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Null Lagrangians, Weak Continuity, and Variational Problems of Arbitrary Order

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We consider the problem of minimizing integral functionals of the form $I(u)=\int_\Omega F(x,\nabla^{[k]}u(x))\,dx$, where $\Omega\subset\mathbb{R}^p$, $u:\Omega\to\mathbb{R}^q$ and $\nabla^{[k]}u$ denotes the set of all partial derivatives of u with orders $\leqslant k$. The method is based on a characterization of null Lagrangians $L(\nabla^k u)$ depending only on derivatives of order k. Applications to elasticity and other theories of mechanics are given.

1. INTRODUCTION

In this paper we study minimization problems for integral functionals of the form

$$I(u)=\int_\Omega F(x,\nabla^{[k]}u(x))\,dx, \tag{1.1}$$

where $\Omega\subset\mathbb{R}^p$ is a bounded open set, $u:\Omega\to\mathbb{R}^q$, and $\nabla^{[k]}u$ denotes the set of all partial derivatives u_I^i of u of all orders $|I|\leqslant k$. The admissible functions u are required to satisfy suitable boundary conditions or other constraints. It is well known that if I is sequentially weakly lower semicontinuous in the Sobolev space $W^{k,\alpha}(\Omega)$, $1<\alpha<\infty$, and if F satisfies certain growth conditions, then the direct method of the calculus of variations can be used to prove the existence of a minimizer for I. The lower semicontinuity of integrals of the form (1.1) has been studied by Morrey [28, 29] in the case $k=1$, and by Meyers [27] for arbitrary k. These authors showed that if F is continuous then a necessary and sufficient condition for I to be sequentially

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l.s.c. with respect to weak* convergence in $W^{k,\infty}(\Omega)$ is that F be *quasiconvex*, that is,

$$\int_D F(x_0, c^{[k-1]}, c + \nabla^k\phi(x))\, dx \geqslant m(D)\, F(x_0, c^{[k-1]}, c) \qquad (1.2)$$

for every fixed $x_0 \in \Omega$, $c^{[k-1]}$, c, all bounded open subsets $D \subset \mathbb{R}^p$ and all $\phi \in C_0^\infty(\Omega)$. (In (1.2) we have used the notation $F(x, \nabla^{[k]}u) = F(x, \nabla^{[k-1]}u, \nabla^k u)$, where $\nabla^k u$ denotes the set of all partial derivatives of u of order k. See Section 3 for an alternative definition of quasiconvexity.) Under additional growth conditions on F they proved that quasiconvexity is sufficient for sequential weak lower semicontinuity of I in $W^{k,p}(\Omega)$, where $1 < p < \infty$. Quasiconvexity of F does *not* imply that $F(x, c^{[k-1]}, \cdot)$ is convex, except in the special cases $p = 1$, k arbitrary and $k = q = 1$. Despite the importance of these results, they cannot be said to solve the general problem of lower semicontinuity, since the quasiconvexity condition (1.2) is only marginally more transparent than lower semicontinuity itself. Furthermore, the growth conditions imposed in order to prove weak semicontinuity in $W^{k,p}(\Omega)$ compare unfavourably with those known to be sufficient in the case when $F(x, c^{[k-1]}, \cdot)$ is convex. In this paper we make some contributions to the general problem, though we do not solve it.

Following work of one of the authors [3] on nonlinear elasticity, we base our attack on the simpler problem of characterizing those functions $L(\nabla^k u)$ that are sequentially weakly continuous from $W^{k,p}(\Omega)$ to $\mathscr{D}'(\Omega)$. In Theorem 3.4 we give several necessary and sufficient conditions for L to have this property. In particular, L is sequentially weakly continuous (for large enough p) if and only if L is a *null Lagrangian*, i.e., the Euler–Lagrange equations

$$\sum_{|I|=k} (-D)^I \frac{\partial L}{\partial u_I^i} (\nabla^k u) = 0, \qquad i = 1,..., q,$$

are satisfied identically for all u. In the case $k = 1$ this result was proved in [4]. Also, in the case $k = 1$, $L(\nabla u)$ is a null Lagrangian if and only if L is an affine combination of the minors of ∇u of all orders p, $1 \leqslant r \leqslant \min(p, q)$. This follows from the results of Landers [25], Ericksen [19], Edelen [14, 15], Rund [38, 39] and de Franchis [13], who consider more general Lagrangians $L(x, u, \nabla u)$. (The trivial example $L = uu_x$ shows that there are null Lagrangians which are not of the form $L(\nabla u)$.) The characterization of the null Lagrangians $L(\nabla^k u)$ for $k > 1$ is substantially more difficult than for $k = 1$; we show (Theorem 4.1) that $L(\nabla^k u)$ is null if and only if L is an affine combination of Jacobians of the form

$$\frac{\partial(u_{I_1}^{v_1},..., u_{I_r}^{v_r})}{\partial(x^{k_1},..., x^{k_r})}, \qquad (1.3)$$

where $1 \leqslant k_i \leqslant p$, $1 \leqslant v_i \leqslant q$, $|I_i| = k-1$, and $r \geqslant 1$. Thus, surprisingly, there are no new sequentially weakly continuous functions $L(\nabla^k u)$ over and above those obtained by applying the result for $k=1$ to the map $x \to \nabla^{k-1}u(x)$. The main tool in the proof of Theorem 4.1 is a transform similar to one introduced by Gel'fand and Dikii [21] in their study of the algebraic properties of differential equations. This enables us, using ideas of compensated compactness due to Murat [31–33] and Tartar [41, 42], to reduce the above theorem to a question about the properties of (ordinary) polynomials. This can be answered using some powerful techniques of algebraic geometry.

Following [3] we say that an integrand $G(\nabla^k u)$ is *polyconvex* if there exists a convex function Φ such that

$$G(\nabla^k u) = \Phi(J(\nabla^k u)) \qquad \text{for all} \quad \nabla^k u,$$

where $J(\nabla^k u)$ denotes the set of all the Jacobians (1.3). We give (Theorem 5.2) necessary and sufficient conditions for G to be polyconvex, and prove (Theorem 5.3) that if G is polyconvex then G is quasiconvex. Polyconvexity is equivalent in the case $k=1$ to a sufficient condition for quasiconvexity introduced by Morrey [28, 29]. Our main existence theorem (Theorem 5.5) establishes the existence of minimizers for $I(u)$ in various classes of admissible functions, the principal hypotheses being that $F(x, c^{[k-1]}, \cdot)$ is polyconvex and that F satisfies a coercivity condition. The proof of this theorem is a simple consequence of our sequential weak continuity results and of a lower semicontinuity result for convex integrands due essentially to Eisen [16]. This lower semicontinuity result does not assume coercivity, and thus enables us to make our coercivity hypothesis on F rather than on the associated convex function of the Jacobians, as was done in [3–5]. Although quasiconvexity does not imply polyconvexity, at the present time the polyconvex functions form the only general class of quasiconvex functions known, so that the restriction to polyconvex integrands may not be serious. For these integrands our results substantially improve those of Morrey and Meyers.

It is known (cf. Theorem 3.3) that if $G(\nabla^k u)$ is quasiconvex then G satisfies the Legendre–Hadamard condition, which, if G is C^2, takes the form

$$\sum_{i,j=1}^{q} \sum_{|I|=|J|=k} \frac{\partial^2 G(H)}{\partial H_I^i \, \partial H_J^j} a^i a^j b_I b_J \geqslant 0, \qquad a \in \mathbb{R}^q, \quad b \in \mathbb{R}^p. \tag{1.4}$$

(For the notation see Section 2.) In the case $k=1$ it is a still unsolved problem of Morrey [29] to decide whether the converse is true. We give a simple example (Example 3.5) to show that this is not the case for $k>1$. That (1.4) is a necessary condition for lower semicontinuity is a special case of a result from the theory of compensated compactness (Murat [31–33], Tartar [41, 42]). Some of the connections between that theory and the problems considered in this paper are discussed at the end of Section 3. (Compensated compactness has beautiful applications to other aspects of the theory of nonlinear partial differential equations.)

In Section 6 we apply our results to establish the existence of energy minimizing deformations in conservative problems of nonlinear elasticity. The cases of classical hyperelasticity, Cosserat continua, and materials of grade N are considered. The assumption of polyconvexity can in each case be interpreted as a constitutive hypothesis on the stored-energy function of the material.

2. Notation

Let Ω be a bounded open subset of $\mathbb{R}^p$. If $u: \Omega \to \mathbb{R}^q$ we write $u(x) = (u^1(x),..., u^q(x))$, $x = (x^1,..., x^p)$. If $I = (i_1,..., i_p)$, $J = (j_1,..., j_p)$ are multi-indices we write $I \pm J = (i_1 \pm j_1,..., i_p \pm j_p)$, $|I| = i_1 + \cdots + i_p$, $I! = i_1! \cdots i_p!$, $\binom{I}{J} = I!/J!(I-J)!$, $I \leqslant J$ if $i_r \leqslant j_r$ for $r = 1,..., p$, $u_I^i = \partial_1^{i_1} \cdots \partial_p^{i_p}(u^i)$, $\partial_j = \partial/\partial x^j$, $x_I = x^I = (x^1)^{i_1} \cdots (x^p)^{i_p}$, and $(-D)^I = (-\partial_1)^{i_1} \cdots (-\partial_p)^{i_p}$. Occasionally we will write $\phi_{,j}$ for $\partial_j \phi$.

Let $X = X(p, q, k)$ denote the $q \times \binom{p+k-1}{k}$-dimensional space of real matrices $V = (V_I^i)$, $1 \leqslant i \leqslant q$, $|I| = k$, and let $Y = Y(p, q, k)$ denote the $q \times \binom{p+k}{k}$-dimensional space of real matrices $V = (V_I^i)$, $1 \leqslant i \leqslant q$, $|I| \leqslant k$. Let $\nabla^k u = (u_I^i)$, $1 \leqslant i \leqslant q$, $|I| = k$, and $\nabla^{[k]} u = (u_I^i)$, $1 \leqslant i \leqslant q$, $|I| \leqslant k$, so that for each $x \in \Omega$ we have $\nabla^k u(x) \in X$, $\nabla^{[k]} u(x) \in Y$.

