial u}\,\big(\,a, u(a), v(a)\,\big)\,\frac{\partial L}{\partial q}\,\big(\,a, u(a), v(a), u'(a), v'(a)\,\big) = 0
\end{aligned}
\tag{4.7}
$$

holds. The other endpoint $x = b$ is treated similarly.

In this situation, null Lagrangians have the form

$$N(x, u, v, p, q) = \mathrm{D}_x A(x, u, v) = \frac{\partial A}{\partial x} + p\,\frac{\partial A}{\partial u} + q\,\frac{\partial A}{\partial v} \tag{4.8}$$

for some function A that depends only on x, u, v. Replacing L by $L + N$ in the preceding computation, we arrive at the modified boundary function

$$B(x) = \left(\frac{\partial L}{\partial p} + \frac{\partial A}{\partial u}\right)\varphi(x) + \left(\frac{\partial L}{\partial q} + \frac{\partial A}{\partial v}\right)\psi(x). \tag{4.9}$$

Thus, given a set of general first order boundary conditions

$$u'(a) = \alpha\big(\,a, u(a), v(a)\,\big), \qquad v'(a) = \beta\big(\,a, u(a), v(a)\,\big), \tag{4.10}$$

the boundary term $B(a) = 0$ for all allowable variations if and only if

$$\frac{\partial A}{\partial u}\big(\,a, u(a), v(a)\,\big) = P\big(\,a, u(a), v(a)\,\big), \qquad \frac{\partial A}{\partial v}\big(\,a, u(a), v(a)\,\big) = Q\big(\,a, u(a), v(a)\,\big), \tag{4.11}$$

where

$$\begin{aligned} P(x, u, v) &= -\,\frac{\partial L}{\partial p}\big(\,x, u, v, \alpha(x, u, v), \beta(x, u, v)\,\big), \\ Q(x, u, v) &= -\,\frac{\partial L}{\partial q}\big(\,x, u, v, \alpha(x, u, v), \beta(x, u, v)\,\big). \end{aligned} \tag{4.12}$$

Thus, we can construct a null Lagrangian that makes the boundary conditions (4.10) variationally admissible if and only if we can solve (4.11) for $A(x, u, v)$, which requires the compatibility constraint

$$\frac{\partial P}{\partial v} = \frac{\partial Q}{\partial u} \tag{4.13}$$

at $x = a$. We conclude that, in contrast to the case of a single dependent variable, not every set of uncoupled boundary conditions is variationally admissible for a first order problem in several dependent variables. Boundary conditions satisfying the constraint (4.13) are also known as *self-adjoint* boundary conditions, [**19**].

Finally, in the case of a single fixed boundary condition (4.6), an easy calculation shows that, with the addition of the null Lagrangian (4.8), the right hand side of the associated natural boundary condition (4.7) becomes the Jacobian determinant between A and H:

$$\frac{\partial A}{\partial u}\big(\,a, u(a), v(a)\,\big)\,\frac{\partial H}{\partial v}\big(\,a, u(a), v(a)\,\big) - \frac{\partial A}{\partial v}\big(\,a, u(a), v(a)\,\big)\,\frac{\partial H}{\partial u}\big(\,a, u(a), v(a)\,\big),$$

and can thus, by suitable choice of $A(x, u, v)$, be made equal to any function of $u(a), v(a)$. We conclude that, as in the single dependent variable problem, there are no constraints on the inhomogeneous term in the single natural boundary condition.

5. Multidimensional First Order Variational Problems.

Next, we turn to first order variational problems that involve several independent variables. In this section, we restrict our attention to the case of two independent variables, which we call x, y, and a single dependent variable u; later sections will deal with more general situations.

Thus, we consider a functional

$$J[u] = \iint_D L(x, y, u, u_x, u_y)\, dx\, dy, \tag{5.1}$$

given by a double integral over a prescribed bounded open domain $D \subset \mathbb{R}^2$, assumed to have smooth (or, possibly, piecewise smooth) boundary, denoted by ∂D. We adopt subscript notation to denote partial derivatives, so $u_x = \dfrac{\partial u}{\partial x}$, $u_y = \dfrac{\partial u}{\partial y}$, and use the variables p, q to respectively represent the first order partial derivatives of u. The *Lagrangian* $L(x, y, u, p, q)$ is assumed to be a sufficiently smooth function of its five arguments. Our goal is to find the function(s) $u = f(x, y)$ that minimize the value of $J[u]$ when subject to a set of prescribed boundary conditions on ∂D.

Let $\Phi(\varepsilon, x, y)$, defined for $\varepsilon \in \mathbb{R}$, $(x, y) \in D$, denote a variation in the function u, where $\Phi(0, x, y) \equiv 0$ and we set $\varphi(x, y) = \dfrac{\partial \Phi}{\partial \varepsilon}(0, x, y)$. We thus introduce the scalar function

$$h(\varepsilon) \equiv J[u + \varepsilon v] = \iint_D L(x, y, u + \Phi, u_x + \Phi_x, u_y + \Phi_y)\, dx\, dy.$$

If u is a local minimizer, then, subject to appropriate smoothness assumptions, it must satisfy

$$0 = h'(0) = \iint_D \left(\varphi(x, y)\, \frac{\partial L}{\partial u} + \varphi_x(x, y)\, \frac{\partial L}{\partial p} + \varphi_y(x, y)\, \frac{\partial L}{\partial q} \right) dx\, dy, \tag{5.2}$$

where the derivatives of L are all evaluated at x, y, u, u_x, u_y. The next step is to remove the derivatives from φ through an integration by parts based on the divergence form of Green's Theorem, [**34**]:

$$\iint_D (\operatorname{div} \mathbf{v})\, dx\, dy = \oint_{\partial D} \mathbf{v} \cdot \mathbf{n}\, ds, \tag{5.3}$$

in which $\mathbf{v}$ is a C^1 vector field, $\mathbf{n}$ is the unit outward normal to the boundary ∂D of the domain, oriented in the counterclockwise direction, and ds is the element of arc length thereon. We set

$$\mathbf{v} = \varphi\, \mathbf{V}, \qquad \text{where} \qquad \mathbf{V} = \left(\frac{\partial L}{\partial p}, \frac{\partial L}{\partial q} \right). \tag{5.4}$$

We conclude that

$$h'(0) = \oint_{\partial D} \varphi\, (\mathbf{V} \cdot \mathbf{n})\, ds + \iint_D \varphi(x, y)\, E(L)\, dx\, dy, \tag{5.5}$$

where

$$E(L) = \frac{\partial L}{\partial u} - \mathrm{D}_x\left(\frac{\partial L}{\partial p}\right) - \mathrm{D}_y\left(\frac{\partial L}{\partial q}\right) \tag{5.6}$$

is the Euler–Lagrange expression associated with the Lagrangian L, with $\mathrm{D}_x, \mathrm{D}_y$ denoting the total derivatives with respect to the independent variables. Taking φ to be of compact support in D and using a two-dimensional version of the Fundamental Lemma 2.1, we conclude that any sufficiently smooth minimizer, or, more generally, critical function, $u(x,y)$ must satisfy the *Euler–Lagrange equation* $E(L) = 0$. Assuming that not all the second order partial derivatives of L with respect to the variables p, q vanish, the Euler–Lagrange equation is a second order partial differential equation.

