Pseudo-Stabilization of Prolonged Group Actions
I. The Order Zero Case

Peter J. Olver[†]
School of Mathematics
University of Minnesota
Minneapolis, MN 55455
U.S.A.
olver@umn.edu
http://www.math.umn.edu/~olver

Abstract. It is shown that every point transformation group whose prolonged orbit dimensions pseudo-stabilize at order 0 is equivalent, under a change of variables, to the elementary similarity group consisting of translations and dilatations.

In the study of differential invariants and symmetry groups of differential equations, the geometry of the prolonged group actions plays an essential role. The purpose of this paper is to further elucidate one aspect of this geometry, namely to provide a complete characterization of groups which "pseudo-stabilize" at the lowest order. This result will help explain an observation in [**5**], which was based on Lie's classification of transformation groups in the plane, [**1**], [**3**], that, in the two-dimensional case, there is essentially only one group which has this anomalous behavior. In this paper, this observation will

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be extended to groups in an arbitrary number of independent and dependent variables. The reader should be familiar with the fundamentals of the Lie theory of prolongation of transformation groups on jet bundles, as presented, for instance in my books [**4**], [**5**]. I shall employ the same basic notation here.

Let G be a connected r-dimensional local transformation group acting on an open subset $M \subset X \times U \simeq \mathbb{R}^p \times \mathbb{R}^q$ of the space (bundle) coordinatized by p independent and q dependent variables. The space of infinitesimal generators of G — its Lie algebra — will be denoted by $\mathfrak{g}$. The action of G on the sections $u = f(x)$ of M induces an action of G on the associated n^{th} order jet bundle $\mathrm{J}^n = \mathrm{J}^n M$, known as the n^{th} prolonged transformation group, and denoted by $G^{(n)}$. The generic or maximal dimension of the orbits of $G^{(n)}$ is known as the *prolonged orbit dimension* (of order n) and denoted by s_n. By definition, $G^{(n)}$ acts semi-regularly (meaning all orbits have the same dimension) on the open subset $V^n \subset \mathrm{J}^n$ consisting of all points contained in orbits of maximal dimension. The orbit dimensions satisfy the elementary inequalities, [**5**],

$$s_{n-1} \leq s_n \leq s_{n-1} + q \binom{p+n-1}{n}. \tag{1}$$

In particular, they form a nondecreasing sequence $s_0 \leq s_1 \leq s_2 \leq \cdots \leq r$, that is bounded by the dimension of G, and hence eventually stabilizes: $s_m = s_\infty$ for all m sufficiently large. We will call s_∞ the stable orbit dimension, and the minimal order n for which $s_n = s_\infty$ the *order of stabilization* of the group. The following fundamental result is due to Ovsiannikov, [**6**].

Theorem 1. *The stable orbit dimension of a transformation group G is equal to the dimension of G if and only if G acts locally effectively.*

Here "locally effectively" means that the only group element in some neighborhood of the identity which acts trivially on M is the identity itself. If G does not act effectively, we can replace it by the quotient group G/G_M, where $G_M = \{\, g \mid g \cdot x = x \text{ for all } x \in M \,\}$ is the global isotropy subgroup, which *does* act effectively on M in essentially the same way as G itself. Consequently, there is no loss in generality in assuming that all our group actions are (locally) effective, and hence $s_\infty = r = \dim G$ in all cases.

Perhaps surprisingly, it is not necessarily true that if $s_k = s_{k+1}$ then the orbit dimension stabilizes at order k. This introduces the possibility that the orbit dimensions "pseudo-stabilize", in the following sense:

Definition 2. A transformation group G acting on M is said to *pseudo-stabilize* at *order* k if its prolonged orbit dimensions satisfy $s_k = s_{k+1} < s_{k+2}$.

A key result, proved in [**5**; Theorem 5.37], which generalizes a theorem of Ovsiannikov, [**6**; p. 313], is that there can be at most *one* such pseudo-stabilization.