The space of r-times continuously differentiable functions $u: \Omega \to \mathbb{R}^q$ (resp. $u: \bar{\Omega} \to \mathbb{R}^q$) is denoted $C^r(\Omega)$ (resp. $C^r(\bar{\Omega})$). The subspace of $C^r(\Omega)$ consisting of infinitely differentiable functions with compact support in Ω is denoted $C_0^\infty(\Omega)$. If $q = 1$ we write $C_0^\infty(\Omega) = \mathscr{D}(\Omega)$. Lebesgue measure in $\mathbb{R}^p$ is denoted by m or dx. If $1 \leqslant s \leqslant \infty$ then $L^s(\Omega)$ is the Banach space of (equivalence classes of) Lebesgue measurable real-valued functions v on Ω with norm

$$\begin{aligned} \|v\|_s &= \left(\int_\Omega |v(x)|^s \, dx \right)^{1/s}, \qquad 1 \leqslant s < \infty, \\ &= \operatorname*{ess\,sup}_{x \in \Omega} |v(x)|, \qquad s = \infty. \end{aligned}$$

$W^{r,s}(\Omega)$ denotes the Banach space consisting of functions $u: \Omega \to \mathbb{R}^q$ all of whose distributional derivatives u_I^i with $|I| \leqslant r$ belong to $L^s(\Omega)$. The norm in $W^{r,s}(\Omega)$ is given by

$$\|u\|_{r,s} = \sum_{i=1}^{q} \sum_{|I| \leqslant r} \|u_I^i\|_s.$$

If $1 \leqslant s < \infty$ then $W_0^{r,s}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{r,s}(\Omega)$. Weak and weak* convergence are written $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$, respectively. Weak* convergence in $W^{r,\infty}(\Omega)$ is *defined* by $u^{(n)} \overset{*}{\rightharpoonup} u$ in $W^{r,\infty}(\Omega)$ if and only if $u^{(n)i}_I \overset{*}{\rightharpoonup} u^i_I$ in $L^\infty(\Omega)$ for $1 \leqslant i \leqslant q$, $|I| \leqslant r$. $W_0^{r,\infty}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{r,\infty}(\Omega)$ with respect to weak* convergence. $\mathscr{D}'(\Omega)$ denotes the space of real-valued distributions on Ω. A sequence of functions $v^{(n)} \in L^1(\Omega)$ converges to $v \in L^1(\Omega)$ in $\mathscr{D}'(\Omega)$ if and only if $\int_\Omega v^{(n)}\phi\, dx \to \int_\Omega v\phi\, dx$ for all $\phi \in \mathscr{D}(\Omega)$.

3. Null Lagrangians, Quasiconvexity, and Sequential Weak Continuity

Let p, q and k be fixed positive integers, and let $Y = Y(p, q, k)$ be defined as in Section 2. A continuous function $L\colon Y \to \mathbb{R}$ is a *null Lagrangian* if

$$\int_\Omega L(\nabla^{[k]}(u+\phi)(x))\, dx = \int_\Omega L(\nabla^{[k]}u(x))\, dx \tag{3.1}$$

for every bounded open set $\Omega \subset \mathbb{R}^p$ and for all $u \in C^k(\bar{\Omega})$, $\phi \in C_0^\infty(\Omega)$. By making a suitable change of variables it is easily shown that if (3.1) holds for $\Omega = \Omega_0$ and for all u, ϕ then it holds for all Ω, u, ϕ. Since

$$\begin{aligned}\frac{d}{dt}&\int_\Omega L(\nabla^{[k]}(u+t\phi)(x))\, dx \\ &= \sum_{i=1}^{q} \sum_{|I| \leqslant k} \int_\Omega \frac{\partial L}{\partial u^i_I}(\nabla^{[k]}(u+t\phi)(x))\, \phi^i_I(x)\, dx,\end{aligned}$$

it follows that if $L \in C^1(Y)$ then L is a null Lagrangian if and only if

$$\sum_{i=1}^{q} \sum_{|I| \leqslant k} \int_\Omega \frac{\partial L}{\partial u^i_I}(\nabla^{[k]}u)\, \phi^i_I\, dx = 0 \tag{3.2}$$

for all $u \in C^k(\bar{\Omega})$, $\phi \in C_0^\infty(\Omega)$; i.e., if and only if the Euler–Lagrange equations

$$\sum_{|I| \leqslant k} (-D)^I \frac{\partial L}{\partial u^i_I} = 0, \qquad i = 1, \ldots, q,$$

are identically satisfied in the sense of distributions for all $u \in C^k(\bar{\Omega})$.

We now show that any C^1 null Lagrangian is a divergence. Other results to this effect have appeared in the literature (e.g., Lawruk and Tulczyjew [26], Gel'fand and Dikii [21]) but they are not so precise.

THEOREM 3.1. *Let $L \in C^1(Y)$ be a null Lagrangian. Then*

$$L(\nabla^{[k]}u) = L(0) - \sum_{i=1}^{q} \sum_{|I| \leqslant k} \sum_{0<J\leqslant I} \binom{I}{J} (-D)^J (u^i_{I-J} K^I_i(\nabla^{[k]}u)) \tag{3.3}$$

in the sense of distributions for all $u \in C^k(\Omega)$, where

$$K^I_i(\nabla^{[k]}u) \overset{\text{def}}{=} \int_0^1 \frac{\partial L}{\partial u^i_I} (t\nabla^{[k]}u)\, dt.$$

Proof. If u is smooth and $\phi \in \mathscr{D}(\Omega)$, then by (3.2)

$$\sum_{i=1}^{q} \sum_{|I|\leqslant k} \int_\Omega \frac{\partial L}{\partial u^i_I} (t\nabla^{[k]}u)(u^i\phi)_I\, dx = 0. \tag{3.4}$$

By approximation, (3.4) holds if $u \in C^k(\bar{\Omega})$. By Leibniz's formula,

$$(u^i\phi)_I = \sum_{J \leqslant I} \binom{I}{J} u^i_{I-J}\phi_J. \tag{3.5}$$

Also

$$L(\nabla^{[k]}u) = L(0) + \sum_{i=1}^{q} \sum_{|I|\leqslant k} \int_0^1 u^i_I \frac{\partial L}{\partial u^i_I} (t\nabla^{[k]}u)\, dt. \tag{3.6}$$

Combining (3.4), (3.5) and (3.6) we obtain (3.3). ∎

Recall that a function $f: \mathbb{R}^p \to \mathbb{R}^l$ is called *almost periodic* (a.p.) if it is the uniform limit of finite trigonometrical polynomials $\sum_{j=1}^{m} a_j e^{i\xi_j \cdot x}$. It follows from the definition that if f is a.p. and if $A \subset \mathbb{R}^p$ is a bounded open set, then the mean value

$$M[f] \overset{\text{def}}{=} \lim_{n\to\infty} \frac{1}{m(A)} \int_A f(nx)\, dx$$

exists and is independent of A. A continuous real-valued function of an a.p. function is a.p. The range $\mathscr{R}(f)$ of values of an a.p. function f is compact.

DEFINITION. Let $U \subset X$ be open. A function $G: U \to \mathbb{R}$ is *quasiconvex* if

$$M[G(\nabla^k v)] \geqslant G(M[\nabla^k v])$$

for any $v \in C^k(\mathbb{R}^p)$ such that $\nabla^k v$ is a.p. and $\mathscr{R}(\nabla^k v) \subset U$.

THEOREM 3.2. *Let $F: \Omega \times Y \to \mathbb{R}$ be continuous. A necessary and sufficient condition for*

$$I(u) \stackrel{\text{def}}{=} \int_\Omega F(x, \nabla^{[k]}u(x))\,dx$$

to be sequentially weak lower semicontinuous on $W^{k,\infty}(\Omega)$ (i.e., $u^{(n)} \stackrel{*}{\rightharpoonup} u$ in $W^{k,\infty}(\Omega)$ implies $I(u) \leqslant \underline{\lim}_{n\to\infty} I(u^{(n)})$) is that $F(x_0, c^{[k-1]}, \cdot)$ be quasiconvex on X for each fixed $x_0 \in \Omega$, $c^{[k-1]} \in Y(p, q, k-1)$.*

Remarks. Let $Q = \prod_{i=1}^{p} (a_i, b_i)$ be an open cell in $\mathbb{R}^n$. If $G: X \to \mathbb{R}$ is quasiconvex, then clearly

$$\frac{1}{m(Q)} \int_Q G(\nabla^k v(x))\,dx \geqslant G\left(\frac{1}{m(Q)} \int_Q \nabla^k v(x)\,dx\right) \tag{3.7}$$

whenever $v \in C^k(\bar{Q})$ and $\nabla^k v$ is periodic with respect to Q, i.e., there exists a continuous function $f(x)$, periodic with respect to Q, such that $\nabla^k v = f$ in $\bar{Q}$. (Indeed (3.7) holds for certain other open simplices; e.g., regular hexagons in $\mathbb{R}^2$.) In particular, let $D \subset \mathbb{R}^p$ be a bounded open set and choose $Q \supset D$. Let $c = (c_I^i) \in X$ and $\phi \in C_0^\infty(D)$. Define

$$v^i(x) = \sum_{|I|=k} \frac{1}{I!} c_I^i x^I + \phi^i(x).$$

Since $\nabla^k v$ is periodic, it follows from (3.7) that

$$\int_D G(c + \nabla^k \phi(x))\,dx \geqslant \int_D G(c)\,dx = m(D)\,G(c). \tag{3.8}$$

That (3.8) holds for fixed c and any $\phi \in C_0^\infty(D)$ is the definition of quasiconvexity used by Meyers [27], generalizing that of Morrey [28] for the case $k = 1$. Meyers showed that a necessary and sufficient condition for I to be sequentially weak* lower semicontinuous on $W^{k,\infty}(\Omega)$ is that $F(x_0, c^{[k-1]}, \cdot)$ be quasiconvex in the sense of (3.8) for each fixed x_0 and $c^{[k-1]}$. It follows from Theorem 3.2 that the two definitions are equivalent.

Proof of Theorem 3.2. By Meyers' result and the above remarks, we have only to show that quasiconvexity of $F(x_0, c^{[k-1]}, \cdot)$ is necessary for lower semicontinuity, and this only for the case of an integrand G depending only on $\nabla^k u$. (Here we use the sufficiency result of Meyers [27, Theorem 2], but a direct argument can be given, in the spirit of [27, Theorem 1, Lemma 1], to show that lower semicontinuity of I implies lower semicontinuity of $\int_\Omega F(x_0, c^{[k-1]}, \nabla^k u(x))\,dx$ for each fixed $x_0, c^{[k-1]}$.)

Let $v \in C^k(\mathbb{R}^p)$ be such that $\nabla^k v$ is almost periodic. Define $v^{(n)}(x) = n^{-k} v(nx)$. By Taylor's theorem $\|v^{(n)}\|_{k,\infty}$ is uniformly bounded. Hence a subsequence $v^{(\mu)} \overset{*}{\rightharpoonup} g$ in $W^{k,\infty}(\Omega)$, and clearly $\nabla^k g = M[\nabla^k v]$ a.e. Similarly, $G(\nabla^k v^{(\mu)}) \overset{*}{\rightharpoonup} M[G(\nabla^k v)]$ in $L^\infty(\Omega)$. By the lower semicontinuity we deduce that

$$M[G(\nabla^k v)] \geqslant G(M[\nabla^k v])$$

as required. ∎

Remark. There is an obvious modification of Theorem 3.2 for the case when F is defined on a suitable subset of $\Omega \times Y$.

We next define a (nonconvex) cone $\Lambda = \Lambda(p, q, k) \subset X$ by

$$\Lambda = \{a \otimes^k b : a \in \mathbb{R}^q, b \in \mathbb{R}^p\},$$

where $a \otimes^k b$ denotes the element of X with components $(a \otimes^k b)^i_I = a^i b_I$.