Suppose first that we have imposed a *fixed boundary condition*

$$u(x,y) = g(x,y) \qquad \text{for} \qquad (x,y) \in S \subseteq \partial D$$

on a subset[†] S of the boundary. This implies $\varphi(x,y) = 0$ for $(x,y) \in S$, and hence that part of the boundary integral in (5.5) will vanish. This leaves an integral over the remainder $T = \partial D \setminus S$, the free boundary where the value of the minimizer u is unspecified. Since $\varphi(x,y)$ is no longer constrained on T, a version of the Fundamental Lemma 2.1 applies to the boundary integral, and implies that $\mathbf{V} \cdot \mathbf{n} = 0$ on T, which plays the role of the *natural boundary condition*. We conclude that either

$$u(x,y) = g(x,y) \qquad \text{or} \qquad \mathbf{V} \cdot \mathbf{n} = \left(\frac{\partial L}{\partial p}, \frac{\partial L}{\partial q}\right) \cdot \mathbf{n} = 0$$

at each point $(x,y) \in \partial D$.

As before, we seek to extend the range of variationally admissible boundary conditions by including a null Lagrangian. In the present situation, first order null Lagrangians have the form of a divergence, [**32**]:

$$N(x,y,u,p,q) = \mathrm{D}_x A(x,y,u) + \mathrm{D}_y B(x,y,u) = \frac{\partial A}{\partial x} + p\,\frac{\partial A}{\partial u} + \frac{\partial B}{\partial y} + q\,\frac{\partial B}{\partial u}\,, \tag{5.7}$$

where $\mathbf{w} = (A, B)$ is a general vector field that depends on x, y and the dependent variable u, but not on derivatives of u. Applying Green's formula (5.3), the integral of any such null Lagrangian depends only on its boundary values:

$$\iint_D N(x,y,u,u_x,u_y)\,dx\,dy = \oint_{\partial D} \mathbf{w} \cdot \mathbf{n}\; ds. \tag{5.8}$$

Adding the null Lagrangian to the original L does not alter the Euler–Lagrange equation, but does change the natural boundary conditions to

$$(\mathbf{V} + \mathbf{w}) \cdot \mathbf{n} = 0. \tag{5.9}$$

[†] For simplicity we assume $S \subset \partial D$ is either empty, or all of ∂D, or a union of one or more disjoint curves contained in ∂D, including their endpoints.

The additional term $\mathbf{w}\cdot\mathbf{n}$ can be arranged, through a suitable choice of the functions A, B, to equal minus any desired function $\beta(x,y,u)$ of a given boundary point $(x,y) \in T \subset \partial D$ and the value of the function $u(x,y)$ there, i.e., we can fix

$$\mathbf{w}\cdot\mathbf{n} = -\,\beta(x,y,u) \qquad \text{for} \qquad (x,y) \in T \subset \partial D. \tag{5.10}$$

The corresponding null Lagrangian (5.7) can be the divergence of any vector field $\mathbf{w} = \big(A(x,y,u), B(x,y,u)\big)$ whose normal component on T equals $-\beta = \mathbf{w}\cdot\mathbf{n}$. (The explicit formula for $\mathbf{w}$ will depend upon the shape of the boundary.) The modified natural boundary conditions take the form

$$\mathbf{V}\cdot\mathbf{n} = \beta(x,y,u), \qquad\qquad (x,y) \in T \subset \partial D. \tag{5.11}$$

these constitute the most general variationally admissible "free boundary conditions" for such a variational problem. In particular, we cannot impose any constraints on the tangential components of $\mathbf{V}$ on the boundary.

Example 5.1. Consider the Dirichlet minimization problem

$$J[u] = \iint_D \left[\, \tfrac{1}{2}\left(u_x^2 + u_y^2\right) - f(x,y)\,u \,\right] dx\,dy = \iint_D \left[\, \tfrac{1}{2}\,\|\,\nabla u\,\|^2 - f(x,y)\,u \,\right] dx\,dy, \tag{5.12}$$

with Lagrangian

$$L = \tfrac{1}{2}(p^2 + q^2) - f(x,y)\,u.$$

The Euler–Lagrange equation (5.6) is the two-dimensional Poisson equation

$$-\,\Delta u = -\,u_{xx} - u_{yy} = f. \tag{5.13}$$

Referring back to (5.4), we find

$$\mathbf{V} = (u_x, u_y) = \nabla u,$$

and hence the natural boundary condition $\mathbf{V}\cdot\mathbf{n} = 0$ is simply the homogeneous Neumann condition $\partial u/\partial\mathbf{n} = 0$. Addition of a null Lagrangian will replace this by the variationally admissible Robin-type boundary condition

$$\frac{\partial u}{\partial \mathbf{n}} = \beta(x,y,u), \tag{5.14}$$

where β is any function depending on the boundary point and the value of u thereon, cf. (5.11). However, we cannot impose a variationally admissible natural boundary condition involving the tangential derivative of u at the boundary. (That being said, for the Dirichlet condition $u = g$ on ∂D, the tangential derivative of u coincides with the tangential derivative of g, and so is constrained by the condition $\partial u/\partial\mathbf{t} = \partial g/\partial\mathbf{t}$.)

Example 5.2. Minimal surfaces, [**21**, **29**], are characterized as (local) minimizers of the *surface area functional*

$$J[u] = \iint_D \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}\; dx\,dy = \iint_D \sqrt{1 + \|\,\nabla u\,\|^2}\; dx\,dy \tag{5.15}$$

with Lagrangian $L = \sqrt{1+p^2+q^2}$. Here we are assuming, for simplicity, that our minimal surface is non-parametric, meaning that it can be described as the graph of a function $z = u(x,y)$ parametrized by $(x,y) \in D$. (In differential geometric terms, this amounts to working on a Monge patch.) Minimal surfaces model soap films in equilibrium, in which the surface tension serves to minimize the area, and thus forms the simplest elastic model for a two-dimensional liquid, [**17**].

Note that

$$\frac{\partial L}{\partial u} = 0, \qquad \frac{\partial L}{\partial p} = \frac{p}{\sqrt{1+p^2+q^2}}\,, \qquad \frac{\partial L}{\partial q} = \frac{q}{\sqrt{1+p^2+q^2}}\,,$$

and hence the Euler–Lagrange equation (5.6) becomes

$$-\frac{\partial}{\partial x}\frac{u_x}{\sqrt{1+u_x^2+u_y^2}} - \frac{\partial}{\partial y}\frac{u_y}{\sqrt{1+u_x^2+u_y^2}} = \frac{-(1+u_y^2)u_{xx} + 2u_x u_y u_{xy} - (1+u_x^2)u_{yy}}{(1+u_x^2+u_y^2)^{3/2}} = 0.$$

Setting the numerator of the left hand side to 0 produces the justly famous *minimal surface equation*

$$(1+u_y^2)\,u_{xx} - 2\,u_x\,u_y\,u_{xy} + (1+u_x^2)\,u_{yy} = 0. \tag{5.16}$$

On a free boundary, the minimal surface must satisfy the natural boundary conditions that the normal component $\mathbf{V}\cdot\mathbf{n}$ of the rescaled gradient vector field

$$\mathbf{V} = \left(\frac{u_x}{\sqrt{1+u_x^2+u_y^2}}, \frac{u_y}{\sqrt{1+u_x^2+u_y^2}}\right) = \frac{\nabla u}{\sqrt{1+\|\nabla u\|^2}}$$

vanishes, which is just the homogeneous Neumann boundary condition $\partial u/\partial\mathbf{n} = 0$.