Theorem 3. *Suppose that the maximal orbit dimensions of the prolonged group actions satisfy $s_k = s_{k+1}$ and, also, $s_n = s_{n+1}$ for some $n > k$. Then $s_m = s_n$ for all $m \geq n$.*

Example 4. Let $r \geq 3$. Let $x, u \in \mathbb{R}$. Consider the $r = k+3$-dimensional group

$$G_k : \qquad (x,u) \longmapsto (\lambda x + a, \lambda^{k+1} u + P_k(x)), \tag{2}$$

in which $\lambda > 0$ and a are constants, and $P_k(x)$ an arbitrary polynomial of degree $\leq k$. The infinitesimal generators are the vector fields

$$\partial_x, \quad \partial_u, \quad x\partial_u, \quad \ldots, \quad x^k\partial_u, \quad x\partial_x + (k+1)u\partial_u.$$

An easy computation, cf. [**5**; Example 5.8], demonstrates that the prolonged maximal orbit dimensions are given by $s_0 = 2$, $s_1 = 3, \ldots, s_k = s_{k+1} = k+2$, $s_{k+2} = s_k = \cdots = k+3$. Thus, the orbit dimensions pseudo-stabilize at order k, and finally stabilize at order $k+2$. In the simplest case $k = 0$, where the group G_0 reduces to the elementary similarity group $(x,u) \to (\lambda x + a, \lambda u + b)$, containing translations and a one-parameter group of elementary dilatations.

Remarkably, in the planar (i.e., one independent variable and one dependent variable) case, these are the only examples of transformation groups that pseudo-stabilize. Two transformation group actions are called *equivalent* if there is a local change of variables or point transformation† $(x,u) \mapsto (\bar{x}, \bar{u}) = \Phi(x,u)$ mapping one to the other, so that $g \cdot (\bar{x}, \bar{u}) = \Phi(g \cdot (x,u))$. In [**1**], [**3**], Lie classified all possible transformation groups without fixed points acting on a two-dimensional space, up to local equivalence. Lie's classification was applied by the author in [**5**; Theorem 5.24] to prove the following result.

Theorem 5. *If G is a connected r-dimensional transformation group acting on an open $M \subset \mathbb{R}^2$ with no fixed points, whose prolonged orbit dimensions pseudo-stabilize at order k, then G is locally equivalent to the transformation group G_{k+3} given in* (2) *with $r = k+3$.*

Thus, up to changes of variables, there is precisely one planar transformation group that pseudo-stabilizes at a given order! In higher dimensions, there is no corresponding classification of transformation groups; although Lie did claim, [**2**], to have completed the three-dimensional classification, he never published the details. Therefore, attempts to generalize Theorem 5 must rely on an alternative approach. In fact, I do not know of any examples of transformation groups whose orbits pseudo-stabilize beyond fairly elementary multi-dimensional generalizations of the groups in Example 4. The main result of this paper is to prove that, in the order 0 case, this is, in fact, the case — there is, up to equivalence, only one example of a transformation group whose prolonged orbit dimensions pseudo-stabilize at order 0. In other words, the first prolongation $G^{(1)}$ has the same maximal orbit dimensions as G itself does but these are strictly less than the dimension of G itself, $s_0 = s_1 < r = \dim G$. (As always, we assume G acts locally effectively.)

Theorem 6. *Let G be a connected r-dimensional group of point transformations acting locally effectively and semi-regularly on an open subset $M \subset X \times U \simeq \mathbb{R}^p \times \mathbb{R}^q$.*

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† In this paper we are restricting our attention to point transformations, since pseudo-stabilization at order 0 does not make sense for contact transformation groups.

If the prolonged orbit dimensions pseudo-stabilize at order 0*, then G has dimension $r = p+q+1$, and, moreover, there exists a local change of variables mapping G to the elementary similarity group*

$$G_0 : \qquad (x,u) \longmapsto (\lambda x + a, \lambda u + b), \qquad a \in \mathbb{R}^p, \quad b \in \mathbb{R}^q, \quad \lambda \in \mathbb{R}. \tag{3}$$

Remark: In particular, an intransitive transformation group cannot pseudo-stabilize at order 0.

