Invariant Theory of Biforms

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Abstract

A biform is a function depending on two vector-valued variables which is a homogeneous polynomial function in each variable separately. In this paper, the foundations of classical invariant theory and the symbolic method as developed for forms are extended to the theory of biforms. Specific results for biquadratics in the plane, including an interesting identification of discriminants, are presented in detail. Applications of these results to the equivalence problem for quadratic Lagrangians and the determination of canonical elastic moduli are indicated.

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1. Introduction.

Classical invariant theory is concerned with the properties of homogeneous polynomials or *forms*, which are are not affected by linear changes of variables. Of particular interest is the explicit determination of particular functions depending on the coefficients of the underlying form, known as invariants and covariants, whose value does not change (except for a determinantal factor) under the prescribed changes of variables. Invariants and covariants will determine fundamental geometric properties of the form which do not depend on the particular coordinates it is written in. A basic constructive method developed in classical invariant theory for producing covariants is the powerful *symbolic method* or *umbral calculus* of Aronhold. Applications include solutions to the *equivalence problem*, which asks when are two different polynomials actually the same under a linear change of variables, and the *canonical form problem*, which seeks to determine a collection of simple "canonical forms" for polynomials of a given degree, with the property that every polynomial is equivalent to precisely one of the canonical forms on the list. References for classical invariant theory include the books by Grace and Young, [1], and Gurevich, [2], and the recent paper of Kung and Rota, [3].

The purpose of this note is to extend classical invariant theory to the study of homogeneous polynomial functions of two vector-valued variables. Specifically, a *biform* of *bidegree* (m,n) is a real-valued function $Q(\mathbf{x},\mathbf{u})$ depending on $\mathbf{x} \in \mathbb{R}^p$ and $\mathbf{u} \in \mathbb{R}^q$ which, for fixed $\mathbf{u}$, is a homogeneous polynomial function of degree m in $\mathbf{x}$, and, for fixed $\mathbf{x}$, is a homogeneous polynomial function of degree n in $\mathbf{u}$. The underlying group which provides the appropriate changes of variable is thus the Cartesian product $GL(p,\mathbb{R}) \times GL(q,\mathbb{R})$; two biforms are *equivalent* if they can be mapped to each other by a suitable group element. (Of course, one can also study complex-valued biforms with respect to complex changes of variables.) Turnbull, [7], treated the case of bilinear biforms, i.e. $m = n = 1$, but I am unaware of any attempt to treat the covariants of more general biforms. All of the general results in classical invariant theory as well as the powerful symbolic method of Aronhold are readily generalized to the case of biforms, and this is discussed in the first half of the paper. The second half of the paper is devoted to the invariant theory of binary biquadratics, i.e. $p = q = 2$, $m = n = 2$, which is of importance for planar elasticity.

The principal motivation for this enterprise comes from the study of quadratic variational problems

$$\mathcal{L}[\mathbf{u}] = \int \sum a^{\alpha\beta}_{IJ} \frac{\partial^n u^\alpha}{\partial x^I} \frac{\partial^n u^\beta}{\partial x^J} dx,$$

in particular those arising in linear elasticity. The results of this paper provide the key to the solution of the *equivalence problem*, which is to determine when two quadratic Lagrangians are equivalent under a linear change of variables $\mathbf{x} \to A\mathbf{x}$, $\mathbf{u} \to B\mathbf{u}$. Essentially, the Lagrangian equivalence problem is translated into the biform equivalence problem by replacing the Lagrangian by its *symbol*

$$Q(\mathbf{x},\mathbf{u}) = \sum a^{\alpha\beta}_{IJ}\ \mathbf{x}^I\,\mathbf{x}^J\,u^\alpha\,u^\beta,$$

which is a biform of bidegree $(2n,2)$. Indeed, [5] determines a complete list of canonical forms for first order quadratic Lagrangians in the plane, i.e. $n = 1$, $p = q = 2$, a result that has important applications in linear elasticity, cf. [6].

The invariant theory of higher order biforms, as well as biforms in more than two dimensions, is completely undeveloped. The most interesting case is that of a ternary biquadratic, i.e. $p = q = 3$, $m = n = 2$, as this directly relates to the study of quadratic variational problems in three-dimensional space, and, in particular, the study of linear three-dimensional elasticity.

Many of the complicated explicit computations of invariants and covariants were initially done on an Apollo workstation using the symbolic manipulation language SMP. Listings of the SMP programs that implement the calculations of the symbolic method, both for ordinary forms as well as biforms, are available from the author.