However, an inhomogeneous Neumann boundary condition is not variationally admissible for the surface area functional, and so a solution to the minimal surface equation that is subject to such a boundary condition cannot be characterized variationally. Indeed, according to (5.9), any variationally admissible boundary condition must, at each point on ∂D, be either a fixed boundary condition $u = \alpha$ or of the form

$$0 = (\mathbf{V}+\mathbf{w})\cdot\mathbf{n} = \frac{\partial u/\partial\mathbf{n}}{\sqrt{1+\|\nabla u\|^2}} - \beta(x,y,u),$$

where $\beta = \mathbf{w}\cdot\mathbf{n}$ is the normal component of $\mathbf{w} = (A,B)$, or, equivalently,

$$\frac{\partial u}{\partial\mathbf{n}} = \beta(x,y,u)\,\sqrt{1+\|\nabla u\|^2}\,, \qquad (x,y)\in\partial D. \tag{5.17}$$

As usual, we can impose fixed boundary conditions on part of the boundary and the modified natural boundary conditions on the remainder, which, when $\beta \neq 0$, requires modification of the surface area functional by addition of a suitable null Lagrangian. The boundary condition (5.17) appears in the theory of equilibrium configurations of capillary surfaces, [**17**; (7.3)], in which β is, typically, independent of u and represents the cosine of the contact angle between the fluid film and a bounding surface. Dependence of the contact angle on u would allow for patterning of the surface.

6. Two-Dimensional Elasticity.

Let us now consider the case of a first order variational problem involving two independent variables, x, y, and two dependent variables, u, v. Thus, we consider a functional

$$J[u,v] = \iint_D L(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u})\, dx\, dy = \iint_D L(x, y, u, v, u_x, u_y, v_x, v_y)\, dx\, dy. \tag{6.1}$$

where $D \subset \mathbb{R}^2$ is as before. The *Lagrangian* will be written in vectorial form

$$L(\mathbf{x}, \mathbf{u}, \mathbf{P}), \qquad \text{where} \qquad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \mathbf{P} = \nabla \mathbf{u} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

Of particular importance are the variational principles arising in two-dimensional hyperelastostatics, [**3**, **23**, **28**], in which $\mathbf{x}$ represents the reference coordinates in a planar elastic body $D \subset \mathbb{R}^2$, while $\mathbf{u}(\mathbf{x})$ represents the deformation of the body, or, in the linear approximation, a small displacement. The Jacobian matrix $\mathbf{P} = \nabla \mathbf{u}$ is known as the *deformation gradient*, and the Lagrangian L represents the *stored energy* of the deformed body (often denoted by W). Frame indifference implies that the stored energy is independent of $\mathbf{u}$ and has a rotational invariance, but these additional constraints will not play a role in our analysis of the boundary conditions. (See Ericksen, [**15**], for a discussion of the effect of such invariances on the form of the null Lagrangians.)

Applying variations $\boldsymbol{\varphi}(\mathbf{x}) = (\,\varphi(x,y), \psi(x,y)\,)^T$ to both components of $\mathbf{u}$, and performing the usual variational calculations leads to the first variation integral

$$\begin{aligned} &\oint_{\partial D} (\,\mathbf{T}\,\boldsymbol{\varphi}\,) \cdot \mathbf{n}\, ds + \iint_D \left[\,\boldsymbol{\varphi} \cdot \mathbf{E}(L)\,\right] dx\, dy \\ &\qquad = \oint_{\partial D} (\,\varphi\, \mathbf{T}_u + \psi\, \mathbf{T}_v\,) \cdot \mathbf{n}\, ds + \iint_D \left[\,\varphi\, E_u(L) + \psi\, E_v(L)\,\right] dx\, dy, \end{aligned} \tag{6.2}$$

which must vanish for all allowable variations. Here

$$\mathbf{T} = (\mathbf{T}_u, \mathbf{T}_v) = \left(\frac{\partial L}{\partial \mathbf{P}}\right)^T = \begin{pmatrix} \partial L/\partial u_x & \partial L/\partial v_x \\ \partial L/\partial u_y & \partial L/\partial v_y \end{pmatrix} \tag{6.3}$$

is known as the *first Piola–Kirchhoff stress tensor* in elasticity, while the components of $\mathbf{E}(L) = (\,E_u(L), E_v(L)\,)^T$ are the Euler–Lagrange expressions; the corresponding second order Euler–Lagrange equations

$$\begin{aligned} E_u(L) &= \frac{\partial L}{\partial u} - \mathrm{D}_x\left(\frac{\partial L}{\partial u_x}\right) - \mathrm{D}_y\left(\frac{\partial L}{\partial u_y}\right) = 0, \\ E_v(L) &= \frac{\partial L}{\partial v} - \mathrm{D}_x\left(\frac{\partial L}{\partial v_x}\right) - \mathrm{D}_y\left(\frac{\partial L}{\partial v_y}\right) = 0, \end{aligned} \tag{6.4}$$

are a consequence of the vanishing of the double integral in (6.2) under variations with compact support.

The boundary integral in (6.2) vanishes provided either the variations vanish on ∂D, or the corresponding boundary terms vanish. Thus, the variationally admissible boundary conditions include the fixed (Dirichlet) conditions

$$u(x,y) = f(x,y), \qquad v(x,y) = g(x,y), \qquad (x,y) \in \partial D, \tag{6.5}$$

or, alternatively, the vanishing of the boundary *tractions*

$$\mathbf{T}_u \cdot \mathbf{n} = \mathbf{T}_v \cdot \mathbf{n} = 0, \qquad (x,y) \in \partial D. \tag{6.6}$$

There is also a hybrid class of boundary conditions, similar to that discussed at the end of Section 4, in which one imposes just one fixed condition, say

$$H\big(x,y,u(x,y),v(x,y)\big) = 0 \qquad \text{for} \qquad (x,y) \in \partial D, \tag{6.7}$$

and the second is the appropriate linear combination of homogeneous traction conditions (6.6) that cause the variational boundary integral to vanish, namely,

$$\frac{\partial H}{\partial v}\,\mathbf{T}_u - \frac{\partial H}{\partial u}\,\mathbf{T}_v = \mathbf{0}, \qquad (x,y) \in \partial D. \tag{6.8}$$

Geometrically, (6.7) requires that each point on the boundary of the deformed body is constrained to lie on a prescribed curve, which is allowed to vary from point to point. (A more physically realizable version of such conditions arises in the equilibrium configuration of the capillary surface of a fluid in a container, [**17**].) Moreover, one can decompose the boundary into subsets with different boundary conditions (fixed, traction, mixed) imposed on the sub-boundaries.