Proof: Note first that the similarity group G_0 has trivial action on the first order derivative coordinates $u_i^\alpha = \partial u^\alpha / \partial x^i$, but scales the higher derivatives via powers of λ, and hence the fact that the group G_0 pseudo-stabilizes is elementary.

Conversely, given a group G that pseudo-stabilizes at order 0, we let $\mathfrak{g}$ denote the Lie algebra of infinitesimal generators on M. Let[†] $1 \leq s = s_0 \leq p + q$ denote the dimension of the orbits of G on M. Note that $s < r = \dim G$, since otherwise G would immediately stabilize at order 0, contrary to our hypothesis. Therefore, we can (locally[‡]) choose generators $\mathbf{v}_1, \ldots, \mathbf{v}_s$ which span the tangent space to the orbit through each point. Let $\mathbf{v} \in \mathfrak{g}$ be any other generator; we can write

$$\mathbf{v} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_s \mathbf{v}_s, \tag{4}$$

for certain *uniquely determined* coefficient functions $\lambda_1(x,u), \ldots, \lambda_s(x,u)$. We shall assume that $\mathbf{v}$ does not belong to the subspace of $\mathfrak{g}$ spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_s$, which occurs if and only if not all of the coefficients λ_κ in (4) are constant. Note that there exists at least one such vector field $\mathbf{v}$ since G must have dimension strictly greater than s.

Since we are assuming that G pseudo-stabilizes at order 0, the maximal dimension of the orbits of its first prolongation $G^{(1)}$ also equals s, and hence the prolonged vector fields $\mathbf{v}_1^{(1)}, \ldots \mathbf{v}_s^{(1)}$ must also span the tangent space to the orbit through each point of $V^1 \subset \mathrm{J}^1$. Since the natural projection $\pi \colon \mathrm{J}^1 \to M$ maps prolonged vector fields back to their progenerators, $d\pi(\mathbf{v}_i^{(1)}) = \mathbf{v}_i$, equation (4) and the fact that the coefficients λ_κ are unique imply that the first order prolongations also have the *same* linear relationship:

$$\mathbf{v}^{(1)} = \lambda_1 \mathbf{v}_1^{(1)} + \cdots + \lambda_s \mathbf{v}_s^{(1)}. \tag{5}$$

The crucial point here is that the coefficients $\lambda_\kappa(x,u)$, which are the same as in (4), do *not* depend on the derivative coordinates u_i^α.

In terms of our local coordinates, the standard prolongation formula, [**5**; Theorem 4.16], shows that any prolonged vector field has the form

$$\mathbf{v}^{(1)} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x,u) \frac{\partial}{\partial u^\alpha} + \sum_{\alpha=1}^{q} \sum_{i=1}^{p} \chi_i^\alpha(x,u^{(1)}) \frac{\partial}{\partial u_i^\alpha}\,, \tag{6}$$

[†] The case $s = 0$ is trivial, since then $G = \{e\}$, and the action stabilizes at order 0.

[‡] Since our considerations are always local, we shall restrict to open subsets when necessary.

where the coefficients of the first order derivative coordinates are explicitly given by

$$\chi_i^\alpha = \sum_{i=1}^{p} \left\{ \frac{\partial \varphi^\alpha}{\partial x^i} + \sum_{\beta=1}^{q} u_i^\beta \frac{\partial \varphi^\alpha}{\partial u^\beta} - \sum_{i=1}^{p} u_j^\alpha \left[\frac{\partial \xi^j}{\partial x^i} + \sum_{\beta=1}^{q} u_i^\beta \frac{\partial \xi^j}{\partial u^\beta} \right] \right\}. \tag{7}$$