2. The Symbolic Method for Biforms.

Consider a general biform

$$Q(\mathbf{x},\mathbf{u}) = \sum_{I,J} \binom{m}{I} \binom{n}{J} a_{IJ}\ \mathbf{x}^I\,\mathbf{u}^J \qquad (1)$$

of bidegree (m,n), defined for $\mathbf{x} = (x^1, \ldots, x^p) \in \mathbb{R}^p$, $\mathbf{u} = (u^1, \ldots, u^q) \in \mathbb{R}^q$. Here the sum is over all multi-indices $I = (i_1, \ldots, i_p)$ of degree $m = i_1 + \ldots + i_p$, and $J = (j_1, \ldots, j_q)$

of degree $n = j_1 + \ldots + j_q$, and $\mathbf{x}^I = (x^1)^{i_1} \cdot \ldots \cdot (x^p)^{i_p}$. Under the linear change of variables

$$\mathbf{x} \to A \cdot \tilde{\mathbf{x}}, \qquad \mathbf{u} \to B \cdot \tilde{\mathbf{u}},$$

determined by the group element $\mathbf{A} = (A,B) \in GL(p,\mathbb{R}) \times GL(q,\mathbb{R})$, Q gets mapped into another biform of the same bidegree

$$\tilde{Q}(\tilde{\mathbf{x}}, \tilde{\mathbf{u}}) = Q(A \cdot \tilde{\mathbf{x}}, B \cdot \tilde{\mathbf{u}});$$

hence the coefficients a_{IJ} of Q get transformed into new coefficients $\tilde{a}_{IJ}$ of $\tilde{Q}$. The explicit formulas for the $\tilde{a}_{IJ}$, while easy to write down, are not particularly useful.

Definition 1. A *covariant* of *biweight* (g,h) of the biform Q is a polynomial function $J(\mathbf{a},\mathbf{x},\mathbf{u})$ depending on the coefficients $\mathbf{a} = (a_{IJ})$ and the independent and dependent variables $\mathbf{x}$, $\mathbf{u}$, which, up to a determinantal factor, does not change under the action of the group $GL(p,\mathbb{R}) \times GL(q,\mathbb{R})$:

$$J(\tilde{\mathbf{a}},\tilde{\mathbf{x}},\tilde{\mathbf{u}}) = (\det A)^g (\det B)^h \, J(\mathbf{a},\mathbf{x},\mathbf{u}), \qquad \mathbf{A} = (A,B) \in GL(p,\mathbb{R}) \times GL(q,\mathbb{R}).$$

An *invariant* $I(\mathbf{a})$ is just a covariant which does not depend on the variables $\mathbf{x}$ or $\mathbf{u}$.

One can also define *contravariants* and *mixed concomitants* of general biforms, cf. [2]; however, for simplicity we will concentrate just on covariants and invariants here, and avoid the introduction of additional contragredient variables.

The simplest biform $Q(\mathbf{x},\mathbf{u})$ of bidegree (m,n) is the "bipower"

$$(\alpha\, \mathbf{x})^m (\varphi\, \mathbf{u})^n \tag{2}$$

of two linear monomials

$$(\alpha\, \mathbf{x}) = (\alpha^1 x^1 + \ldots + \alpha^p x^p) \qquad \text{and} \qquad (\varphi\, \mathbf{u}) = (\varphi^1 u^1 + \ldots + \varphi^q u^q).$$

The coefficients of (2), which are the power products $a_{IJ} = \alpha^I \varphi^J$, will serve to determine the symbolic expressions for the covariants of more general biforms of the given bidegree. Mimicing the symbolic method for ordinary forms, we are lead to introduce an *alphabet*

$$\mathcal{A} = \{(\alpha_1,\varphi_1), (\alpha_2,\varphi_2), (\alpha_3,\varphi_3), \ldots\},$$

whose elements are *ordered pairs of umbral letters*: $\alpha_i = (\alpha_i^1,\ldots,\alpha_i^p)$ and $\varphi_i = (\varphi_i^1,\ldots,\varphi_i^q)$; the letters α_i and φ_i in any pair are said to be *linked*. The coefficient a_{IJ} of Q will thus have the symbolic expression as a product of powers

$$a_{IJ} \sim \alpha^I \varphi^J,$$

where (α,φ) are any of the linked pairs of umbral letters in $\mathcal{A}$. Thus, a symbolic representative of the monomial

$$\Big(\prod_\nu a_{I_\nu J_\nu}\Big)\, \mathbf{x}^K \mathbf{u}^L$$

will be

$$\Big(\prod_\nu \alpha_\nu^{I_\nu} \varphi_\nu^{J_\nu}\Big)\, \mathbf{x}^K \mathbf{u}^L .$$