DEFINITIONS. A function $G: U \to \mathbb{R}$ is Λ-convex if

$$G(tH + (1-t)\bar{H}) \leqslant tG(H) + (1-t)\, G(\bar{H}), \tag{3.9}$$

whenever $t \in [0, 1]$, $H - \bar{H} \in \Lambda$ and the line segment $l(H, \bar{H}) \overset{\text{def}}{=} \{sH + (1-s)\bar{H} : s \in [0, 1]\} \subset U$. G is *strictly Λ-convex* if strict inequality holds in (3.9) when $t \in (0, 1)$, $0 \neq H - \bar{H} \in \Lambda$ and $l(H, \bar{H}) \subset U$. Note that if G is C^2 then G is Λ-convex if and only if the Legendre–Hadamard condition

$$\sum_{i,j=1}^{q} \sum_{|I|=|J|=k} \frac{\partial^2 G(H)}{\partial H^i_I\, \partial H^j_J} a^i a^j b_I b_J \geqslant 0, \qquad a \in \mathbb{R}^q, \quad b \in \mathbb{R}^p$$

holds for all $H \in X$.

THEOREM 3.3. *Let $G: U \to \mathbb{R}$ be continuous and quasiconvex. Then G is Λ-convex.*

Proof. The result is true if G is smooth (Meyers [27, Theorem 7].) If G is continuous and ρ a mollifier on X, the mollified functions $\rho_\epsilon * G$ are smooth and quasiconvex on a slightly smaller domain than U, and hence are Λ-convex here. Letting $\varepsilon \to 0$ we deduce that G is Λ-convex. ∎

We now come to the main result of this section, which gives a number of necessary and sufficient conditions for a function $L(\nabla^k u)$, depending only on kth order derivatives of u, to be a null Lagrangian.

THEOREM 3.4. *Let $L: X \to \mathbb{R}$ be continuous. The following conditions are equivalent*:

(i) *L is a null Lagrangian*

(ii) $\int_\Omega L(c + \nabla^k\phi(x))\,dx = \int_\Omega L(c)\,dx = m(\Omega)\,L(c)$ *for all* $\phi \in C_0^\infty(\Omega)$, *for all constant* $c \in X$, *and for every bounded open subset* $\Omega \subset \mathbb{R}^p$.

(iii) $M[L(\nabla^k v)] = L(M[\nabla^k v])$ *for any* $v \in C^k(\mathbb{R}^p)$ *such that* $\nabla^k v$ *is* a.p.

(iv) *L is a polynomial, and for any integer $r \geqslant 2$, the* rth *order total differential $D^r L$ satisfies*

$$D^r L(H)[a^1 \otimes^k b^1, ..., a^r \otimes^k b^r] = 0, \qquad a^i \in \mathbb{R}^q,\ b^i \in \mathbb{R}^p,\ H \in X,$$

whenever the vectors $b^1, ..., b^r$ are linearly dependent in $\mathbb{R}^p$.

(v) *L is C^1 and*

$$\sum_{|I|=k} (-D)^I \frac{\partial L}{\partial u_I^i}(\nabla^k u) = 0, \qquad i = 1, ..., q$$

in the sense of distributions for all $u \in C^k(\bar{\Omega})$.

(vi) *The map $u \to L(\nabla^k u(\cdot))$ is sequentially weak* continuous from $W^{k,\infty}(\Omega)$ to $L^\infty(\Omega)$. (That is, $u^{(n)} \overset{*}{\rightharpoonup} u$ in $W^{k,\infty}(\Omega)$ implies $L(\nabla^k u^{(n)}) \overset{*}{\rightharpoonup} L(\nabla^k u)$ in $L^\infty(\Omega)$.)*

(vii) *L is a polynomial (of degree s, say) and the map $u \mapsto L(\nabla^k u(\cdot))$ is sequentially weakly continuous from $W^{k,s}(\Omega) \to \mathscr{D}'(\Omega)$. (That is $u^{(n)} \rightharpoonup u$ in $W^{k,s}(\Omega)$ implies $L(\nabla^k u^{(n)}) \to L(\nabla^k u)$ in $\mathscr{D}'(\Omega)$.)*

Proof. Setting $u^i = \sum_{|I|=k} (1/I!)\, c_I^i x^I$ in (3.1) shows that (i) implies (ii). It is easily proved that (vi) holds if and only if the functionals $\pm\int_\Omega \theta(x)\, L(\nabla^k u(x))\,dx$, θ continuous, are sequentially weak* lower semicontinuous on $W^{k,\infty}(\Omega)$. Hence, by Theorem 3.2 and the subsequent remark, conditions (ii), (iii) and (vi) are equivalent. Furthermore, (ii) implies that L is Λ-affine (i.e., equality holds in (3.9)). It is easily shown that the orthogonal complement in X of $\operatorname{Span}\Lambda$ is zero, so that $\operatorname{Span}\Lambda = X$ and hence L is a polynomial of degree $s \leqslant \dim X$. Let $\phi, \psi \in C_0^\infty(\Omega)$, $\varepsilon > 0$. Then by (ii),

$$\frac{d}{d\varepsilon}\int_\Omega L(\varepsilon\nabla^k\phi + \nabla^k\psi)\,dx\,\Big|_{\varepsilon=0} = \sum_{i=1}^{q} \sum_{|I|=k} \int_\Omega \frac{\partial L}{\partial u_I^i}(\nabla^k\psi)\,\phi_I^i\,dx = 0.$$

Choosing $\psi = \psi^{(n)}$ with $\psi^{(n)} \to u$ in $C^k(\operatorname{supp}\phi)$, we obtain (v). We have already shown that (v) implies (i). That (vii) implies (vi) is obvious.

Suppose (v) holds. We establish (vii). We have already proved that L is a polynomial of degree s, say. Let $u^{(n)} \rightharpoonup u$ in $W^{k,s}(\Omega)$. We prove by induction on s that $L(\nabla^k u^{(n)}) \to L(\nabla^k u)$ in $\mathscr{D}'(\Omega)$. If $s = 1$ then L is affine and so the result is trivial. Suppose the result is true for $s - 1$ and that L is homogeneous of degree s. It follows from (ii) that $\partial L/\partial u_I^i$ is a null Lagrangian of degree $s - 1$ for any i, I. Since $(\partial L/\partial u_I^i)(\nabla^k u^{(n)})$ is bounded on $L^{s'}(\Omega)$, where $1/s + 1/s' = 1$, it follows by the induction hypothesis that

$$\frac{\partial L}{\partial u_I^i}(\nabla^k u^{(n)}) \rightharpoonup \frac{\partial L}{\partial u_I^i}(\nabla^k u) \qquad \text{in } L^{s'}(\Omega). \tag{3.10}$$

We now apply Theorem 3.1. Note first that by approximation (3.3) holds for any $u \in W^{k,s}(\Omega)$. Also

$$K_i^I(\nabla^k u^{(n)}) = \frac{1}{s}\frac{\partial L}{\partial u_I^i}(\nabla^k u^{(n)}).$$

Without loss of generality we may assume that $\partial\Omega$ is smooth. Thus, by the compactness of the imbedding $W^{k,s}(\Omega) \to W^{k-1,s}(\Omega)$,

$$u_{I-J}^{(n)i} \to u_{I-J}^i \qquad \text{in } L^s(\Omega) \tag{3.11}$$

for any $0 < J \leqslant I$. Combining (3.10) and (3.11) we obtain

$$u_{I-J}^{(n)i} K_i^I(\nabla^k u^{(n)}) \rightharpoonup u_{I-J}^i K_i^I(\nabla^k u) \qquad \text{in } L^1(\Omega).$$

Thus

$$(-D)^J u_{I-J}^{(n)i} K_i^I(\nabla^k u^{(n)}) \to (-D)^J u_{I-J}^i K_i^I(\nabla^k u) \qquad \text{in } \mathscr{D}'(\Omega),$$

and so by (3.3)

$$L(\nabla^k u^{(n)}) \to L(\nabla^k u) \qquad \text{in } \mathscr{D}'(\Omega)$$

as required. Hence (vii) holds.

Let (iii) hold. We prove (iv) using an argument of Tartar [42]. If $r = 2$ the result follows simply from Theorem 3.3. Suppose the result is true for $r - 1$; we prove it holds for r. If $\operatorname{rank}\{b^1, ..., b^r\} < r - 1$ then $\{b^1, ..., b^{r-1}\}$ are linearly dependent, so that

$$D^{r-1}L(H)[a^1 \otimes^k b^1, ..., a^{r-1} \otimes^k b^{r-1}] = 0. \tag{3.12}$$

Differentiating (3.12) in the direction $a^r \otimes^k b^r$ we obtain

$$D^r L(H)[a^1 \otimes^k b^1, ..., a^r \otimes^k b^r] = 0. \tag{3.13}$$

Suppose that $\operatorname{rank}\{b^1,\dots,b^r\} = r-1$. Without loss of generality we may suppose that

$$b^r = b^1 + \cdots + b^{r-1}.$$

Let

$$w(x) = \sum_{j=1}^{r} t_j a^j \sin(b^j \cdot x),$$

where the t_j are real parameters. Then

$$\nabla^k w(x) = \pm \sum_{j=1}^{r} t_j a^j \otimes^k b^j \begin{matrix}\sin\\ \cos\end{matrix}(b^j \cdot x),$$

and so $M[\nabla^k w] = 0$. By (iii)

$$M[L(H + \nabla^k w)] = L(M[H + \nabla^k w]) = L(H). \tag{3.14}$$

The left-hand side of (3.14) is a polynomial in the t_j. Equating the coefficient of $t_1 \cdots t_r$ to zero, we obtain

$$M\left[D^r L(H)[a^1 \otimes^k b^1,\dots, a^r \otimes^k b^r] \begin{matrix}\sin\\ \cos\end{matrix}(b^1 \cdot x) \cdots \begin{matrix}\sin\\ \cos\end{matrix}(b^r \cdot x)\right] = 0.$$

Since

$$M\left[\begin{matrix}\sin\\ \cos\end{matrix}(b^1 \cdot x) \cdots \begin{matrix}\sin\\ \cos\end{matrix}(b^r \cdot x)\right] \neq 0,$$

this gives (3.13).