We now substitute (6), along with the corresponding formulae for the $\mathbf{v}_\kappa$, whose coefficients are denoted by ξ_κ^i, φ_κ^α, and $(\chi_\kappa)_i^\alpha$, into (5). Since the $\lambda_\kappa(x,u)$ do not depend on the derivative coordinates, we deduce that the coefficients of the first prolonged vector fields are related by the system of linear equations

$$\xi^i = \sum_{\kappa=1}^{s} \lambda_\kappa \xi_\kappa^i, \qquad \varphi^\alpha = \sum_{\kappa=1}^{s} \lambda_\kappa \varphi_\kappa^\alpha, \qquad \chi_i^\alpha = \sum_{\kappa=1}^{s} \lambda_\kappa (\chi_\kappa)_i^\alpha, \qquad \begin{array}{l} i = 1, \ldots, p, \\ \alpha = 1, \ldots, q. \end{array} \tag{8}$$

If we differentiate the first two sets of equations (8) and substitute into the third using the prolongation formula (7), we find that the coefficients λ_κ must satisfy the following system of partial differential equations:

$$\begin{array}{ll} \displaystyle\sum_{\kappa=1}^{s} \frac{\partial \lambda_\kappa}{\partial x^i} \varphi_\kappa^\alpha = 0, & \\ \displaystyle\sum_{\kappa=1}^{s} \frac{\partial \lambda_\kappa}{\partial u^\alpha} \varphi_\kappa^\alpha = \sum_{\kappa=1}^{s} \frac{\partial \lambda_\kappa}{\partial x^i} \xi_\kappa^i, & \displaystyle\sum_{\kappa=1}^{s} \frac{\partial \lambda_\kappa}{\partial u^\beta} \varphi_\kappa^\alpha = 0, \qquad \alpha \neq \beta, \\ & \displaystyle\sum_{\kappa=1}^{s} \frac{\partial \lambda_\kappa}{\partial x^i} \xi_\kappa^j = 0, \qquad i \neq j. \\ \displaystyle\sum_{\kappa=1}^{s} \frac{\partial \lambda_\kappa}{\partial u^\alpha} \xi_\kappa^i = 0, & \end{array} \tag{9}$$

These hold for all $i \neq j = 1, \ldots, p$, and $\alpha \neq \beta = 1, \ldots, q$. Note that in the second set of equations there is no summation on α or i, and hence the function

$$\mu = \sum_{\kappa=1}^{s} \frac{\partial \lambda_\kappa}{\partial u^\alpha} \varphi_\kappa^\alpha = \sum_{\kappa=1}^{s} \frac{\partial \lambda_\kappa}{\partial x^i} \xi_\kappa^i, \tag{10}$$

is independent of both α and i.

Now let

$$\Lambda = \left(\frac{\partial \lambda_\kappa}{\partial x^i}, \frac{\partial \lambda_\kappa}{\partial u^\alpha} \right)$$

denote the $s \times (p+q)$ Jacobian matrix of the coefficients $\lambda_1, \ldots, \lambda_s$. Let

$$\Psi = \left(\xi_\kappa^i, \varphi_\kappa^\alpha \right)$$

denote the $s \times (p+q)$ matrix of coefficients of the vector fields $\mathbf{v}_1, \ldots, \mathbf{v}_s$. (The coefficients of $\mathbf{v}_\kappa$ form the κ^{th} row of Ψ.) Using (10), the linear system (9) can be re-expressed in a simple matrix format as

$$\Psi^T \Lambda = \mu \mathbb{1}, \tag{11}$$

where $\mathbb{1}$ is the $(p+q) \times (p+q)$ identity matrix. First note that Ψ is nonsingular, since the $\mathbf{v}_\kappa$ are linearly independent at each point by construction. Thus, if $\mu \equiv 0$, then (11) would imply that the Jacobian matrix $\Lambda \equiv 0$ would be identically zero, which would imply that all of the coefficients λ_κ are constant. However, this is contrary to our original assumption that $\mathbf{v}$ does not lie in the subspace spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_s$. Therefore the function μ cannot vanish identically if pseudo-stabilization is to occur.