As noted in [**13**], other types of boundary conditions that arise in applications, e.g., the *dead load* boundary conditions in which the boundary tractions are equal to a fixed function of the reference coordinates, [**3, 28**], require the introduction of a suitable null Lagrangian into the variational functional. The most general first order null Lagrangian is a linear combination of the basic null Lagrangians depending only on $\nabla \mathbf{u}$, which are all the subdeterminants thereof, [**5, 15, 33**]:

$$N(\mathbf{x},\mathbf{u},\nabla\mathbf{u}) = \lambda + \boldsymbol{\alpha}\cdot\nabla u + \boldsymbol{\beta}\cdot\nabla v + \mu\,(u_x v_y - u_y v_x). \tag{6.9}$$

The coefficients λ, μ are scalars, while $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are vectors, all depending on $(\mathbf{x},\mathbf{u})$. The condition that N be a null Lagrangian imposes the following constraints:

$$\frac{\partial \lambda}{\partial u} = \nabla\cdot\boldsymbol{\alpha}, \qquad \frac{\partial \lambda}{\partial v} = \nabla\cdot\boldsymbol{\beta}, \qquad \frac{\partial \boldsymbol{\beta}}{\partial u} - \frac{\partial \boldsymbol{\alpha}}{\partial v} = \nabla^\perp \mu, \tag{6.10}$$

which are readily found by computing its Euler–Lagrange expressions. Here

$$\nabla = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \end{pmatrix}, \qquad \nabla^\perp = \begin{pmatrix} \partial/\partial y \\ -\,\partial/\partial x \end{pmatrix}, \tag{6.11}$$

are, respectively, the gradient and skew-gradient with respect to the reference coordinates.

For simplicity let us parametrize the boundary curve by arc length s, and write

$$\partial D = \{\, (x(s), y(s) \mid\ 0 \leq s \leq \ell \,\}, \tag{6.12}$$

where ℓ is the length of ∂D and closure implies $x(0) = x(\ell),\ y(0) = y(\ell)$. The unit tangent and unit (outwards) normal vectors are

$$\mathbf{t} = (x_s, y_s), \qquad \mathbf{n} = (y_s, -x_s), \qquad x_s^2 + y_s^2 = 1. \tag{6.13}$$

The "null Piola–Kirchhoff stresses" associated with (6.9) are

$$\mathbf{S}_u = \begin{pmatrix} \partial N/\partial u_x \\ \partial N/\partial u_y \end{pmatrix} = \boldsymbol{\alpha} + \mu\, \nabla^\perp v, \qquad \mathbf{S}_v = \begin{pmatrix} \partial N/\partial v_x \\ \partial N/\partial v_y \end{pmatrix} = \boldsymbol{\beta} - \mu\, \nabla^\perp u. \tag{6.14}$$

On ∂D, their normal components are the "null tractions"

$$\mathbf{S}_u \cdot \mathbf{n} = \boldsymbol{\alpha} \cdot \mathbf{n} + \mu\, \nabla v \cdot \mathbf{t} = \boldsymbol{\alpha} \cdot \mathbf{n} + \mu\, \frac{\partial v}{\partial s}\,, \qquad \mathbf{S}_v \cdot \mathbf{n} = \boldsymbol{\beta} \cdot \mathbf{n} - \mu\, \nabla u \cdot \mathbf{t} = \boldsymbol{\beta} \cdot \mathbf{n} - \mu\, \frac{\partial u}{\partial s}\,,$$

where the second terms in each expression are multiples of the tangential (arc length) derivatives of the indicated deformations. Thus, the most general variationally admissible traction boundary conditions have the form

$$\mathbf{T}_u \cdot \mathbf{n} = \boldsymbol{\alpha} \cdot \mathbf{n} + \mu\, \nabla v \cdot \mathbf{t}, \qquad \mathbf{T}_v \cdot \mathbf{n} = \boldsymbol{\beta} \cdot \mathbf{n} - \mu\, \nabla u \cdot \mathbf{t}, \tag{6.15}$$

where the coefficients $\boldsymbol{\alpha}, \boldsymbol{\beta}, \mu$ satisfy the constraints (6.10). In particular, if they depend only on the reference coordinates x, y, then the third constraint implies that μ is constant, and we deduce the generalized *dead load* boundary conditions

$$\mathbf{T}_u \cdot \mathbf{n} = f(x, y) + \mu\, \nabla v \cdot \mathbf{t}, \qquad \mathbf{T}_v \cdot \mathbf{n} = g(x, y) - \mu\, \nabla u \cdot \mathbf{t}, \tag{6.16}$$

where $f = \boldsymbol{\alpha} \cdot \mathbf{n},\ \ g = \boldsymbol{\beta} \cdot \mathbf{n}$, are arbitrary. The variational problem must be modified by adding in the appropriate null Lagrangian (6.9) to the stored energy. On the other hand, if the coefficients in (6.15) only depend on the deformation $\mathbf{u} = (u, v)$, then $\mu(u, v)$ is arbitrary, but the variationally admissible "load vectors" $\boldsymbol{\alpha}(u, v)$, $\boldsymbol{\beta}(u, v)$ are constrained by

$$\frac{\partial \boldsymbol{\beta}}{\partial u} = \frac{\partial \boldsymbol{\alpha}}{\partial v}\,.$$

The hybrid case, in which only one fixed boundary condition is imposed, is left to the reader to complete.

7. Three-Dimensional Elasticity.

The fully three-dimensional case, which includes three-dimensional hyperelastostatics, [**3**, **23**, **28**], is handled similarly albeit with additional computational challenges, primarily due to the larger number of null Lagrangians. We begin with a functional

$$J[u, v] = \iiint_D L(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u})\; dx\, dy\, dz, \tag{7.1}$$

over a solid domain $D \subset \mathbb{R}^3$ with smooth boundary surface ∂D, oriented using the unit outwards normal $\mathbf{n}$. The *Lagrangian* (stored energy) L depends on the reference coordinates $\mathbf{x} = (\, x, y, z\,)^T$, the deformation (displacement) $\mathbf{u}(\mathbf{x}) = (\, u(x, y, z), v(x, y, z), w(x, y, z)\,)^T$, and the deformation gradient $\mathbf{P} = \nabla \mathbf{u}$, which is the 3×3 Jacobian matrix whose entries are the first order partial derivatives of the dependent variables u, v, w with respect to the independent variables x, y, z. As before, frame indifference, material symmetries, and constitutive laws impose a variety of constraints on the Lagrangian, but these do not affect our presentation.