The *umbral space* $\mathcal{U}$ is the space of polynomials $P(\alpha_1,\ldots,\alpha_k,\varphi_1,\ldots,\varphi_k,\mathbf{x},\mathbf{u})$, depending on the symbolic letter pairs $(\alpha_1,\varphi_1), \ldots, (\alpha_k,\varphi_k)$, which are homogeneous of degree m in the α's and of degree n in the φ's. The symbolic representative of a polynomial $J(\mathbf{a},\mathbf{x},\mathbf{u})$ is not unique, as we can always interchange the symbolic letter pairs. (If we replace α_i by α_j, then we must also replace φ_i by φ_j.) A symbolic polynomial is *symmetric* if it is unchanged by such interchanges.

Proposition 2. Each homogeneous polynomial $J(\mathbf{a},\mathbf{x},\mathbf{u})$ of degree k in the coefficients $\mathbf{a} = (a_{IJ})$ of a biform Q has a unique symmetric symbolic representative in $\mathcal{U}$ depending on the first k letter pairs $(\alpha_1,\varphi_1), \ldots, (\alpha_k,\varphi_k)$ in the alphabet $\mathcal{A}$.

In the invariant theory of ordinary forms, there are two types of symbolic factors or brackets, cf. [2; §18.2]. (As above, we are leaving aside contravariants, which introduce yet a third type of bracket factor.) For biforms, there are correspondingly four kinds of brackets: two kinds of determinantal ones only involving only umbral letters:

$$[\alpha_1 \ldots \alpha_p] = \det(\alpha_j^i) \quad \text{and} \quad [\varphi_1 \ldots \varphi_q] = \det(\varphi_j^i),$$

and two linear ones involving the variables $\mathbf{x}$ and $\mathbf{u}$:

$$(\alpha\, \mathbf{x}) = \sum \alpha^i x^i \qquad \text{and} \qquad (\varphi\, \mathbf{u}) = \sum \varphi^i u^i .$$

Note that we are not allowed to mix the two types of umbral letters in either type of bracket. A bracket polynomial, i.e. a polynomial in the four bracket expressions, is said to be *homogeneous* if the same number of bracket factors of each of the four kinds appear in each constituent monomial. The *biweight* (g,h) of a homogeneous bracket polynomial is the number of each kind of determinantal bracket factor in any constituent monomial. An easy extension of the proof of the First Fundamental Theorem for ordinary forms, cf. [2; §17], immediately leads to a version for biforms:

Theorem 3. A homogeneous polynomial $J(\mathbf{a},\mathbf{x},\mathbf{u})$ is a covariant of the biform $Q(\mathbf{x},\mathbf{u})$ of biweight (g,h) if and only if its symmetric umbral representative is a homogeneous bracket polynomial of biweight (g,h). Conversely, any homogeneous bracket polynomial of biweight (g,h) is an umbral representative of a covariant of biweight (g,h).

Finally, we note that there is a straightforward extension of the Basis Theorem of Hilbert, cf. [2; theorem 21.3], [3; theorem 6.1], that shows that the space of covariants of a biform of a given bidegree is finitely generated.

Theorem 4. Let Q be a biform of a given bidegree. Then there is a finite set of covariants $C_1, \ldots, C_N$ with the property that any other covariant can be expressed as a polynomial $P(C_1, \ldots, C_N)$ in these basic covariants.

It should be relatively easy to adapt the procedure of Gordan, cf. [1], [2], to actually compute the generating set of covariants of biforms of low degree. However, even for the simplest nontrivial case of a binary biquadratic, this appears to be very complicated, and remains an open problem. However, since there is a complete classification of the canonical forms for binary biquadratics in [5] this question is of less direct interest for the applications to variational problems and elasticity.

3. Elementary Covariants of Binary Biquadratics.

We now illustrate the general considerations of the previous section with the first nontrivial case of a biform, i.e. a binary biquadratic $(p = q = 2,\ m = n = 2)$, and construct some of the basic covariants using the symbolic method. The general binary biquadratic is

$$Q(\mathbf{x},\mathbf{u}) = a_{11}^{11}x^2u^2 + 2a_{12}^{11}xyu^2 + a_{22}^{11}y^2u^2 + 2a_{11}^{12}x^2uv + 4a_{12}^{12}xyuv + \\ + 2a_{22}^{12}y^2uv + a_{11}^{22}x^2v^2 + 2a_{12}^{22}xyv^2 + a_{22}^{22}y^2v^2,$$