If G acts intransitively, then the orbit dimension satisfies $s < p + q$. The matrix on the left hand side of (11) has rank at most s, which requires that $\mu \equiv 0$ (since otherwise the right hand side would have rank $p + q$), which forces a contradiction. Therefore we have now proven that pseudo-stabilization at order 0 requires G to act transitively, and to have dimension $r > s = p + q$, and that (11) is an equation for square matrices of size $p + q$. Furthermore, the Jacobian matrix Λ must be nonsingular, which implies that the coefficients $\lambda_\kappa(x, u)$ are functionally independent. Moreover, again by invertibility of Ψ, (11) implies that

$$\Lambda \Psi^T = \mu \mathbb{1}. \tag{12}$$

However, the (κ, ν) entry of the latter matrix equation reads

$$\sum_{i=1}^{p} \xi^i_\kappa \frac{\partial \lambda_\nu}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha_\kappa \frac{\partial \lambda_\nu}{\partial u^\alpha} = \begin{cases} 0, & \kappa \neq \nu, \\ \mu, & \kappa = \nu, \end{cases} \tag{13}$$

which is equivalent to the conditions

$$\mathbf{v}_\kappa(\lambda_\nu) = \mu \cdot \delta^\kappa_\nu, \qquad \kappa, \nu = 1, \ldots, p + q, \tag{14}$$

where δ^κ_ν is the standard Kronecker delta.

Since the λ_κ are functionally independent, we can use them as new local coordinates $z^1 = \lambda_1(x, u), \ldots, z^{p+q} = \lambda_{p+q}(x, u)$, on M. In terms of these new coordinates, formulae (14) and (4) imply that the generators of the Lie algebra have the form

$$\mathbf{v}_1 = \sigma(z) \frac{\partial}{\partial z^1}, \quad \ldots, \quad \mathbf{v}_{p+q} = \sigma(z) \frac{\partial}{\partial z^{p+q}}, \quad \mathbf{v} = \sigma(z) \left(z^1 \frac{\partial}{\partial z^1} + \cdots + z^{p+q} \frac{\partial}{\partial z^{p+q}} \right), \tag{15}$$

for some function $\sigma(z)$, which is just the function μ rewritten in the new coordinates, i.e., $\mu(x, u) = \sigma(\lambda_1(x, u), \ldots, \lambda_{p+q}(x, u))$.

Let us now show that G can have precisely one additional linearly independent vector field $\mathbf{v}$ besides the given vector fields $\mathbf{v}_1, \ldots, \mathbf{v}_{p+q}$ that span TM, proving that $\dim G = r = p + q + 1$. If $\widetilde{\mathbf{v}} = \sum \widetilde{\lambda}_\kappa \mathbf{v}_\kappa$ is another such generator, then the same argument as led to (14) will require that

$$\mathbf{v}_\kappa(\widetilde{\lambda}_\nu) = \widetilde{\mu} \cdot \delta^\kappa_\nu \tag{16}$$

for some function $\widetilde{\mu}$. In view of our new local coordinate formulae (15) for the vector fields $\mathbf{v}_\kappa$, equation (16) for $\kappa \neq \nu$ implies that $\widetilde{\lambda}_\kappa$ is a function of z^κ alone. Moreover, equation (16) for $\kappa = \nu$ implies that $\partial \widetilde{\lambda}_\kappa / \partial z^\kappa$ is independent of κ and hence a constant. Therefore, each $\widetilde{\lambda}_\kappa = b z^\kappa + c_\kappa$ is an affine function of the corresponding $z^\kappa = \lambda_\kappa$, so that $\widetilde{\mathbf{v}} = b\mathbf{v} + \sum_\kappa c_\kappa \mathbf{v}_\kappa$ lies in the span of the vector fields (15), proving the claim.