Applying variations $\boldsymbol{\varphi}(\mathbf{x}) = (\,\varphi(x,y,z), \psi(x,y,z), \chi(x,y,z)\,)^T$ to the three components of $\mathbf{u}$, and performing the usual variational calculations leads to the first variation integral

$$\oint_{\partial D} (\,\mathbf{T}\,\boldsymbol{\varphi}\,) \cdot \mathbf{n}\; ds + \iiint_D \left[\,\boldsymbol{\varphi} \cdot \mathbf{E}(L)\,\right] dx\,dy\,dz, \tag{7.2}$$

where

$$\mathbf{T} = (\mathbf{T}_u, \mathbf{T}_v, \mathbf{T}_w) = \left(\frac{\partial L}{\partial \mathbf{P}}\right)^T \tag{7.3}$$

is the *first Piola–Kirchhoff stress tensor*, while

$$\mathbf{E}(L) = (\,E_u(L), E_v(L), E_w(L)\,)^T = \mathbf{0} \tag{7.4}$$

is the usual system of second order Euler–Lagrange equations.

The boundary integral vanishes provided either each variation vanishes on ∂D, or the corresponding traction vanishes. Thus, the variationally admissible boundary conditions include the fixed (Dirichlet) conditions

$$\mathbf{u}(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} \in \partial D, \tag{7.5}$$

or, alternatively, the vanishing of the boundary surface tractions:

$$\mathbf{T} \cdot \mathbf{n} = (\,\mathbf{T}_u \cdot \mathbf{n}, \mathbf{T}_v \cdot \mathbf{n}, \mathbf{T}_w \cdot \mathbf{n}\,)^T = \mathbf{0}, \qquad \mathbf{x} \in \partial D. \tag{7.6}$$

One can also impose hybrid boundary conditions with one or two fixed conditions, and two or one complementary homogeneous traction conditions, say when the boundary deformations are restricted to prescribed curves or surfaces.

As before — see also [**33**] — the basic null Lagrangians depending only on $\nabla \mathbf{u}$ are all the subdeterminants (of sizes $1 \times 1, 2 \times 2$ and 3×3) of the deformation gradient†, and the general first order null Lagrangian can be written using the vector triple product:

$$\begin{aligned} N(\mathbf{x}, \mathbf{u}, \nabla \mathbf{u}) = \lambda + \boldsymbol{\alpha} \cdot \nabla u + \boldsymbol{\beta} \cdot \nabla v + \boldsymbol{\gamma} \cdot \nabla w + \boldsymbol{\rho} \cdot \nabla v \times \nabla w \\ + \boldsymbol{\sigma} \cdot \nabla w \times \nabla u + \boldsymbol{\tau} \cdot \nabla u \times \nabla v + \mu \nabla u \cdot \nabla v \times \nabla w. \end{aligned} \tag{7.7}$$

The coefficients λ, μ are scalars, while $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\tau}$ are vectors, all depending on $(\mathbf{x}, \mathbf{u})$, and satisfying the following constraints:

$$\begin{gathered} \frac{\partial \lambda}{\partial u} = \nabla \cdot \boldsymbol{\alpha}, \qquad \frac{\partial \lambda}{\partial v} = \nabla \cdot \boldsymbol{\beta}, \qquad \frac{\partial \lambda}{\partial w} = \nabla \cdot \boldsymbol{\gamma}, \qquad \frac{\partial \boldsymbol{\beta}}{\partial w} - \frac{\partial \boldsymbol{\gamma}}{\partial v} = \nabla \times \boldsymbol{\rho}, \\ \frac{\partial \boldsymbol{\gamma}}{\partial u} - \frac{\partial \boldsymbol{\alpha}}{\partial w} = \nabla \times \boldsymbol{\sigma}, \qquad \frac{\partial \boldsymbol{\alpha}}{\partial v} - \frac{\partial \boldsymbol{\beta}}{\partial u} = \nabla \times \boldsymbol{\tau}, \qquad \frac{\partial \boldsymbol{\rho}}{\partial u} = \frac{\partial \boldsymbol{\sigma}}{\partial v} = \frac{\partial \boldsymbol{\tau}}{\partial w} = \nabla \mu. \end{gathered} \tag{7.8}$$

† See [**6**] for the generalization of this result to higher order null Lagrangians.

The “null Piola–Kirchhoff stresses”

$$\begin{aligned}\mathbf{S}_u &= \boldsymbol{\alpha} + \boldsymbol{\sigma} \times \nabla w - \boldsymbol{\tau} \times \nabla v + \mu \, \nabla v \times \nabla w, \\ \mathbf{S}_v &= \boldsymbol{\beta} + \boldsymbol{\tau} \times \nabla u - \boldsymbol{\rho} \times \nabla w + \mu \, \nabla w \times \nabla u, \\ \mathbf{S}_w &= \boldsymbol{\gamma} + \boldsymbol{\rho} \times \nabla v - \boldsymbol{\sigma} \times \nabla u + \mu \, \nabla u \times \nabla v,\end{aligned} \tag{7.9}$$

are thus found by computing the derivatives of the null Lagrangian with respect to the deformation gradient variables. The corresponding “null tractions” on the boundary are obtained by taking their dot products with the unit normal $\mathbf{n}$, and these can be appended to the homogeneous traction boundary conditions without affecting their variational admissibility. The first terms, when $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ depend only on $\mathbf{x}$, produce dead load boundary conditions; the last terms are related to pressure loading on the boundary of the deformed body since the associated null Lagrangian $\det \nabla \mathbf{u} = \nabla u \cdot \nabla v \times \nabla w$ measures the local change in volume. Other options appear to be less well known and would be worth investigating for their potential physical relevance.

Remark: Each vector triple product arising in a null traction, say $\nabla u \times \nabla v \cdot \mathbf{n}$, corresponds geometrically to the volume of the tetrahedron with sides $\nabla u, \nabla v, \mathbf{n}$ or, equivalently, the area of the parallelogram whose sides are the orthogonal projections of the vectors $\nabla u, \nabla v$ onto the tangent space of the boundary surface.

8. Multidimensional Second Order Variational Problems.

While the natural boundary conditions for first order variational problems involving several independent variables are reasonably straightforward, the same cannot be said of second and higher order problems. Forsyth, [**18**; Chapter 11] devotes his penultimate chapter to their study, but his presentation of the boundary conditions is incomplete, and fails to deal with the additional complications that we investigate here. On the other hand, Timoshenko, [**42**], derives the proper natural boundary conditions for the particular Lagrangian arising in linear plate mechanics, but does not extend his analysis to general variational problems. We will derive Timoshenko’s boundary conditions from our general formulas. In addition, the range of natural boundary conditions can be extended using null Lagrangians, although the details are computationally challenging and not particularly enlightening to the author. As in Section 5, for simplicity, we restrict our attention to the case of two independent variables and one dependent variable.