where $\mathbf{x} = (x^1,x^2) = (x,y)$ and $\mathbf{u} = (u^1,u^2) = (u,v)$ are both in $\mathbb{R}^2$. Note that the symbolic forms of the coefficients of Q are

$$a^{ij}_{k\ell} \sim \alpha^i \alpha^j \varphi^k \varphi^\ell ,$$

where $(\alpha,\varphi) = ((\alpha^1,\alpha^2),(\varphi^1,\varphi^2))$ are any of the umbral letter pairs in our alphabet. Consider first the covariant with symbolic bracket expression

$$[\alpha_1\ \alpha_2]^2\ (\varphi_1\ \mathbf{u})^2\ (\varphi_2\ \mathbf{u})^2, \tag{2}$$

where (α_1,φ_1), (α_2,φ_2) are two distinct umbral letter pairs. It represents a polynomial which is a quadratic function of the coefficients $a^{ij}_{k\ell}$, and a quartic polynomial in $\mathbf{u}$. A short computation shows that it is the same as the discriminant

$$\Delta_{\mathbf{x}}(\mathbf{u}) = \frac{1}{4}(Q_{xx} \cdot Q_{yy} - Q_{xy}^2) = \frac{1}{8}\ \frac{\partial^2(Q,Q)}{\partial(x,y)^2}$$

of the quadratic polynomial Q with respect to the variables $\mathbf{x}$, which is a covariant of biweight (2,0). (The subscripts on Q indicate partial derivatives, and the second expression is the hyperjacobian notation introduced in [4], which generalizes the classical transvectant notation.) Similarly, the covariant with symbolic bracket expression

$$[\varphi_1\ \varphi_2]^2\ (\alpha_1\ \mathbf{x})^2\ (\alpha_2\ \mathbf{x})^2 \tag{3}$$

represents the $\mathbf{u}$-discriminant

$$\Delta_{\mathbf{u}}(\mathbf{x}) = \frac{1}{4}(Q_{uu} \cdot Q_{vv} - Q_{uv}^2) = \frac{1}{8}\ \frac{\partial^2(Q,Q)}{\partial(u,v)^2},$$

which has biweight (0,2). These discriminants have the usual properties of a discriminant of an ordinary quadratic polynomial; for instance, $\Delta_{\mathbf{x}}(\mathbf{u}_0) = 0$ implies that $Q(\mathbf{x},\mathbf{u}_0)$ is a perfect square, etc. The symbolic form

$$[\alpha_1\ \alpha_2]\ [\varphi_1\ \varphi_2]\ (\alpha_1\ \mathbf{x})\ (\alpha_2\ \mathbf{x})\ (\varphi_1\ \mathbf{u})\ (\varphi_2\ \mathbf{u}),$$

represents a mixed biquadratic covariant of biweight (2,2), which has the explicit formula

$$C_2 = \frac{1}{4}(Q_{xu}\,Q_{yv} - Q_{xv}\,Q_{yu}) = \frac{1}{8}\,\frac{\partial^2(Q,Q)}{\partial(u,v)\,\partial(x,y)}.$$

The simplest invariant of Q has bracket expression

$$[\alpha_1\,\alpha_2]^2\,[\varphi_1\,\varphi_2]^2. \tag{4}$$

In terms of the coefficients of Q, it has the explicit formula

$$I_2 = 2a_{11}^{11}a_{22}^{22} - 4a_{12}^{11}a_{12}^{22} + 2a_{22}^{11}a_{11}^{22} - 4a_{11}^{12}a_{22}^{12} + 4(a_{12}^{12})^2,$$

and has biweight $(2,2)$. There is a single cubic invariant, and it has bracket expression

$$[\alpha_1\,\alpha_2]\,[\alpha_1\,\alpha_3]\,[\alpha_2\,\alpha_3]\,[\varphi_1\,\varphi_2]\,[\varphi_1\,\varphi_3]\,[\varphi_2\,\varphi_3]\,. \tag{5}$$