To complete the proof of Theorem 3.4 we prove that (iv) implies (ii). It suffices to show that if L satisfies (iv) and $r \geqslant 2$ then

$$\int_\Omega D^r L(c)[\nabla^k \phi,\dots, \nabla^k \phi]\, dx = 0 \tag{3.15}$$

for all constant $c \in X$, $\phi \in C_0^\infty(\Omega)$; for if (3.15) holds then (ii) follows by integration. Let

$$\frac{\partial^r L(c)}{\partial c^{i_1}_{I_1} \cdots \partial c^{i_r}_{I_r}} = A^{I_1 \cdots I_r}_{i_1 \cdots i_r}.$$

Following Murat [33], we use Plancherel's identity to compute the left-hand side of (3.15). Defining the Fourier transform $\hat{\phi}$ of ϕ by

$$\hat{\phi}(\xi) = \int_{\mathbb{R}^p} \phi(x)\, e^{-2\pi i(\xi \cdot x)}\, dx,$$

we thus have

$$\int_{\mathbb{R}^p} \sum_{i_j, I_l} A^{I_1\cdots I_r}_{i_1\cdots i_r} \phi^{i_1}_{I_1} \cdots \phi^{i_r}_{I_r}\, dx$$

$$= \int_{\mathbb{R}^p} \sum A^{I_1\cdots I_r}_{i_1\cdots i_r} \hat{\phi}^{i_1}_{I_1} \overline{(\phi^{i_2}_{I_2} \cdots \phi^{i_r}_{I_r})^{\wedge}}\, d\xi^1$$

$$= \int_{\mathbb{R}^p} \sum A^{I_1\cdots I_r}_{i_1\cdots i_r} \hat{\phi}^{i_1}_{I_1}(\xi^1) \left(\int_{\mathbb{R}^p} \overline{\hat{\phi}^{i_2}_{I_2}(\xi^2)}\, \overline{(\phi^{i_3}_{I_3} \cdots \phi^{i_r}_{I_r})^{\wedge} (\xi^1 - \xi^2)}\, d\xi^2 \right) d\xi^1$$

$$= \int_{\mathbb{R}^p} \cdots \int_{\mathbb{R}^p} \sum A^{I_1\cdots I_r}_{i_1\cdots i_r} \hat{\phi}^{i_1}_{I_1}(\xi^1)\, \overline{\hat{\phi}^{i_2}_{I_2}(\xi^2)} \cdots$$

$$\cdots \overline{\hat{\phi}^{i_{r-1}}_{I_{r-1}}(\xi^{r-1})}\, \overline{\hat{\phi}^{i_r}_{I_r}\left(\xi^1 - \sum_{j=1}^{r-1} \xi^j\right)}\, d\xi^1 \cdots d\xi^{r-1},$$

where the integrations are to be performed in the order shown. Since $\hat{\phi}^{i_1}_{I_1}(\xi^1) = (-2\pi i)^k \xi^1_{I_1} \hat{\phi}^{i_1}(\xi^1)$ etc., and $\xi^1, \ldots, \xi^{r-1}$, $\xi^1 - \sum_{j=1}^{r-1} \xi^j$ are linearly dependent, it follows from (iv) that the integrand is zero. Hence (ii) holds. ∎

Remarks 1. Because the characterization of null Lagrangians as affine combinations of kth order Jacobians (see Section 4) is not straightforward, we have chosen to give proofs of the various equivalences in Theorem 3.4 that are independent of it. Some of these proofs are new even for the case $k = 1$ (cf. [4, Theorem 3.1]).

2. In the case $k = 1$ it is shown in [4, Theorem 3.1] that conditions (i), (ii), (v)–(vii) are equivalent also to the condition that L be Λ-affine, that is,

$$\sum_{i,j=1}^{q} \sum_{|I|=|J|=k} \frac{\partial^2 L(H)}{\partial H^i_I\, \partial H^j_J} a^i a^j b_I b_J = 0, \qquad a \in \mathbb{R}^q, \quad b \in \mathbb{R}^p, \tag{3.16}$$

holds for all $H \in X$. For general k this is false, as is shown by the following example. Of course, by (iv), (3.16) is a necessary condition for L to be a null Lagrangian, for all k, and it is sufficient if L is quadratic.

EXAMPLE 3.5. Let $p = k = 2$, $q = 3$, let (x, y) correspond to (x^1, x^2), and let

$$L(\nabla^2 u) = \sum_{j,k,l=1}^{3} \varepsilon_{jkl} u^j_{xx} u^k_{xy} u^l_{yy},$$

where ε_{jkl} denotes the alternating symbol. It is easily verified that L is Λ-affine but not a null Lagrangian. Since L is an odd function it follows also that L is not quasiconvex. Hence the converse of Theorem 3.3 is false for general k. It is an unsolved problem of Morrey [29, p. 122] to decide if Λ-convexity implies quasiconvexity when $k=1$.

To complete this section we exhibit the connection between the work of Murat [31–33] and Tartar [41, 42] and Theorem 3.4. These authors have considered in particular the following question. Let $\Omega \subset \mathbb{R}^p$ be open, let a_{ijk} be real constants, and let $s>1$, $\nu>1$ be given. For what continuous functions $F:\mathbb{R}^N \to \mathbb{R}$ do the conditions

$$v_1^{(n)},\ldots, v_N^{(n)} \rightharpoonup v_1,\ldots, v_N \qquad \text{in } L^s(\Omega),$$

$$\sum_{j,k} a_{ijk} \frac{\partial v_j^{(n)}}{\partial x^k} \text{ bounded in } L^\nu(\Omega) \qquad \text{for } i=1,\ldots, M,$$

imply that

$$F(v_1^{(n)},\ldots, v_N^{(n)}) \to F(v_1,\ldots, v_N)$$

in the sense of distributions?

Let $\mathscr{V}=\{(\lambda,\xi): \lambda \in \mathbb{R}^n,\ \xi \in \mathbb{R}^p,\ \sum_{j,k} a_{ijk}\lambda_j \xi_k = 0 \text{ for } i=1,\ldots, M\}$, $\mathscr{V}^0 = \{(\lambda,\xi)\in \mathscr{V}: \xi \neq 0\}$, and $\Lambda = \{\lambda \in \mathbb{R}^N: (\lambda,\xi)\in \mathscr{V}^0 \text{ for some } \xi\} = \mathrm{Proj}_{\mathbb{R}^N} \mathscr{V}^0$. Then (Tartar [42, Theorem 18]) a necessary condition for F to be weakly continuous in the above sense is that for any $r \geqslant 2$

$$D^r F(v)[\lambda^1,\ldots,\lambda^r] = 0 \text{ for all } v \in \mathbb{R}^N,$$

if $(\lambda^i,\xi^i)\in \mathscr{V}^0$ with rank $(\xi^1,\ldots,\xi^r) < r$. This condition is sufficient (Murat [33]) for suitable s, ν provided $F: (L^s(\Omega))^N \to L^1_{\mathrm{loc}}(\Omega)$ and

$$\dim\{\lambda \in \mathbb{R}^N: (\lambda,\xi)\in \mathscr{V}^0\} \qquad \text{is independent of } \xi. \tag{3.17}$$

In our case, the functions v_j are the u_I^i, $1 \leqslant i \leqslant p$, $|I|=k$. We know that

$$\frac{\partial u^i_{J,l}}{\partial x^j} - \frac{\partial u^i_{J,j}}{\partial x^l} = 0 \tag{3.18}$$

for all i, j, l and $|J|=k-1$, where $u^i_{J,l} = \partial_l u^i_J$, or equivalently, J, l denotes the multi-index $(j_1,\ldots, j_l+1,\ldots, j_r)$. The cone $\mathscr{V}$ corresponding to the differential relations (3.18) is

$$\mathscr{V} = \{H \in X: \text{there exists } b \in \mathbb{R}^p \text{ such that } H^i_{J,l} b_j - H^i_{J,j} b_l = 0$$
$$\text{if } 1 \leqslant i \leqslant q,\ 1 \leqslant j,\ l \leqslant p \text{ and } |J| = k-1\}.$$

If $H \in \mathscr{V}$ then the vector with lth component $H^i_{J,l}$ is parallel to b, and so

$H^i_{J,l} = H^{1,i}_J b_l$ for suitable constants $H^{1,i}_J$. Clearly $H^{1,i}_{K,j} b_l = H^{1,i}_{K,l} b_j$ if $|K| = k-2$, so that $H^{1,i}_{K,j} = H^{2,i}_K b_j$. It follows by induction that

$$\mathscr{V} = \{(a \otimes^k b, b) \colon a \in \mathbb{R}^q, b \in \mathbb{R}^p\}.$$

Hence Λ is as previously defined, and condition (iv) of Theorem 3.4 corresponds exactly to those mentioned above. Note also that

$$\dim\{\lambda \in \mathbb{R}^N \colon (\lambda, b) \in \mathscr{V}^0\} = q$$

is independent of b.

Now suppose that $u^{(n)i}_I \in L^s(\Omega)$ satisfy (3.18), but that we do not know that $u^{(n)i}_I = \partial_1^{i_1} \cdots \partial_p^{i_p} u^i$ for functions u^i, where $I = (i_1, ..., i_p)$. Suppose further that

$$u^{(n)i}_I \rightharpoonup u^i_I \text{ in } L^s(\Omega),\ |I| = k.$$

We claim that for any open ball B with $\bar{B} \subset \Omega$ there exist functions $v^{(n)}$, $v \in W^{k,s}(B)$ such that

$$v^{(n)} \rightharpoonup v \qquad \text{in } W^{k,s}(B),$$

and

$$u^{(n)i}_I = v^{(n)i}_I, \qquad u^i_I = v^i_I, \qquad \text{whenever } |I| = k.$$

To prove this, let ρ be a mollifier, and consider the mollified functions $u^{(n)i}_{\epsilon\, I} \stackrel{\text{def}}{=} \rho_\epsilon * u^{(n)i}_I$. These satisfy (3.18) in B and are smooth. Since B is simply connected, it follows by induction that there exist smooth functions $w^{(n)}_\epsilon \colon B \to \mathbb{R}^q$ such that $w^{(n)i}_{\epsilon\, I} = u^{(n)i}_{\epsilon\, I}$ for all i, $|I| \leqslant k$. Given n, ε there exists a polynomial $\phi^{(n)}_\epsilon$ of degree less than k such that

$$\int_B \nabla^{[k-1]} w^{(n)}_\epsilon \, dx = \int_B \nabla^{[k-1]} \phi^{(n)}_\epsilon \, dx.$$

Let $v^{(n)}_\epsilon = w^{(n)}_\epsilon - \phi^{(n)}_\epsilon$. The inequality

$$\int_B |\nabla^{[k-1]} z|^s \, dx \leqslant C\left(\left|\int_B \nabla^{[k-1]} z \, dx\right|^s + \int_B |\nabla^k z|^s dx\right) \tag{3.19}$$

holds for all $z \in W^{k,s}(B)$; this can be proved using the argument in Morrey [29, p. 82] and the fact that if the right-hand side of (3.19) is zero then so is z. Applying (3.19) to $v^{(n)}_\epsilon$ we see that $\|v^{(n)}_\epsilon\|_{k,s}$ is bounded for each n, independently of ε. Standard arguments imply that a subsequence $v^{(n)}_{\varepsilon_j} \rightharpoonup v^{(n)}$ in $W^{k,s}(B)$ as $\varepsilon_j \to 0$, that $\nabla^k v^{(n)} = (u^{(n)i}_I)$, and that

$$\int_B |\nabla^{[k-1]} v^{(n)}|^s \, dx \leqslant C \int_B |\nabla^k v^{(n)}|^s \, dx. \tag{3.20}$$

The right-hand side of (3.20) is bounded independently of n. Since $\int_B \nabla^{[k-1]} v^{(n)} \, dx = 0$, it follows easily that $v^{(n)} \rightharpoonup v$ in $W^{k,s}(B)$ and $v^i_I = u^i_I$ whenever $|I| = k$, as claimed.