As before, $D \subset \mathbb{R}^2$ is a bounded open domain with smooth connected boundary ∂D, which we orient in the usual counterclockwise direction, although the latter assumptions can easily be weakened. We consider a functional of the form

$$J[u] = \iint_D L(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) \; dx \, dy, \tag{8.1}$$

where the *Lagrangian* L depends on derivatives of order ≤ 2 of the scalar-valued function $u(x, y)$. We perform the usual variational calculations, starting with the variation of the

Lagrangian $h(\varepsilon) = J[u + \Phi]$, with

$$h'(0) = \iint_D \left(\varphi \frac{\partial L}{\partial u} + \varphi_x \frac{\partial L}{\partial u_x} + \varphi_y \frac{\partial L}{\partial u_y} + \varphi_{xx} \frac{\partial L}{\partial u_{xx}} + \varphi_{xy} \frac{\partial L}{\partial u_{xy}} + \varphi_{yy} \frac{\partial L}{\partial u_{yy}} \right) dx\, dy. \tag{8.2}$$

We then integrate each term on the right hand side that involves a derivative of the variation φ by parts, leading to the Euler–Lagrange expression

$$E(L) = \frac{\partial L}{\partial u} - \mathrm{D}_x \left(\frac{\partial L}{\partial u_x} \right) - \mathrm{D}_y \left(\frac{\partial L}{\partial u_y} \right) + \mathrm{D}_x^2 \left(\frac{\partial L}{\partial u_{xx}} \right) + \mathrm{D}_x \mathrm{D}_y \left(\frac{\partial L}{\partial u_{xy}} \right) + \mathrm{D}_y^2 \left(\frac{\partial L}{\partial u_{yy}} \right) \tag{8.3}$$

multiplying φ plus a divergence, which, by Green's formula (5.3), can be replaced by the corresponding boundary integral. Applying the Fundamental Lemma, we conclude that any sufficiently smooth critical function must satisfy the fourth order *Euler–Lagrange equation* $E(L) = 0$.

The divergence terms that result from integrating the second and third terms in (8.2) by parts are given in (5.5–6). For the fourth and sixth terms, the result is straightforward:

$$\begin{aligned} \varphi_{xx} \frac{\partial L}{\partial u_{xx}} &= \varphi\, \mathrm{D}_x^2 \left(\frac{\partial L}{\partial u_{xx}} \right) + \mathrm{D}_x \left(\varphi_x \frac{\partial L}{\partial u_{xx}} - \varphi\, \mathrm{D}_x \frac{\partial L}{\partial u_{xx}} \right), \\ \varphi_{yy} \frac{\partial L}{\partial u_{yy}} &= \varphi\, \mathrm{D}_y^2 \left(\frac{\partial L}{\partial u_{yy}} \right) + \mathrm{D}_y \left(\varphi_y \frac{\partial L}{\partial u_{yy}} - \varphi\, \mathrm{D}_y \frac{\partial L}{\partial u_{yy}} \right). \end{aligned} \tag{8.4}$$

However, for the fifth term, there is an ambiguity, as we can either start with the x derivative or with the y derivative, leading to the alternative expressions

$$\varphi_{xy} \frac{\partial L}{\partial u_{xy}} = \varphi\, \mathrm{D}_x \mathrm{D}_y \left(\frac{\partial L}{\partial u_{xy}} \right) + \begin{cases} \mathrm{D}_x \left(\varphi_y \dfrac{\partial L}{\partial u_{xy}} \right) + \mathrm{D}_y \left(-\varphi\, \mathrm{D}_x \dfrac{\partial L}{\partial u_{xy}} \right), \\[2ex] \mathrm{D}_x \left(-\varphi\, \mathrm{D}_y \dfrac{\partial L}{\partial u_{xy}} \right) + \mathrm{D}_y \left(\varphi_x \dfrac{\partial L}{\partial u_{xy}} \right). \end{cases} \tag{8.5}$$

The divergence terms produce ostensibly different boundary integrals, and thus potentially different natural boundary conditions! Forsyth, [**18**], takes the average of the two expressions in (8.5), but this is just his particular convention, and he stops his presentation before having to deal with the resolution of the problem. His final formula, at the end of Section 337, does not properly encapsulate what the natural boundary conditions should be. Our goal is to ascertain what further analysis is needed.

Observe that the difference between the two expressions in (8.5) is a *null divergence*, [**2**, **31**, **33**], namely a vector field (A, B) that satisfies

$$\mathrm{D}_x A + \mathrm{D}_y B \equiv 0, \tag{8.6}$$

which, assuming it is defined on a topologically trivial domain, is valid if and only if the vector field is a skew gradient:

$$A = \mathrm{D}_y C, \qquad B = -\mathrm{D}_x C, \tag{8.7}$$

for some function C; see [**32**; § 5.4].

The key point underlying the resolution of the difficulty is that, on the boundary, the intrinsic components of the gradient of φ have different effects, and there is a second integration by parts that can be employed. Let $\mathbf{n}, \mathbf{t}$ denote the unit normal and unit tangent vectors at a point $(x, y) \in \partial D$. We can write

$$\nabla \varphi = \varphi_{\mathbf{n}}\, \mathbf{n} + \varphi_{\mathbf{t}}\, \mathbf{t}, \tag{8.8}$$

where the coefficients $\varphi_{\mathbf{n}}, \varphi_{\mathbf{t}}$ are its *normal* and *tangential derivatives*. For a general variation, the normal derivative can be specified independently of φ itself; however, its tangential derivative can be realized as the derivative of the restriction of φ to ∂D, and hence is not independent thereof. Moreover, because the boundary ∂D is a closed curve, a term involving $\varphi_{\mathbf{t}}$ can be integrated by parts without affecting the value of the overall boundary integral.

As in (6.12, 13), we parametrize the boundary curve ∂D by arc length, noting that we can identify the derivative in the tangential direction $\mathbf{t}$ with the arc length derivative: $\partial/\partial \mathbf{t} = \partial/\partial s$. Substituting (6.13) into (8.8), we deduce that

$$\varphi_x = \varphi_{\mathbf{t}}\, x_s + \varphi_{\mathbf{n}}\, y_s = \varphi_s\, x_s + \varphi_{\mathbf{n}}\, y_s, \qquad \varphi_y = \varphi_{\mathbf{t}}\, y_s - \varphi_{\mathbf{n}}\, x_s = \varphi_s\, y_s - \varphi_{\mathbf{n}}\, x_s, \tag{8.9}$$

and so

$$\varphi_{\mathbf{t}} = \varphi_s = \frac{d}{ds}\, \varphi(x(s), y(s)) = \varphi_x\, x_s + \varphi_y\, y_s. \tag{8.10}$$

In this manner, referring back to (8.2), we deduce a formula of the form

$$h'(0) = \iint_D \big[\operatorname{div} \mathbf{W} + \varphi(x,y)\, E(L) \big]\, dx\, dy = \oint_{\partial D} (\mathbf{W} \cdot \mathbf{n}\,)\, ds + \iint_D \varphi(x,y)\, E(L)\, dx\, dy,$$

where the boundary terms, resulting from the preceding integration by parts, are assembled using (5.4), (8.4), along with the first expression in (8.5). (As the reader can check, the second expression leads to the same final formula for the boundary integral, as does Forsyth's average of the two.) Thus, $\mathbf{W} = (W_1, W_2)$, where

$$\begin{aligned} W_1 &= \varphi \left(\frac{\partial L}{\partial u_x} - \mathrm{D}_x \frac{\partial L}{\partial u_{xx}} \right) + \varphi_x \left(\frac{\partial L}{\partial u_{xx}} \right) + \varphi_y \left(\frac{\partial L}{\partial u_{xy}} \right), \\ W_2 &= \varphi \left(\frac{\partial L}{\partial u_y} - \mathrm{D}_x \frac{\partial L}{\partial u_{xy}} - \mathrm{D}_y \frac{\partial L}{\partial u_{yy}} \right) + \varphi_y \left(\frac{\partial L}{\partial u_{yy}} \right). \end{aligned}$$