It is of biweight $(3,3)$, and the explicit formula is

$$I_3 = a_{11}^{11}a_{12}^{12}a_{22}^{22} - a_{11}^{11}a_{22}^{12}a_{12}^{22} - a_{12}^{11}a_{11}^{12}a_{22}^{22} + a_{12}^{11}a_{22}^{12}a_{11}^{22} + a_{22}^{11}a_{11}^{12}a_{12}^{22} - a_{22}^{11}a_{12}^{12}a_{11}^{22},$$

where we have dropped a common factor of 12. Still more complicated is the fourth order invariant with symbolic expression

$$[\alpha_1\,\alpha_2]^2\,[\alpha_3\,\alpha_4]^2\,[\varphi_1\,\varphi_3]^2\,[\varphi_2\,\varphi_4]^2\,. \tag{6}$$

and biweight $(4,4)$. A fairly lengthy computation shows that, up to a factor of 2, it has the explicit expression

$$I_4 = a_{11}^{11}a_{12}^{12}a_{22}^{22} - a_{11}^{11}a_{22}^{12}a_{12}^{22} - a_{12}^{11}a_{11}^{12}a_{22}^{22} + a_{12}^{11}a_{22}^{12}a_{11}^{22} + a_{22}^{11}a_{11}^{12}a_{12}^{22} - a_{22}^{11}a_{12}^{12}a_{11}^{22}.$$

4. Quartic Invariants and Biquadratics.

We can indicate some further developments in the the invariant theory of the binary biquadratic by applying the technique of *composition* of covariants. If Q is any (bi)form, and J is a polynomial covariant for Q, then we can regard J itself as a (bi)form, whose coefficients are certain polynomial combinations of the coefficients of Q. Any covariant K, which depends directly on the coefficients of J, is then, by composition, a covariant of Q, denoted by $K \circ J$.

We recall that for a binary quartic

$$f(\mathbf{x}) = a\cdot x^4 + 4b\cdot x^3y + 6c\cdot x^2y^2 + 4d\cdot xy^3 + e\cdot y^4,$$

the Hessian

$$H = (f, f)^{(2)} = \frac{1}{144}\frac{\partial^2(\Delta_u, \Delta_u)}{\partial(x,y)^2},$$

is a covariant of weight 2, and the two important invariants

$$i = (f, f)^{(4)} = \frac{1}{576}\frac{\partial^4(f, f)}{\partial(x, y)^4} = 2a\cdot e - 8b\cdot d + 6c^2,$$

and

$$j = (f,H)^{(4)} = \frac{1}{576}\frac{\partial^4(f, H)}{\partial(x, y)^4} = 6 \det \begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix},$$

have weights 4 and 6 repectively, cf. [1; §89]. (Here, $(f,g)^{(k)}$ denotes the classical k^{th} transvectant of f and g.)

Since the discriminant $\Delta_{\mathbf{u}}(\mathbf{x})$ of a binary biquadratic form is a binary quartic form in the variables $\mathbf{x} = (x,y)$, the corresponding covariants are, by composition, covariants of the original biquadratic Q. Thus we have the Hessian of the **u**-discriminant

$$H_{\mathbf{u}}(\mathbf{x}) = H \circ \Delta_{\mathbf{u}} = \frac{1}{144}\frac{\partial^2(\Delta_u, \Delta_u)}{\partial(x,y)^2},$$

which is again a binary quartic in $\mathbf{x}$, and a covariant of biweight (2,4), as well as the two invariants

$$i_{\mathbf{u}} = i \circ \Delta_{\mathbf{u}} = \frac{1}{576}\frac{\partial^4(\Delta_u, \Delta_u)}{\partial(x, y)^4} \quad \text{and} \quad j_{\mathbf{u}} = j \circ \Delta_{\mathbf{u}} = \frac{1}{576}\frac{\partial^4(\Delta_u, H_u)}{\partial(x, y)^4},$$

which have biweights (4,4) and (6,6) respectively. Similarly, we can compute covariants of the **x**-discriminant, leading to the alternative Hessian

$$H_{\mathbf{x}}(\mathbf{u}) = H \circ \Delta_{\mathbf{x}} = \frac{1}{144}\frac{\partial^2(\Delta_{\mathbf{x}}, \Delta_{\mathbf{x}})}{\partial(u, v)^2},$$

of biweight (4,2), and the two invariants

$$i_{\mathbf{x}} = i \circ \Delta_{\mathbf{x}} = \frac{1}{576}\frac{\partial^4(\Delta_{\mathbf{x}}, \Delta_{\mathbf{x}})}{\partial(u, v)^4} \quad \text{and} \quad j_{\mathbf{x}} = j \circ \Delta_{\mathbf{x}} = \frac{1}{576}\frac{\partial^4(\Delta_{\mathbf{x}}, H_{\mathbf{x}})}{\partial(u,v)^4},$$

of biweights (4,4) and (6,6) respectively. The two Hessians are easily seen to be different quartic polynomials (even if one identifies the variables $\mathbf{x}$ and $\mathbf{u}$). Remarkably, the i and j invariants of the two discriminants are the *same* invariants of the original biquadratic polynomial Q. This is a key result for the canonical form problem, treated in [5].