It follows from the above that $L(\nabla^k u)$ is sequentially weakly continuous from $W^{k,s}(\Omega) \to \mathscr{D}'(\Omega)$ if and only if $L(u^i_I)$ is weakly continuous in the sense of Murat and Tartar (for functions satisfying (3.18)). The sufficiency result of Murat, which does not assume (3.18) and of course applies to many other situations of interest, does not give as sharp a result as Theorem 3.4 (vii) since, for example, when L is a polynomial of degree 3 it implies only that $L(\nabla^k u)$ is sequentially weakly continuous from $W^{k,4}(\Omega) \to \mathscr{D}'(\Omega)$.

4. A Characterization of Null Lagrangians

This section continues the investigation of null Lagrangians begun in Section 3, with the goal being an explicit characterization of all null Lagrangians $L: X \to \mathbb{R}$ depending exclusively on kth order derivatives. By Theorem 3.4 these Lagrangians are precisely those which define sequentially weakly continuous maps from $W^{k,s}(\Omega)$ to $\mathscr{D}'(\Omega)$. It will be shown that L is null if and only if it is an affine combination of null Lagrangians of a particularly simple form, the Jacobian determinants, which we now define. Let $K = (k_1, \ldots, k_r)$, $1 \leqslant k_i \leqslant p$, and let $\alpha = (\nu_1, I_1; \ldots; \nu_r, I_r)$, where $1 \leqslant \nu_i \leqslant q$ and $|I_i| = k - 1$. Then the corresponding kth order Jacobian determinant $J^\alpha_K: X \to \mathbb{R}$ is given by the formula

$$J^\alpha_K(\nabla^k u) = \frac{\partial(u^{\nu_1}_{I_1}, \ldots, u^{\nu_r}_{I_r})}{\partial(x^{k_1}, \ldots, x^{k_r})} = \det\left(\frac{\partial u^{\nu_i}_{I_i}}{\partial x^{k_j}}\right). \tag{4.1}$$

The main theorem to be proved in this section is then as follows:

Theorem 4.1. *Let $L \in C(X)$. Then L is a null Lagrangian if and only if L is an affine combination of* kth *order Jacobian determinants, i.e.,*

$$L = C^0_0 + \sum_{\alpha, K} C^\alpha_K J^\alpha_K$$

for suitable constants C^α_K.

We first dispose of the easy result that each Jacobian determinant is a null Lagrangian. The formula

$$\frac{\partial(\phi^1,\ldots,\phi^m)}{\partial(x^1,\ldots,x^m)}=\sum_{s=1}^{m}(-1)^{s+1}\frac{\partial}{\partial x^s}\left(\phi^1\frac{\partial(\phi^2,\ldots,\phi^m)}{\partial(x^1,\ldots,\hat{x}^s,\ldots,x^m)}\right) \tag{4.2}$$

is well known (cf. Morrey [29, Lemma 4.4.6]). In (4.2) the notation $\hat{x}^s$ indicates that the term x^s is omitted. Substitution of the appropriate derivatives of the u^i for the ϕ^j and possible relabelling of the x^j shows that each Jacobian determinant is a divergence and, hence, by Theorem 3.4 (condition (ii)), a null Lagrangian.

To prove the converse we first fix some notation. Let $\mathscr{L}^r=\mathscr{L}^r(p,q,k)$ denote the space of all Lagrangians $L\colon X\to\mathbb{R}$ which are homogeneous polynomials of degree r. By Theorem 3.4 every null Lagrangian is a polynomial, and clearly we can look at each homogeneous piece of a null Lagrangian separately, so that it suffices to prove the theorem for $L\in\mathscr{L}^r$. Let W^r denote the space of all r-linear functions

$$Q\colon \underbrace{X\times X\times\cdots\times X}_{r\ \text{factors}}\to\mathbb{R}.$$

Each $L\in\mathscr{L}^r$ gives rise to a function in W^r via polarization, or, equivalently, as the rth order total differential $D^rL(H)$, where H is an arbitrary point in X. Equality of mixed partial derivatives implies that $D^rL(H)$ is actually in the subspace W_0^r of symmetric r-linear functions. Conversely, given a symmetric function $Q\in W_0^r$, Euler's identity for homogeneous polynomials shows that there is a unique $L\in\mathscr{L}^r$ with $Q=D^rL(H)$, namely $L(H)=(1/r!)\,Q(H,\ldots,H)$ for $H\in X$.

Next let $Z^{r,k}$ denote the space of all functions

$$P\colon \underbrace{(\mathbb{R}^q\times\mathbb{R}^p)\times\cdots\times(\mathbb{R}^q\times\mathbb{R}^p)}_{r\ \text{factors}}\to\mathbb{R},$$

which are linear on each of the factors $\mathbb{R}^q$ and are homogeneous polynomials of degree k on each of the factors $\mathbb{R}^p$. Given $Q\in W^r$, there is an induced function $P=\gamma(Q)\in Z^{r,k}$ defined by

$$P(a^1,b^1;\ldots;a^r,b^r)=Q(a^1\otimes^k b^1,\ldots,a^r\otimes^k b^r).$$

Since Q is r-linear and Span $\Lambda=X$ it follows that $\gamma\colon W^r\to Z^{r,k}$ is a linear isomorphism. Similarly if $Z_0^{r,k}$ denotes the space of symmetric P's in $Z^{r,k}$, then $\gamma\colon W_0^r\to Z_0^{r,k}$ restricts to a linear isomorphism.

We are now in a position to describe the *transform* of a homogeneous

Lagrangian. If $L \in \mathscr{L}^r$, then its transform is the polynomial function $\mathscr{F}(L) \in Z_0^{r,k}$ defined by

$$\mathscr{F}(L) = \gamma\left(\frac{1}{r!} D^r L(H)\right),$$

where $H \in X$ is arbitrary. This transform was introduced in the case $q = 1$ by Gel'fand and Dikii [21], and for $q > 1$ is closely related to a generalization of Gel'fand and Dikii's transform used by Shakiban in her thesis [40, 35] to study the Euler operator and conservation laws using algebraic techniques. In direct analogy with the Fourier transform of classical analysis, this transform converts differential operations into algebraic operations, which can be studied by known techniques. The preceding constructions immediately imply the following fundamental theorem:

THEOREM 4.2. *The transform $\mathscr{F}$ gives a linear isomorphism between $\mathscr{L}^r$ and $Z_0^{r,k}$.*

It will be of use to give an explicit formula for the transform of a Lagrangian. If π is a permutation of the integers $\{1,..., r\}$ there is an induced map $\hat{\pi}: Z^{r,k} \to Z^{r,k}$ given by

$$\hat{\pi}P(a^1, b^1;...; a^r, b^r) = P(a^{\pi(1)}, b^{\pi(1)};...; a^{\pi(r)}, b^{\pi(r)}).$$

The "symmetrizing" map is $\sigma = (1/r!) \sum \hat{\pi}$, the sum being over all permutations π of $\{1,..., r\}$. Note that $Z_0^{r,k} = \operatorname{im} \sigma$. Note also that $\sigma \circ \sigma = \sigma$; i.e., σ is a projection.

LEMMA 4.3. *The action of the transform on a monomial is given by*

$$\mathscr{F}(u_{I_1}^{\nu_1} \cdots u_{I_r}^{\nu_r}) = \sigma(a_{\nu_1}^1 b_{I_1}^1 \cdots a_{\nu_r}^r b_{I_r}^r). \tag{4.3}$$

The proof is an easy computation. Using this formula, we can readily compute some transforms.

EXAMPLE 4.4. Consider the second order Jacobian

$$J = \frac{\partial(u_x, u_y)}{\partial(x, z)} = u_{xx} u_{yz} - u_{xy} u_{xz}.$$

Here $p = 3$, $q = 1$, so $a^i \in \mathbb{R}$, $b^i = (b_1^i, b_2^i, b_3^i) \in \mathbb{R}^3$, $i = 1, 2$, where (x, y, z) corresponds to (x^1, x^2, x^3). Then

$$\begin{aligned}\mathscr{F}(J) &= \sigma(a^1(b_1^1)^2 a^2 b_2^2 b_3^2 - a^1 b_1^1 b_2^1 a^2 b_1^2 b_3^2)\\ &= \tfrac{1}{2}(a^1(b_1^1)^2 a^2 b_2^2 b_3^2 + a^1 b_2^1 b_3^1 a^2 (b_1^2)^2 - a^1 b_1^1 b_2^1 a^2 b_1^2 b_3^2 - a^1 b_1^1 b_3^1 a^2 b_1^2 b_2^2)\\ &= \tfrac{1}{2} a^1 a^2 (b_1^1 b_2^2 - b_2^1 b_1^2)(b_1^1 b_3^2 - b_3^1 b_1^2).\end{aligned}$$

(The last factorization, as we will shortly see, is not accidental.)

The next step is to describe the transform of an arbitrary Jacobian determinant. Let $a^1,\dots,a^r \in \mathbb{R}^q$, $b^1,\dots,b^r \in \mathbb{R}^p$. Given a multi-index $K=(k_1,\dots,k_r)$, $1 \leqslant k_j \leqslant p$, let B_K denote the $r \times r$ matrix with (i,j)th entry $b^i_{k_j}$. Similarly, given a collection of indices and multi-indices as in (4.1), let $(A \otimes B)_\alpha$ denote the $r \times r$ matrix with (i,j)th entry $a^i_{\nu_j} b^i_{I_j}$.