Using (6.13, 9, 10), we thus find

$$\begin{aligned} \mathbf{W} \cdot \mathbf{n} = \varphi &\left[y_s \left(\frac{\partial L}{\partial u_x} - \mathrm{D}_x \frac{\partial L}{\partial u_{xx}} \right) - x_s \left(\frac{\partial L}{\partial u_y} - \mathrm{D}_x \frac{\partial L}{\partial u_{xy}} - \mathrm{D}_y \frac{\partial L}{\partial u_{yy}} \right) \right] \\ &+ \varphi_s \left[x_s y_s \frac{\partial L}{\partial u_{xx}} + y_s^2 \frac{\partial L}{\partial u_{xy}} - x_s y_s \frac{\partial L}{\partial u_{yy}} \right] + \varphi_{\mathbf{n}} \left[y_s^2 \frac{\partial L}{\partial u_{xx}} - x_s y_s \frac{\partial L}{\partial u_{xy}} + x_s^2 \frac{\partial L}{\partial u_{yy}} \right]. \end{aligned}$$

The term involving the tangential derivative $\varphi_s = \varphi_{\mathbf{t}}$ can be integrated by parts, using the formula

$$\oint_{\partial D} (f\, g_s)\, ds = \oint_{\partial D} (-\, f_s\, g)\, ds,$$

valid for any functions f, g on ∂D. (No boundary terms appear because $\partial(\partial D) = \varnothing$.) Thus, we can rewrite

$$\oint_{\partial D} (\mathbf{W} \cdot \mathbf{n}\,)\, ds = \oint_{\partial D} (F\,\varphi + G\,\varphi_{\mathbf{n}})\, ds,$$

where

$$\begin{aligned}
F &= y_s \left(\frac{\partial L}{\partial u_x} - \mathrm{D}_x \frac{\partial L}{\partial u_{xx}} \right) + \big(x_s y_{ss} + y_s x_{ss} + x_s^2\, y_s \mathrm{D}_x + x_s y_s^2\, \mathrm{D}_y \,\big) \left(\frac{\partial L}{\partial u_{yy}} - \frac{\partial L}{\partial u_{xx}} \right) \\
&\qquad - x_s \left(\frac{\partial L}{\partial u_y} - \mathrm{D}_y \frac{\partial L}{\partial u_{yy}} \right) + \big(x_s x_{ss} - y_s y_{ss} + x_s^3\, \mathrm{D}_x - y_s^3\, \mathrm{D}_y \,\big) \frac{\partial L}{\partial u_{xy}} \\
&= \mathbf{H} \cdot \mathbf{n} + \frac{d}{ds} \left[x_s y_s \left(\frac{\partial L}{\partial u_{yy}} - \frac{\partial L}{\partial u_{xx}} \right) + \frac{1}{2}\,(x_s^2 - y_s^2)\, \frac{\partial L}{\partial u_{xy}} \right], \qquad (8.11) \\
\mathbf{H} &= \left(\frac{\partial L}{\partial u_x} - \mathrm{D}_x \frac{\partial L}{\partial u_{xx}} - \frac{1}{2}\, \mathrm{D}_y \frac{\partial L}{\partial u_{xy}}\,,\ \frac{\partial L}{\partial u_y} - \frac{1}{2}\, \mathrm{D}_x \frac{\partial L}{\partial u_{xy}} - \mathrm{D}_y \frac{\partial L}{\partial u_{yy}} \right)^T, \\
G &= y_s^2\, \frac{\partial L}{\partial u_{xx}} - x_s y_s\, \frac{\partial L}{\partial u_{xy}} + x_s^2\, \frac{\partial L}{\partial u_{yy}}\,.
\end{aligned}$$

where we have used the identities

$$x_s^2 + y_s^2 = 1, \qquad x_s x_{ss} + y_s y_{ss} = 0, \qquad (8.12)$$

to simplify and make the final result evidently symmetric under a 90° rotation of x and y, as it must be, even though the intervening calculation was not. Indeed, the final formulas are invariant under completely general orientation-preserving changes of variables in x, y.

Thus, at each point in ∂D, either $\varphi = 0$, which means that u must satisfy an inhomogeneous Dirichlet boundary condition $u = f$, or the natural boundary condition $F = 0$ holds. Furthermore, either $\varphi_{\mathbf{n}} = 0$, which means that u must satisfy an inhomogeneous Neumann boundary condition $\partial u / \partial \mathbf{n} = g$, or the natural boundary condition $G = 0$ holds. There are thus four possibilities, similar to what we found for a second order scalar variational problem. The boundary curve ∂D could then be divided into up to four sub-curves (which need not individually be connected) on each of which one of these four possibilities is imposed. This produces the complete set of variationally admissible boundary conditions for the fourth order Euler–Lagrange equations before we admit the addition of a null Lagrangian, which we now investigate. The fact that the first natural boundary condition turns out to be cubic in the components of the tangent (or the normal) to the curve and also involves its second derivatives (curvature) is striking.

Example 8.1. The equations of *linear plate mechanics* are based on the Lagrangian

$$L = \tfrac{1}{2}\,u_{xx}^2 + \nu\,u_{xx}u_{yy} + \tfrac{1}{2}\,u_{yy}^2 + (1-\nu)u_{xy}^2 = \tfrac{1}{2}\,(u_{xx}+u_{yy})^2 + (\nu-1)(u_{xx}u_{yy} - u_{xy}^2), \tag{8.13}$$

where the constant ν denotes the Poisson ratio of the material, which is assumed to be uniform; see Timoshenko, [**42**], for details. The Euler–Lagrange equation is the *biharmonic equation*

$$E(L) = u_{xxxx} + 2\,u_{xxyy} + u_{yyyy} = \Delta^2 u = 0, \tag{8.14}$$

where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplacian operator, and is independent of ν. This is because the final term in the Lagrangian (8.13) is a multiple of the *Hessian determinant* $H = \nabla^2 u$ of the function u, which is a null Lagrangian:

$$\det H = \det \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix} = u_{xx}u_{yy} - u_{xy}^2 = \mathrm{D}_x(u_x u_{yy}) + \mathrm{D}_y(-\,u_x u_{xy}). \tag{8.15}$$

Consequently the simplified Lagrangian

$$\widetilde{L} = \tfrac{1}{2}\,(u_{xx}+u_{yy})^2 = \tfrac{1}{2}\,u_{xx}^2 + 2\,u_{xx}u_{yy} + \tfrac{1}{2}\,u_{yy}^2 \tag{8.16}$$

has exactly the same Euler–Lagrange expression: $E(\widetilde{L}) = E(L) = \Delta^2 u$. However, the two Lagrangians possess quite different natural boundary conditions, and only the former leads to physically meaningful expressions.