Theorem 5. Let Q be a binary biquadratic form. Let $\Delta_{\mathbf{x}}(\mathbf{u})$ and $\Delta_{\mathbf{u}}(\mathbf{x})$ be the two discriminants, which are quartic forms in $\mathbf{u}$ and $\mathbf{x}$ respectively. Then the invariants of these two quartic forms are the same:

$$i_{\mathbf{x}} = i \circ \Delta_{\mathbf{x}} = i_{\mathbf{u}} = i \circ \Delta_{\mathbf{u}}, \qquad j_{\mathbf{x}} = j \circ \Delta_{\mathbf{x}} = j_{\mathbf{u}} = j \circ \Delta_{\mathbf{u}}.$$

Corollary 6. The cross ratios of the roots of the two discriminants $\Delta_{\mathbf{x}}(\mathbf{u})$ and $\Delta_{\mathbf{u}}(\mathbf{x})$ of a biquadratic are the same.

This follows immediately from the result [1; page 205] that the ratio i^3/j^2 essentially determines the cross ratio of the four roots of the quartic.

Corollary 7. The two discriminants $\Delta_{\mathbf{x}}(\mathbf{u})$ and $\Delta_{\mathbf{u}}(\mathbf{x})$ of a binary biquadratic either both have all simple roots, or both have repeated roots.

Indeed, the discriminant of a quartic, whose vanishing indicates the presence of repeated roots, is given by $\Delta = \frac{1}{27}(i^3 - 6j^2)$, cf. [1; page 198]. Therefore theorem 5 implies that the two quartics $\Delta_{\mathbf{x}}(\mathbf{u})$ and $\Delta_{\mathbf{u}}(\mathbf{x})$ have the same discriminant, and the corollary follows immediately. Note that it is *not* asserted that $\Delta_{\mathbf{x}}(\mathbf{u})$ and $\Delta_{\mathbf{u}}(\mathbf{x})$ have identical root multiplicities! For example, the biquadratic form $Q = x^2u^2 + xyv^2$ has $\mathbf{u}$-discriminant $\Delta_{\mathbf{u}} = -4x^3y$, which has a triple root at 0 and a simple root at ∞, whereas the $\mathbf{x}$-discriminant $\Delta_{\mathbf{x}} = v^4$ has a quadruple root at ∞. The complete description of the possible root configurations of a biquadratic form follows from the general classification theorem presented in [5].

There are at least three possible ways to prove theorem 5. One is to explicitly write out the invariants i_u, j_u, i_x and j_x, and compare terms. This was the original version of the proof, and was effected on an Apollo computer using the symbolic manipulation language SMP. The explicit formula for j_u runs to two entire printed pages! A second approach is to write out the formulas in terms of the partial derivatives of the biquadratic form Q; this looks feasible, but I have not attempted to do this. The final approach is to work entirely symbolically, using the transvectant calculus of Gordan. The goal is to obtain the complete symbolic bracket expressions for the two pairs of invariants and prove them the same. Since the two pairs of invariants are constructed in completely analogous fashion, exchanging the roles of the **x** and **u** variables, we are lead to consider the following concept.

Definition 8. For a biform $Q(\mathbf{x},\mathbf{u})$, in which **x** and **u** both lie in $\mathbb{R}^p$, define the *interchange involution* which maps Q to the new biform $\hat{Q}(\mathbf{x},\mathbf{u}) = Q(\mathbf{u},\mathbf{x})$.

The interchange induces an involution on the coefficients of Q, mapping $a^{ij}_{k\ell}$ to $a^{k\ell}_{ij}$, and hence on the covariants themselves. For instance, the interchange involution clearly takes the covariants Δ_u, H_u, i_u, j_u to Δ_x, H_x, i_x, j_x, respectively. Moreover, the interchange involution acts on the symbolic expressions by interchanging the various umbral letter pairs: $\alpha_i \leftrightarrow \varphi_i$. We will call a covariant *interchange symmetric* if it is unchanged by this involution; easy examples are the invariants I_2, I_3, I_4 constructed in section 3, as well as the covariant C_2. Theorem 5 will be proven if we can show that, in addition, the invariants i_u and j_u are interchange-symmetric. More to the point, we need to show that if we have an symbolic representative for i_u, or for j_u, and we interchange all the umbral letter pairs $\alpha_i \leftrightarrow \varphi_i$, then we obtain another symbolic representative for the same invariant i_u or j_u.