Lastly, we must impose the conditions that the vector fields (15) form a Lie algebra. We first compute, for $i < j$,

$$[\mathbf{v}_i, \mathbf{v}_j] = \sigma \cdot \left(\frac{\partial \sigma}{\partial z^i} \frac{\partial}{\partial z^j} - \frac{\partial \sigma}{\partial z^j} \frac{\partial}{\partial z^i} \right).$$

This will be a constant coefficient linear combination of the generators (15) if and only if there exist constants $(b_1^{ij}, \ldots, b_{p+q}^{ij})$ and d^{ij} such that

$$\frac{\partial \sigma}{\partial z^i} = d^{ij} z^j + b_j^{ij}, \qquad - \frac{\partial \sigma}{\partial z^j} = d^{ij} z^i + b_i^{ij}, \qquad 0 = d^{ij} z^k + b_k^{ij}, \quad k \neq i, j. \tag{17}$$

Cross-differentiation of the first two equations implies that $d^{ij} = 0$, and hence

$$\sigma(z) = a_1 z^1 + \cdots + a_{p+q} z^{p+q} + c$$

for some constant c. Secondly, we compute

$$\begin{aligned} [\mathbf{v}_i, \mathbf{v}] &= \sigma(z) \left[\left(z^i \frac{\partial \sigma}{\partial z^i} + \sigma - \sum_j z^j \frac{\partial \sigma}{\partial z^j} \right) \frac{\partial}{\partial z^i} + \sum_{i \neq j} z^j \frac{\partial \sigma}{\partial z^i} \frac{\partial}{\partial z^j} \right] \\ &= \sigma(z) \left[(a_i z^i + b) \frac{\partial}{\partial z^i} + \sum_{i \neq j} a_j z^j \frac{\partial}{\partial z^j} \right]. \end{aligned}$$

Clearly, this vector field will be a constant coefficient linear combination of the vector fields (15) if and only if all the a_i are equal, and so

$$\sigma(z) = a(z^1 + \cdots + z^{p+q}) + c. \tag{18}$$

Now, if $a = 0$, then (18) implies that the vector fields (15) span the Lie algebra of the desired similarity group,

$$\mathbf{v}_1 = \frac{\partial}{\partial z^1}, \quad \ldots, \quad \mathbf{v}_{p+q} = \frac{\partial}{\partial z^1}, \qquad \text{and} \qquad \mathbf{v} = z^1 \frac{\partial}{\partial z^1} + \cdots + z^{p+q} \frac{\partial}{\partial z^{p+q}}.$$

(We have dropped an inessential multiple.) Otherwise, we must perform one last change of variables to convert the Lie algebra (15) into the desired form. We set

$$w^1 = \frac{z^1}{\sigma(z)}, \quad \ldots, \quad w^{p+q-1} = \frac{z^{p+q-1}}{\sigma(z)}, \quad w^{p+q} = \frac{1}{\sigma(z)}.$$

In terms of the new coordinates, we find that

$$\begin{gathered} \mathbf{v}_1 = \frac{\partial}{\partial w^1} - a\,\mathbf{w}, \quad \ldots, \quad \mathbf{v}_{p+q-1} = \frac{\partial}{\partial w^{p+q-1}} - a\,\mathbf{w}, \\ \mathbf{v}_{p+q} = -a\,\mathbf{w}, \qquad \mathbf{v} = c\,\mathbf{w} - \frac{\partial}{\partial w^{p+q}}, \end{gathered} \tag{19}$$

where $\mathbf{w} = \sum w^i \partial_{w^i}$. Clearly the transformed vector fields (19) span the desired Lie algebra of the similarity group (in the new variables). This completes the proof of the theorem. *Q.E.D.*

Of course, Theorem 6 leads one to immediately speculate on the case of higher order pseudo-stabilization. Specifically, is it the case that all such pseudo-stabilizing groups in higher dimension are straightforward analogs of the two-dimensional groups appearing in Example 4? As yet, I have been unable to answer this question, which appears to be quite a bit more difficult to analyze.

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