Let us compute the natural boundary condition terms (8.11), first for the physical plate Lagrangian (8.13). We find

$$\begin{aligned} F &= -\,y_s(u_{xxx}+u_{xyy}) + x_s(u_{xxy}+u_{yyy}) + (\nu-1)\,\frac{d}{ds}\left[\,x_s y_s(u_{yy}-u_{xx}) + (x_s^2 - y_s^2)\,u_{xy}\,\right] \\ &= -\,(\Delta u)_{\mathbf{n}} + (1-\nu)\bigl(\,\mathbf{n}^T H\,\mathbf{t}\,\bigr)_s, \\ G &= (\nu\,x_s^2 + y_s^2)\,u_{xx} + 2\,(\nu-1)\,x_s y_s u_{xy} + (x_s^2 + \nu\,y_s^2)u_{yy} = \Delta u + (\nu-1)\,\mathbf{t}^T H\,\mathbf{t}. \end{aligned} \tag{8.17}$$

Identifying $\mathbf{n} = (y_s, -x_s) = (\cos\alpha, \sin\alpha)$, we recover the formulas in [**42**; eq. (110)]. Physically, G represents the bending moment of the plate at the boundary point. The first term in the second expression for F — the normal derivative of the Laplacian of the displacement — represents the shearing force the plate exerts on its boundary, while the second term — the arc length (tangential) derivative of the twisting moment — represents the reaction of the boundary to the twisting of the plate.

On the other hand, the natural boundary conditions associated with the alternative Lagrangian (8.16) are much simpler:

$$\widetilde{F} = -\,y_s(u_{xxx}+u_{xyy}) + x_s(u_{xxy}+u_{yyy}) = -\,(\Delta u)_{\mathbf{n}}, \qquad \widetilde{G} = u_{xx} + u_{yy} = \Delta u. \tag{8.18}$$

However, they do not represent the proper physical forces exerted on the free boundary of an elastic plate, hence the choice of the more complicated physical Lagrangian (8.13). See [**36**, **39**] for discussion of the role played by the null Lagrangian term in the resolution of the polygon-circle plate paradox, [**4**], in which the energy functionals of simply supported polygonal plates converge to that of a clamped circular plate.

Finally, as noted in (8.15), second order null Lagrangians no longer depend linearly on the second order derivatives of u. According to [**2**], every second order null Lagrangian has the form

$$N = \alpha + \lambda u_{xx} + \mu u_{xy} + \nu u_{yy} + \sigma (u_{xx} u_{yy} - u_{xy}^2) \tag{8.19}$$

where the coefficients $\alpha, \ldots, \sigma$ depend on x, y, u, u_x, u_y, but are not arbitrary functions. Indeed, as always, N must be a divergence:

$$N = \mathrm{D}_x X + \mathrm{D}_y Y \qquad \text{where} \qquad \begin{aligned} X &= A - C\, u_{xy} + D\, u_{yy}, \\ Y &= B + C\, u_{xx} - D\, u_{xy}, \end{aligned} \tag{8.20}$$

with A, B, C, D arbitrary functions of x, y, u, u_x, u_y, and hence

$$\begin{aligned} \alpha &= \frac{\partial A}{\partial x} + u_x \frac{\partial A}{\partial u} + \frac{\partial B}{\partial y} + u_y \frac{\partial B}{\partial u}, \\ \sigma &= \frac{\partial C}{\partial u_y} + \frac{\partial D}{\partial u_x}, \end{aligned} \qquad \begin{aligned} \lambda &= \frac{\partial A}{\partial u_x} + \frac{\partial C}{\partial y} + u_y \frac{\partial C}{\partial u}, \\ \mu &= \frac{\partial A}{\partial u_y} + \frac{\partial B}{\partial u_x} - \frac{\partial C}{\partial x} - u_x \frac{\partial C}{\partial u} - \frac{\partial D}{\partial y} - u_y \frac{\partial D}{\partial u}, \\ \nu &= \frac{\partial B}{\partial u_y} + \frac{\partial D}{\partial x} + u_x \frac{\partial D}{\partial u}. \end{aligned} \tag{8.21}$$

However, there is redundancy in these expressions, because one can add in a null divergence (8.6) to the vector field (X, Y) without affecting the form of N. Indeed, if

$$\begin{aligned} X &= \mathrm{D}_y Z = \frac{\partial Z}{\partial y} + u_y \frac{\partial Z}{\partial u} + u_{xy} \frac{\partial Z}{\partial u_x} + u_{yy} \frac{\partial Z}{\partial u_y}, \\ Y &= -\mathrm{D}_x Z = -\frac{\partial Z}{\partial x} - u_x \frac{\partial Z}{\partial u} - u_{xx} \frac{\partial Z}{\partial u_x} - u_{xy} \frac{\partial Z}{\partial u_y}, \end{aligned} \tag{8.22}$$

then the corresponding N is identically zero. We can thus choose Z in order that either C or D in (8.20) is zero.

As usual, adding the null Lagrangian (8.19) to the original L does not alter the Euler–Lagrange equation, but does affect the natural boundary conditions. The resulting formulas can be found by replacing L by $L + N$ in (8.11), but are quite complicated, and, at least to me, not especially enlightening. We will therefore leave this final step to the motivated reader to try to ascertain to what extent the natural boundary conditions for a second order Lagrangian can be modified through this process.

9. Conclusions.

The flexibility afforded by the addition of a null Lagrangian serves to enlarge the possible variationally admissible boundary conditions for the Euler–Lagrange equations associated with a variational problem. However, not every boundary condition is variationally admissible; the classification of those that are forms the subject of this paper. However, this analysis becomes increasingly complicated as the order of the Lagrangian increases, and we have therefore focussed on the simplest and most important cases for applications. For variational problems arising in continuum mechanics, most of the variationally admissible boundary conditions have an evident physical interpretation, and can be applied in a range of practical problems.

Various further directions of research are indicated. It would be of much interest to extend this analysis to other types of elastic materials, including, for example, general rod, plate, and shell theories, and Cosserat (micropolar) continua, [**3**, **11**, **16**]. We restricted our attention to null Lagrangians that have the same order as the variational problem; higher order null Lagrangians could produce interesting higher order boundary conditions, as arising, for instance, in the modeling of plates supported on an elastic foundation, [**24**]. The impact of these methods on the design of finite element and other numerical approximation schemes for such boundary value problems, [**9**, **41**], is also worthy of development. Another interesting direction would be to extend this analysis to constrained variational problems, including incompressible and inextensible materials. Moreover, it would be worth investigating the effect of null Lagrangians and the associated boundary conditions on the standard variational tests for minima and maxima (second variation, conjugate points, etc.), [**19**, **20**, **26**, **35**]. Finally, we have exclusively studied the boundary value problems arising in statics; extending this analysis to initial-boundary value problems arising in dynamics would also be worth pursuing.

Acknowledgments: Thanks to Aritra Lahiri for asking about the implementation of Robin boundary conditions in the calculus of variations, which served as the initial inspiration to write this paper. Thanks also to Roger Fosdick, Eliot Fried, and Chi-Sing Man for references and insightful comments on an earlier version. I also thank the referees for their careful reading of the paper, corrections of misprints, additional references, and suggestions for improving the exposition.

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