LEMMA 4.5. *If J^α_K is the Jacobian determinant defined by* (4.1), *then its transform is*

$$\mathscr{F}(J^\alpha_K) = \frac{1}{r!}\det(B_K)\det((A \otimes B)_\alpha). \tag{4.4}$$

Proof. We use the notation $u^i_{I,j} = \partial_j(u^i_I)$. Then

$$J^\alpha_K = \sum_\rho \operatorname{sgn}(\rho) \prod_{l=1}^r u^{\nu_l}_{I_l, k_{\rho(l)}},$$

the sum being over all permutations ρ of $\{1,\dots,r\}$. Using formula (4.3) for the transform yields

$$\begin{aligned}\mathscr{F}(J^\alpha_K) &= \sum_\rho \operatorname{sgn}(\rho)\frac{1}{r!}\sum_\pi \prod_{l=1}^r a^{\pi(l)}_{\nu_l} b^{\pi(l)}_{I_l} b^{\pi(l)}_{k_{\rho(l)}}\\ &= \frac{1}{r!}\sum_{\tilde\rho}\sum_\pi \operatorname{sgn}(\tilde\rho)\operatorname{sgn}(\pi)\prod_{l=1}^r a^l_{\nu_{\pi(l)}} b^l_{I_{\pi(l)}} b^l_{k_{\tilde\rho(l)}}.\end{aligned}$$

Separating the sums over $\tilde\rho$ and π completes the proof. ∎

The next step in the proof is to use condition (iv) of Theorem 3.4, namely, that a continuous null Lagrangian satisfies for any $r \geqslant 2$ the identity

$$D^r L(H)[a^1 \otimes^k b^1, \dots, a^r \otimes^k b^r] = 0$$

wherever the vectors $b^1,\dots,b^r$ are linearly dependent. In particular, when L is also a homogeneous polynomial of degree r, then

$$\mathscr{F}(L)(a^1, b^1; \dots; a^r, b^r) = 0 \tag{*}$$

wherever $b^1,\dots,b^r$ are linearly dependent. If $r > p$ then clearly $\mathscr{F}(L) = 0$. Suppose $r \leqslant p$. Then $b^1,\dots,b^r$ are linearly dependent if and only if $\det(B_K) = 0$ for all multi-indices K. Since we are working with polynomials, we can invoke Hilbert's Nullstellensatz on the vanishing of polynomials. See Jacobsen [24, p. 254] for a proof of this celebrated theorem.

THEOREM 4.6. *Suppose* $p(x^1,..., x^m) = p(x)$ *is a polynomial such that* $p(x) = 0$ *for all complex* x *satisfying* $q_1(x) = 0, \ldots, q_k(x) = 0$ *for certain other polynomials* q_j. *Then there is a positive integer* ν *such that* $[p(x)]^\nu = r_1(x)\, q_1(x) + \cdots + r_k(x)\, q_k(x)$ *for suitable polynomials* $r_1,..., r_k$.

(To utilize this theorem, we must of course check that (*) actually holds for all complex linearly dependent vectors $b^1,..., b^r$, but this is easily inferred from the real case by inspection of the coefficients of the resulting polynomial.)

This is almost what we want, except for the appearance of the integer ν. Clearly, if the ideal generated by the polynomials $q_1,..., q_k$ is prime, meaning that if $p \cdot \tilde{p} = \hat{r}_1 q_1 + \cdots + \hat{r}_k q_k$, then either $p = r_1 q_1 + \cdots + r_k q_k$ or $\tilde{p} = \tilde{r}_1 q_1 + \cdots + \tilde{r}_k q_k$, then ν can be taken as 1 in the theorem. For the case of determinantal polynomials it is known that this ideal is prime.

THEOREM 4.7. *The ideal generated by all polynomials* $\det(B_K)$ *corresponding to all multi-indices* K *is prime (over* $\mathbb{C}$).

(The appearance of extra variables a^j does not affect the statement of this theorem.).

Proofs of Theorem 4.7 can be found in Northcott [34, Proposition 2, p. 197] for the case in question and Mount [30] for the more general case of arbitrary sized minors of a rectangular matrix.[1] Theorems 4.6 and 4.7 then imply

LEMMA 4.8. *If* L *is a homogeneous polynomial null Lagrangian, then there exist polynomials* $P_K \in Z^{r,k-1}$ *corresponding to multi-indices* K *such that*

$$\mathscr{F}(L) = \sum_K P_K \det(B_K). \tag{4.5}$$

We now utilize the symmetry properties of the transform $\mathscr{F}(L)$ to determine the general form of the P_K in (4.5). If π is any permutation of $\{1,..., r\}$ then $\hat{\pi}(\det(B_K)) = \operatorname{sgn}(\pi) \det(B_K)$. Therefore, applying the symmetrizing map σ to (4.5) yields

$$\begin{aligned}\mathscr{F}(L) &= \sigma\left(\sum_K P_K \det(B_K)\right)\\ &= \sum_K P'_K \det(B_K),\end{aligned}$$

[1] We would like to thank M. F. Atiyah for leading us to these references.

where

$$P'_K = \frac{1}{r!} \sum_{\pi} \operatorname{sgn}(\pi)\, \hat{\pi}(P_K).$$

Note that for any permutation π,

$$\hat{\pi}(P'_K) = \operatorname{sgn}(\pi)\, P'_K.$$

Theorem 4.1 now follows from Theorem 4.2, Lemma 4.5, Lemma 4.8 and the following easy lemma.

LEMMA 4.9. *Suppose* $P \in Z^{r,k-1}$ *and* $\hat{\pi}(P) = \operatorname{sgn}(\pi)P$ *for all permutations* π *of* $\{1,..., r\}$. *Then*

$$P = \sum_{\alpha} C_\alpha \det((A \otimes B)_\alpha)$$

for some constants C_α.

Proof. Suppose

$$P(a^1, b^1; ...; a^r, b^r) = \sum_{\alpha} c_\alpha (a \otimes b)_\alpha,$$

where, for $\alpha = (\nu_1, I_1; ...; \nu_r, I_r)$,

$$(a \otimes b)_\alpha = a^1_{\nu_1} b^1_{I_1} \cdots a^r_{\nu_r} b^r_{I_r}.$$

Applying $\hat{\pi}$ to P shows that

$$c_\alpha = \operatorname{sgn}(\pi)\, c_{\pi(\alpha)},$$

where $\pi(\alpha) = (\nu_{\pi(1)}, I_{\pi(1)}; ...; \nu_{\pi(r)}, I_{\pi(r)})$. Therefore

$$P = \frac{1}{r!} \sum_{\alpha} c_\alpha \sum_{\alpha} \operatorname{sgn}(\pi)(a \otimes b)_{\pi(\alpha)}$$

$$= \frac{1}{r!} \sum_{\alpha} c_\alpha \det((A \otimes B)_\alpha). \quad \blacksquare$$

EXAMPLES 4.10. Using Theorem 4.1, we now write down the null Lagrangians $L(\nabla^k u)$ for various values of p, q and k.

(a) $p = 1$:

$$L(\nabla^k u) = c_0 + \sum_{i=1}^{q} c_i \partial^k u^i,$$

i.e., L is affine.

(b) $k=1$;

$$L(\nabla u) = \text{affine combination of minors of } \nabla u.$$

This is the case treated in [3–5], Reshetnyak [36, 37].

(c) $k=2, p=2, q=1$,

$$L(u_{xx}, u_{xy}, u_{yy}) = c_0 + c_1 u_{xx} + c_2 u_{xy} + c_3 u_{yy} + c_4(u_{xx}u_{yy} - u_{xy}^2).$$

(d) $k=2, p=2, q=2$; then (cf. [5])

$$L(\nabla^2 u) = c_0 + \sum c^i_{\alpha\beta} u^i_{,\alpha\beta} + \sum_{j=1}^{6} d_j \phi_j,$$

where

$$\phi_1 = u^1_{,11}u^1_{,22} - (u^1_{,12})^2 \qquad \phi_2 = u^2_{,11}u^2_{,22} - (u^2_{,12})^2,$$
$$\phi_3 = u^1_{,11}u^2_{,22} - u^1_{,12}u^2_{,12} \qquad \phi_4 = u^1_{,22}u^2_{,11} - u^1_{,12}u^2_{,12},$$
$$\phi_5 = u^1_{,11}v^2_{,12} - u^1_{,12}u^2_{,11} \qquad \phi_6 = u^1_{,12}u^2_{,22} - u^1_{,22}u^2_{,12}.$$

Note that the ϕ_j are linearly independent.

In general, there are nontrivial linear relations between the Jacobians J^α_K of a given degree r. In the case $k=2$, $q=1$, these relations can be deduced from the basis theorem for quadratic p-relations (Hodge and Pedoe [45, p. 315]) and Lemma 4.5. Writing u_i for $u_{,i}$, let $(j_1,\ldots,j_r \mid k_1,\ldots,k_r) = (J \mid K)$ denote the Jacobian of degree r

$$\frac{\partial(u_{j_1},\ldots,u_{j_r})}{\partial(x^{k_1},\ldots,x^{k_r})}.$$

There are $\frac{1}{2}\binom{p}{r}[\binom{p}{r}+1]$ different such Jacobians, since interchanging two j_i's just changes the sign, and also $(J \mid K) = (K \mid J)$. The result in Hodge and Pedoe says that all the relations between these Jacobians can be obtained from those of the form

$$\begin{aligned}(j_1,\ldots,j_{r-1},k_0 \mid k_1,k_2,\ldots,k_r)&\\ \pm (j_1,\ldots,j_{r-1},k_1 \mid k_0,k_2,\ldots,k_r) &\cdots\\ \pm (j_1,\ldots,j_{r-1},k_m \mid k_0,k_1,\ldots,k_{r-1}) &= 0\end{aligned}$$

(with appropriate $\pm$ signs). For example,

$$\frac{\partial(u_1,u_2)}{\partial(x^3,x^4)} - \frac{\partial(u_1,u_3)}{\partial(x^2,x^4)} + \frac{\partial(u_1,u_4)}{\partial(x^2,x^3)} = 0. \tag{4.6}$$

The problem of determining the number $n_r = n_r(k, p, q)$ of linearly independent Jacobians $J_K^\alpha(\nabla^k u)$ of degree r for general k, p, q seems to be difficult.

The maximum degree of a nonzero Jacobian $J_K^\alpha(\nabla^k u)$ is $R = R(k, p, q)$, where $R = \min(p, q)$ if $k = 1$, and $R = p$ if $k > 1$. If $1 \leqslant r \leqslant R$, choose

$$J^r(\nabla^k u) = (J^{r,1}(\nabla^k u),..., J^{r,N_r}(\nabla^k u))$$

to be a fixed N_r-tuple of Jacobians of degree r with the property that any Jacobian J_K^α of degree r is a linear combination of the $J^{r,i}(\nabla^k u)$. The $J^{r,i}(\nabla^k u)$ can, of course, be chosen to be linearly independent, so that $N_r = n_r$, but we do not insist that this has been done. (But see Lemma 5.1 below.) Note that $J^1(\nabla^k u)$ just consists of the elements u_I^i of $\nabla^k u$.

We now refine slightly the statements in Theorem 3.4 concerning the sequential weak continuity of the J_K^α.

THEOREM 4.11. *Let $2 \leqslant r \leqslant R$, and let $\alpha_i \geqslant 1$ satisfy $\alpha_1 \geqslant (r-1)p/(p+1)$, $(1/\alpha_1) + (1/\alpha_i) \leqslant 1$ for $1 \leqslant i \leqslant r-1$. Let $u^{(n)} \rightharpoonup u$ in $W^{k,\alpha_1}(\Omega)$, and let $J^i(\nabla^k u^{(n)})$ be bounded in $(L^{\alpha_i}(\Omega))^{N_i}$ for $2 \leqslant i \leqslant r-1$. Then $J^i(\nabla^k u^{(n)}) \to J^i(\nabla^k u)$ in $(\mathscr{D}'(\Omega))^{N_i}$ for $1 \leqslant i \leqslant r$.*

Proof. We use the method of [3, Theorem 6.2; 4, Theorem 3.4] to prove by induction that the following statement (P_m) holds for $1 \leqslant m \leqslant r$. Let $S = \{v \in W^{k,\alpha_1}(\Omega): J^i(\nabla^k v) \in (L^{\alpha_i}(\Omega))^{N_i}, 2 \leqslant i \leqslant r-1\}$.