We begin with the simpler of the two invariants i_u, which is the fourth transvectant of the discriminant Δ_u with itself. Thus we must transvect the symbolic expression (2) four times with itself using the umbral letters α_1, α_2. (See [1] for the mechanics of transvection.) We find that the resulting transvectant is a linear combination (the precise numerical coefficients are not important) of two different symbolic bracket expressions. The first has the form

$$[\alpha_1\ \alpha_3]^2\ [\alpha_2\ \alpha_4]^2\ [\varphi_1\ \varphi_2]^2\ [\varphi_3\ \varphi_4]^2 . \tag{7}$$

where (α_i, φ_i), $i = 1,2,3,4$, are the linked letter pairs. This one is just the symbolic form of the invariant I_4, cf. (6), and so is interchange symmetric. We can also see this symbolically: if we interchange each α_i and φ_i in (7), and then use the fact that the letter pairs themselves are all interchangeable, so we can excahnge the pairs (α_2, φ_2) and (α_3, φ_3), we revert to the same symbolic expression (7), which proves the symmetry.

The second contribution to the transvectant for i_u is a multiple of

$$[\alpha_1\, \alpha_3]\, [\alpha_1\, \alpha_4]\, [\alpha_2\, \alpha_3]\, [\alpha_2\, \alpha_4]\, [\varphi_1\, \varphi_2]^2\, [\varphi_3\, \varphi_4]^2 \tag{8}$$

which is not so clearly symmetric under the interchange involution. However, we now appeal to the fundamental syzygy among the bracket factors:

$$[\alpha\, \beta]\, [\gamma\, \delta] = [\alpha\, \gamma]\, [\beta\, \delta] + [\alpha\, \delta]\, [\gamma\, \beta], \tag{9}$$

cf. [3; lemma 3.1]. (Here $\alpha, \beta, \gamma, \delta$ are umbral letters of the same kind.) If we use this relation on the second and third bracket factors in (8), i.e. set $\alpha = \alpha_1$, $\beta = \alpha_3$, $\gamma = \alpha_2$, $\delta = \alpha_4$, we find that (8) is equal to a sum of the bracket polynomials

$$[\alpha_1\, \alpha_3]^2\, [\alpha_2\, \alpha_4]^2\, [\varphi_1\, \varphi_2]^2\, [\varphi_3\, \varphi_4]^2$$

which is just the symbolic expression (6) for the invariant I_4, and

$$A = [\alpha_1\, \alpha_3]\, [\alpha_1\, \alpha_2]\, [\alpha_4\, \alpha_3]\, [\alpha_2\, \alpha_4]\, [\varphi_1\, \varphi_2]^2\, [\varphi_3\, \varphi_4]^2 . \tag{10}$$

Using the syzygy once again, this time on the first and fourth bracket factors of (10), we find that $A = B + C$, where

$$B = -[\alpha_1\, \alpha_2]^2\, [\alpha_3\, \alpha_4]^2\, [\varphi_1\, \varphi_2]^2\, [\varphi_3\, \varphi_4]^2,$$

$$C = [\alpha_1\, \alpha_4]\, [\alpha_1\, \alpha_2]\, [\alpha_4\, \alpha_3]\, [\alpha_2\, \alpha_3]\, [\varphi_1\, \varphi_2]^2\, [\varphi_3\, \varphi_4]^2 .$$

If we exchange (α_3,φ_3) with (α_4,φ_4), we see that $C = -A$, hence $A = -\frac{1}{2}B$. However, B is just the symbolic expression for the square of the invariant I_2, cf. (4), and is obviously interchange-symmetric. Thus (8) is interchange-symmetric. Since both summands (7), (8), in the full invariant i_u have been shown to be interchange-symmetric, i_u itself must be interchange-symmetric, and we conclude that $i_u = i_x$, as desired.

Turning to the invariant j_u, we see that we first need the symbolic form for the Hessian H_u which is the second transvectant of Δ_u with itself. This will be a linear combination of the two bracket expressions

$$[\alpha_2\ \alpha_3]^2\ [\varphi_1\ \varphi_2]^2\ [\varphi_3\ \varphi_4]^2\ (\alpha_1\ x)^2\ (\alpha_4\ x)^2$$

and

$$[\alpha_1\ \alpha_3]\ [\alpha_2\ \alpha_4]\ [\varphi_1\ \varphi_2]^2\ [\varphi_3\ \varphi_4]^2\ (\alpha_1\ x)\ (\alpha_2\ x)\ (\alpha_3\ x)\ (\alpha_4\ x)$$

These in turn must be transvected four times with Δ_u. A long computation shows that there are five distinct bracket expressions entering into the final expression for j_u :

1) $$[\alpha_2\ \alpha_3]^2\ [\alpha_4\ \alpha_5]^2\ [\alpha_6\ \alpha_1]^2\ [\varphi_1\ \varphi_2]^2\ [\varphi_3\ \varphi_4]^2\ [\varphi_5\ \varphi_6]^2,$$

which is clearly interchange-symmetric.