Statement (P_m). If $v \in S$ and $\alpha = (v_1, I_1; ...; v_m, I_m)$, $K = (k_1,..., k_m)$ are multi-indices with $1 \leqslant k_i \leqslant p$, $1 \leqslant v_i \leqslant q$, $|I_i| = k-1$, then the identities

$$\sum_{s=1}^{m} (-1)^{s+1} \frac{\partial w^s}{\partial x^{k_s}} = 0, \tag{4.7}$$

$$J_K^\alpha(\nabla^k v) = \sum_{s=1}^{m} (-1)^{s+1} \frac{\partial}{\partial x^{k_s}} (v_{I_1}^{v_1} w^s), \tag{4.8}$$

where

$$w^s \stackrel{\text{def}}{=} \frac{\partial(v_{I_2}^{v_2},..., v_{I_m}^{v_m})}{\partial(x^{k_1},..., \hat{x}^{k_s},..., x^{k_m})}$$

hold in the sense of distributions, and

$$J^m(\nabla^k u^{(n)}) \to J^m(\nabla^k u) \qquad \text{in } (\mathscr{D}'(\Omega))^{N_m}. \tag{4.9}$$

Note first that (4.7), (4.8) hold if v is smooth (cf. (4.2)). In the case $m = 1$, the $(m-1)$th order Jacobians in (4.7), (4.8) are taken equal to 1 by definition; thus (P_1) clearly holds. Suppose (P_{m-1}) holds. Let $v \in S$, and let

$v^{(j)}$ be a sequence of smooth functions on Ω with $v^{(j)} \to v$ in $W^{k,\alpha_1}(\Omega)$. Without loss of generality we may suppose that the boundary of Ω is smooth. Thus $v^{(j)}{}^{v_2}_{I_2} \to v^{v_2}_{I_2}$ strongly in $L^\gamma(\Omega)$, where $1/\gamma = 1/\alpha_1 - 1/p$ if $\alpha_1 < p$, $\gamma > 1$ arbitrary if $\alpha_1 \geqslant p$. Since $1/\gamma + (m-2)/\alpha_1 \leqslant 1$, it follows from (P_{m-1}) and (4.8) that

$$\frac{\partial(v^{(j)}{}^{v_2}_{I_2},\ldots, v^{(j)}{}^{v_m}_{I_m})}{\partial(x^{k_1},\ldots, \hat{x}^{k_s},\ldots, x^{k_m})} \to w^s$$

in $\mathscr{D}'(\Omega)$. Thus (4.7) holds. To prove (4.8), let ρ be a mollifier and let $w^{(j)s} = \rho_{1/j} * w^s$ be the mollified functions. Let $\phi \in \mathscr{D}(\Omega)$. For large enough j,

$$\sum_{s=1}^{m} (-1)^{s+1} \frac{\partial w^{(j)s}(x)}{\partial x^{k_s}} = 0$$

if $x \in \operatorname{supp}(\phi)$. Therefore

$$\int_\Omega \sum_{s=1}^{m} (-1)^{s+1} \frac{\partial v^{(j)}{}^{v_1}_{I_1}}{\partial x^{k_s}} w^{(j)s}\phi \, dx$$

$$= \int_\Omega \sum_{s=1}^{m} (-1)^s v^{(j)}{}^{v_1}_{I_1} w^{(j)s} \phi_{,k_s} \, dx. \tag{4.10}$$

Since $w^{(j)s} \to w^s$ in $L^{\alpha_{m-1}}(\Omega)$ and $1/\alpha_1 + 1/\alpha_{m-1} \leqslant 1$, passing to the limit in (4.10) gives (4.8). Finally, by (P_{m-1}) we have that

$$\frac{\partial(u^{(n)}{}^{v_2}_{I_2},\ldots, u^{(n)}{}^{v_m}_{I_m})}{\partial(x^{k_1},\ldots, \hat{x}^{k_s},\ldots, x^{k_m})} \rightharpoonup \frac{\partial(u^{v_2}_{I_2},\ldots, u^{v_m}_{I_m})}{\partial(x^{k_1},\ldots, \hat{x}^{k_s},\ldots, x^{k_m})} \quad \text{in } (L^{\alpha_{m-1}}(\Omega))^{N_{m-1}}.$$

Since $u^{(n)}{}^{v_1}_{I_1} \to u^{v_1}_{I_1}$ in $L^{\alpha_1}(\Omega)$, it follows from (4.8) that (4.9) holds. This completes the induction. ∎

5. Polyconvexity, Lower Semicontinuity, and Existence Theorems

Let $J^1,\ldots, J^R$ be defined as in Section 4, and let $1 \leqslant r \leqslant R$. Then $J^{[r]} = (J^1,\ldots, J^r)$ can be regarded as a map from X to $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{N_r} = \mathbb{R}^{\sigma_r}$, where $\sigma_r = \sum_{i=1}^r N_i$. As before, let $U \subset X$ be open.

Lemma 5.1. *$J^{[r]}$ consists of linearly independent Jacobians if and only if the convex hull* $\operatorname{Co} J^{[r]}(U)$ *of* $J^{[r]}(U)$ *is open.*

Proof. Obviously if the Jacobians in $J^{[r]}$ are linearly dependent then $J^{[r]}(U)$ is contained in a proper subspace of $\mathbb{R}^N$, and so $\operatorname{Co} J^{[r]}(U)$ is not open.

Let $J^{[r]}$ consist of linearly independent Jacobians J^{α}_{K}. We claim that for any $H \in U$ and any $\varepsilon > 0$ such that the open ball $B_\epsilon(H)$ with centre H and radius ε lies in U, there exists $\delta > 0$ such that $\operatorname{Co} J^{[r]}(B_\epsilon(H)) \supset B_\delta(J^{[r]}(H))$. If not, there exists a nonzero $\theta \in \mathbb{R}^{\sigma_r}$ such that $\langle J^{[r]}(\bar{H}) - J^{[r]}(H), \theta \rangle \geqslant 0$ for all $\bar{H} \in B_\epsilon(H)$. Set $\bar{H} = H + t\zeta$, where $\zeta \in X$ and t is a real parameter. For small enough t we have

$$\sum_{i=1}^{R} \frac{t^i}{i!} \langle D^i J^{[r]}(H)(\zeta,..., \zeta), \theta \rangle \geqslant 0.$$

Hence

$$\langle D^1 J^{[r]}(H)(\zeta), \theta \rangle = 0 \qquad \text{for all } \zeta \in X.$$

Therefore

$$\langle D^2 J^{[r]}(\zeta, \zeta), \theta \rangle \geqslant 0 \qquad \text{for all } \zeta \in X.$$

Let $\zeta = a^1 \otimes^k b^1 + a^2 \otimes^k b^2$. By Theorems 3.4(iv), 4.1 we obtain

$$\langle D^2 J^{[r]}(H)(a^1 \otimes^k b^1, a^2 \otimes^k b^2), \theta \rangle = 0,$$

and hence, using the fact that $\operatorname{Span} \Lambda = X$,

$$\langle D^2 J^{[r]}(H)(\zeta, \zeta), \theta \rangle = 0 \qquad \text{for all} \quad \zeta \in X.$$

Proceeding inductively, we deduce that for $i = 1,..., R$,

$$\langle D^i J^{[r]}(H)(\zeta,..., \zeta), \theta \rangle = 0 \qquad \text{for all } \zeta \in X.$$

It follows that

$$\langle J^{[r]}(\eta), \theta \rangle = 0 \qquad \text{for all } \eta \in X,$$

contradicting the linear independence of the J^{α}_{K}. This proves the claim.

Let $\phi = \sum_{i=1}^{m} \lambda_i J^{[r]}(H_i) \in \operatorname{Co} J^{[r]}(U)$, where $\lambda_i \geqslant 0$, $\sum_{i=1}^{m} \lambda_i = 1$, $H_i \in U$. Then there exist $\delta_i > 0$ such that the open set $\sum_{i=1}^{m} \lambda_i B_{\delta_i}(J^{[r]}(H_i))$ is contained in $\operatorname{Co} J^{[r]}(U)$. Hence $\operatorname{Co} J^{[r]}(U)$ is open. ∎

Remark. Lemma 5.1 complements [3, Theorem 4.3] in the case $k = 1$. (In [3, p. 358] it is erroneously stated that $J^{[r]}(U)$ is open, but this does not affect the subsequent analysis.)

Definitions. Let $1 \leqslant r \leqslant R$. A function $G: U \to \mathbb{R}$ is *r-polyconvex* if there exists a convex function $\Phi: \operatorname{Co} J^{[r]}(U) \to \mathbb{R}$ such that $G(H) = \Phi(J^{[r]}(H))$ for all $H \in U$. G is *polyconvex* if it is R-polyconvex.

Note that if G is r-polyconvex then it is polyconvex.

THEOREM 5.2. *Let $G: U \to \mathbb{R}$. The following conditions are equivalent*:

(i) *G is r-polyconvex.*

(ii) *For each $H \in U$ there exists $A(H) \in \mathbb{R}^{\sigma_r}$ such that*

$$G(\bar{H}) \geqslant G(H) + \langle J^{[r]}(\bar{H}) - J^{[r]}(H), A(H)\rangle \tag{5.1}$$

for all $\bar{H} \in U$.

(iii) (a) *There exist constants $a \in \mathbb{R}$, $b \in \mathbb{R}^{\sigma_r}$ such that*

$$G(H) \geqslant a + \langle J^{[r]}(H), b\rangle \qquad \text{for all} \quad H \in U, \tag{5.2}$$

and

(b) *if H_i, $\sum_{i=1}^m \lambda_i H_i \in U$, $\lambda_i \geqslant 0$, $\sum_{i=1}^m \lambda_i = 1$, and*

$$J^{[r]}\left(\sum_{i=1}^m \lambda_i H_i\right) = \sum_{i=1}^m \lambda_i J^{[r]}(H_i)$$

then

$$G\left(\sum_{i=1}^m \lambda_i H_i\right) \leqslant \sum_{i=1}^m \lambda_i G(H_i). \tag{5.3}$$

Proof. Let (i) hold. If $J^{[r]}$ consists of linearly independent Jacobians, then by Lemma 5.1 $\operatorname{Co} J^{[r]}(U)$ is open. Since Φ is convex, for each $H \in U$ there exists $A(H) \in \mathbb{R}^{\sigma_r}$ such that

$$\begin{aligned} G(\bar{H}) = \Phi(J^{[r]}(\bar{H})) &\geqslant \Phi(J^{[r]}(H)) + \langle J^{[r]}(\bar{H}) - J^{[r]}(H), A(H)\rangle \\ &= G(H) + \langle J^{[r]}(\bar{H}) - J^{[r]}(H), A(H)\rangle, \end{aligned}$$

so that (ii) holds. If $J^{[r]} = (\hat{J}, L\hat{J})$, where $\hat{J}$ consists of linearly independent Jacobians, L is linear, and where we have rearranged the Jacobians in $J^{[r]}$ if necessary, define $\hat{\Phi}(\eta) = \Phi(\eta, L\eta)$. Then $\hat{\Phi}$ is convex on $\operatorname{Co} \hat{J}(U)$ and we obtain (ii) as before.