2) $$[\alpha_1\ \alpha_3]\ [\alpha_2\ \alpha_3]\ [\alpha_6\ \alpha_1]\ [\alpha_6\ \alpha_2]\ [\alpha_4\ \alpha_5]^2[\varphi_1\ \varphi_2]^2\ [\varphi_3\ \varphi_4]^2\ [\varphi_5\ \varphi_6]^2.$$

Here we use the basic syzygy (9) on the first and fourth factors to rewrite this as a sum of a bracket expression of type 1 and the expression

2a) $$[\alpha_1\ \alpha_2]\ [\alpha_2\ \alpha_3]\ [\alpha_3\ \alpha_6]\ [\alpha_6\ \alpha_1]\ [\alpha_4\ \alpha_5]^2[\varphi_1\ \varphi_2]^2\ [\varphi_3\ \varphi_4]^2\ [\varphi_5\ \varphi_6]^2$$

A second use of the syzygy, this time on the second and fourth factors shows that this equals $-\frac{1}{2}$ times the reducible bracket expression

$$[\alpha_1\ \alpha_2]^2\ [\alpha_3\ \alpha_6]^2\ [\alpha_6\ \alpha_1]^2\ [\varphi_1\ \varphi_2]^2\ [\varphi_3\ \varphi_4]^2\ [\varphi_5\ \varphi_6]^2,$$

which is clearly interchange-symmetric.

3) $$[\alpha_1\ \alpha_3]\ [\alpha_2\ \alpha_3]\ [\alpha_2\ \alpha_5]\ [\alpha_4\ \alpha_5]\ [\alpha_4\ \alpha_6]\ [\alpha_6\ \alpha_1]\ [\varphi_1\ \varphi_2]^2\ [\varphi_3\ \varphi_4]^2\ [\varphi_5\ \varphi_6]^2$$

Performing the syzygy on the second and sixth factors, we get a sum of an expression of type 2) and the expression

3a) $[\alpha_1\ \alpha_2]\,[\alpha_1\ \alpha_3]\,[\alpha_2\ \alpha_5]\,[\alpha_3\ \alpha_6]\,[\alpha_4\ \alpha_5]\,[\alpha_4\ \alpha_6]\,[\varphi_1\ \varphi_2]^2\,[\varphi_3\ \varphi_4]^2\,[\varphi_5\ \varphi_6]^2$

A second syzygy on the second and third factors leads to a sum of the same expression, with a minus sign, and the reducible bracket expression

$$[\alpha_1\ \alpha_2]^2\,[\alpha_3\ \alpha_5]\,[\alpha_3\ \alpha_6]\,[\alpha_4\ \alpha_5]\,[\alpha_4\ \alpha_6]\,[\varphi_1\ \varphi_2]^2\,[\varphi_3\ \varphi_4]^2\,[\varphi_5\ \varphi_6]^2,$$

which is the symbolic representative for the product of the invariants I_2 and (8), which was proved to be interchange-symmetric earlier. Thus 3a) itself is interchange-symmetric.

4) $[\alpha_1\ \alpha_3]\,[\alpha_2\ \alpha_4]\,[\alpha_2\ \alpha_5]\,[\alpha_3\ \alpha_6]\,[\alpha_4\ \alpha_5]\,[\alpha_6\ \alpha_1]\,[\varphi_1\ \varphi_2]^2\,[\varphi_3\ \varphi_4]^2\,[\varphi_5\ \varphi_6]^2$

A syzygy on the fifth and sixth factors shows that this equals a sum of expressions of type 3) and of type 3a), and so is interchange-symmetric.

5) $[\alpha_1\ \alpha_4]\,[\alpha_2\ \alpha_4]\,[\alpha_2\ \alpha_5]\,[\alpha_3\ \alpha_6]\,[\alpha_4\ \alpha_5]\,[\alpha_6\ \alpha_1][\varphi_1\ \varphi_2]^2\,[\varphi_3\ \varphi_4]^2\,[\varphi_5\ \varphi_6]^2$

Performing a syzygy on the first and third factors shows that it equals a sum of types 2a) and 3), and so is interchange-symmetric. Thus the entire invariant $j_{\mathbf{u}}$ is interchange-symmetric, which completes the proof of the theorem.

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