Let (ii) hold. Fixing H in (5.1) gives (iii)(a). If H_i, λ_i satisfy the hypotheses of (iii)(b) then

$$G(H_i) \geqslant G(H) + \langle J^{[r]}(H_i) - J^{[r]}(H), A(H)\rangle.$$

Multiplying by λ_i, summing, and setting $H: \sum_{i=1}^m \lambda_i H_i$ gives (5.3).

Let (iii) hold. Following Busemann, Ewald and Shephard [9], define Φ on $\operatorname{Co} J^{[r]}(U)$ by

$$\Phi(\eta) = \inf \sum_{i=1}^m \lambda_i G(H_i),$$

where the infimum is taken over all $H_i \in U$, $\lambda_i \geqslant 0$, $\sum_{i=1}^m \lambda_i = 1$, satisfying $\eta = \sum_{i=1}^m \lambda_i J^{[r]}(H_i)$. By (iii)(a), $\Phi(\eta) \geqslant a + \langle \eta, b \rangle$, so that Φ: $\operatorname{Co} J^{[r]}(U) \to \mathbb{R}$. It is not hard to show that Φ is convex. By choosing $m = 1$ it follows that

$$G(H) \geqslant \Phi(J^{[r]}(H)) \qquad \text{for all } H \in U.$$

On the other hand, $J^{[r]}(H) = \sum_{i=1}^m \lambda_i J^{[r]}(H_i)$ implies in particular that $H = \sum_{i=1}^m \lambda_i H_i$, so that by (iii)(b)

$$G(H) \leqslant \Phi(J^{[r]}(H)) \qquad \text{for all } H \in U.$$

Hence (iii) implies (i). ∎

Remarks. 1. One can define the notion of polyconvexity with respect to an arbitrary set of Jacobians in the obvious way, and a modified version of Theorem 5.2 holds.

2. Note that (iii)(a) is always satisfied if G is bounded below. When checking r-polyconvexity, only values of $m \leqslant \sum_{i=1}^r n_i + 1$ need be considered; this is a consequence of Carathéodory's theorem (cf. [9]).

THEOREM 5.3. *Let G: $U \to \mathbb{R}$ be polyconvex. Then G is quasiconvex.*

Proof. Write $J = J^{[r]}$. Let $v \in C^k(\mathbb{R}^p)$ be such that $\nabla^k v$ is a.p. and $\mathscr{R}(\nabla^k v) \subset U$. By Theorems 3.4, 4.1,

$$M[J(\nabla^k v)] = J(M[\nabla^k v]).$$

By Jensen's inequality, and the fact that any real-valued convex function is continuous,

$$\begin{aligned} M[G(\nabla^k v)] &= M[\Phi(J(\nabla^k v))] \\ &= \lim_{n \to \infty} \frac{1}{m(A)} \int_A \Phi(J(\nabla^k v(nx)))\, dx \\ &\leqslant \overline{\lim_{n \to \infty}}\, \Phi\left(\frac{1}{m(A)} \int_A J(\nabla^k v(nx))\, dx\right) \\ &= \Phi(M[J(\nabla^k v)]) = \Phi(J(M[\nabla^k v])) = G(M[\nabla^k v]). \end{aligned}$$ ∎

Remark. The converse of Theorem 5.3 is false if $k = 1$ (cf. Morrey [28, p. 26; 3, p. 361]).

Our main existence result, Theorem 5.5 below, is based on our weak continuity results and the following lower semicontinuity theorem. We use the notation $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$.

THEOREM 5.4. *Let* $\Phi: \Omega \times (\mathbb{R}^s \times \mathbb{R}^\sigma) \to \bar{\mathbb{R}}$ *satisfy the following properties*:

(i) $\Phi(\cdot, z, v): \Omega \to \bar{\mathbb{R}}$ *is measurable for every* $(z, v) \in \mathbb{R}^s \times \mathbb{R}^\sigma$,

(ii) $\Phi(x, \cdot, \cdot): \mathbb{R}^s \times \mathbb{R}^\sigma \to \bar{\mathbb{R}}$ *is continuous for almost all* $x \in \Omega$,

(iii) $\Phi(x, z, \cdot): \mathbb{R}^\sigma \to \bar{\mathbb{R}}$ *is convex for almost all* $x \in \Omega$ *and all* $z \in \mathbb{R}^s$.

Let $z^{(n)}, z: \mathbb{R}^s \to \bar{\mathbb{R}}$ *be measurable functions such that* $z^{(n)} \to z$ *almost everywhere* (a.e.) *and let* $v^{(n)} \rightharpoonup v$ *in* $(L^1(\Omega))^\sigma$ *as* $n \to \infty$. *Suppose further that there exists* $\phi \in L^1(\Omega)$ *such that*

$$\Phi(x, z^{(n)}(x), v^{(n)}(x)) \geqslant \phi(x), \qquad \Phi(x, z(x), v(x)) \geqslant \phi(x)$$

for all n and almost all $x \in \Omega$. *Then*

$$\int_\Omega \Phi(x, z(x), v(x))\, dx \leqslant \varliminf_{n\to\infty} \int_\Omega \Phi(x, z^{(n)}(x), v^{(n)}(x))\, dx. \tag{5.4}$$

This theorem is essentially the same as that proved by Eisen [16], with the difference that we allow Φ to take the value $+\infty$. Related results have been given by Berkovitz [8], Cesari [10, 11] and Ekeland and Témam [17, p. 226]. Although the proof is just a simple modification of Eisen's, we include it for the convenience of the reader.

Proof of Theorem 5.4. By considering $\Phi - \phi$ we can suppose that $\Phi(x, z^{(n)}(x), v^{(n)}(x)) \geqslant 0$ and $\Phi(x, z(x), v(x)) \geqslant 0$ almost everywhere. Clearly it suffices also to prove (5.4) for the case when

$$\lim_{n\to\infty} \int_\Omega \Phi(x, z^{(n)}(x), v^{(n)}(x))\, dx = \alpha < \infty. \tag{5.5}$$

We first claim that

$$h^{(n)}(x) \stackrel{\text{def}}{=} \Phi(x, z^{(n)}(x), v^{(n)}(x)) - \Phi(x, z(x), v^{(n)}(x))$$

converges to zero in measure. Choose fixed representatives for $z^{(n)}$, z, $v^{(n)}$ and v. If our claim were false then there would exist $\varepsilon > 0$, $\delta > 0$, and subsequences $z^{(\mu)}$, $v^{(\mu)}$ such that $m(M_\mu) \geqslant \delta$ for all μ, where

$$M_\mu \stackrel{\text{def}}{=} \{x \in \Omega: |\Phi(x, z^{(\mu)}(x), v^{(\mu)}(x)) - \Phi(x, z(x), v^{(\mu)}(x))| \geqslant \varepsilon,$$
$$z^{(\mu)}(x) \to z(x) \text{ and } \Phi(x, ., .) \text{ is continuous}\}.$$

Since $v^{(\mu)} \rightharpoonup v$ in $(L^1(\Omega))^\sigma$, and by (5.5), there exists $K > 0$ such that

$$\int_\Omega |v^{(\mu)}(x)|\, dx \leqslant K, \quad \int_\Omega \Phi(x, z^{(\mu)}(x), v^{(\mu)}(x))\, dx \leqslant K$$

for all μ, and thus $m(N_\mu) < \delta/2$, where

$$N_\mu \overset{\text{def}}{=} \left\{ x \in \Omega \colon |v^{(\mu)}(x)| > \frac{4K}{\delta} \text{ or } \Phi(x, z^{(\mu)}(x), v^{(\mu)}(x)) > \frac{4K}{\delta} \right\}.$$

Let $M'_\mu = M_\mu \backslash N_\mu$. Thus $m(M'_\mu) > \delta/2$ for all μ. This implies, by the selection lemma of Eisen, that for a further subsequence $M \overset{\text{def}}{=} \bigcap_\mu M'_\mu$ is nonempty. If $x \in M$ then

$$|v^{(\mu)}(x)| \leqslant \frac{4K}{\delta}, \qquad 0 \leqslant \Phi(x, z^{(\mu)}(x), v^{(\mu)}(x)) \leqslant \frac{4K}{\delta},$$

$$|\Phi(x, z^{(\mu)}(x), v^{(\mu)}(x)) - \Phi(x, z(x), v^{(\mu)}(x))| \geqslant \varepsilon, \qquad z^{(\mu)}(x) \to z(x),$$

and $\Phi(x, \cdot, \cdot)$ is continuous. Choosing a convergent subsequence $v^{(\mu)}$ gives the contradiction.

Extracting a subsequence from $h^{(n)}$, we may suppose that $h^{(n)} \to 0$ a.e. in Ω. By Mazur's theorem, there exist convex combinations $\sigma^{(j)} = \sum_{n=j}^{\infty} \lambda_n^j v^{(n)}$, where only finitely many λ_n^j are nonzero for each j, such that $\sigma^{(j)}(x) \to v(x)$ a.e. in Ω as $j \to \infty$. Since $\Phi(x, z(x), \cdot)$ is convex,

$$\Phi(x, z(x), \sigma^{(j)}(x)) + \sum_j^\infty \lambda_n^j h^{(n)}(x) \leqslant \sum_j^\infty \lambda_n^j \Phi(x, z^{(n)}(x), v^{(n)}(x)) \tag{5.6}$$

for almost all $x \in \Omega$ and large enough j. Taking the $\underline{\lim}_{j \to \infty}$ of (5.6), integrating over Ω, and applying Fatou's lemma, we obtain (5.4). ∎

We now consider the problem of minimizing the functional

$$I(u) = \int_\Omega F(x, \nabla^{[k]} u(x))\, dx \tag{5.7}$$

over a suitable class of admissible functions u. Roughly speaking, our hypotheses will be that $F(x, c^{[k-1]}, \cdot)$ is r-polyconvex for fixed x, $c^{[k-1]}$, and satisfies suitable growth conditions.

We suppose, then, that $F\colon \Omega \times Y \to \bar{\mathbb{R}}$ satisfies for some r, $1 \leqslant r \leqslant R$,

(H1) there exists a function $\Phi\colon \Omega \times Y^{[k-1]} \times \mathbb{R}^{\sigma_r} \to \bar{\mathbb{R}}$, where $Y^{[k-1]} = Y(p, q, k-1)$, such that

$$F(x, c^{[k-1]}, H) = \Phi(x, c^{[k-1]}, J^{[r]}(H)) \tag{5.8}$$

for almost all $x \in \Omega$ and all $c^{[k-1]} \in Y^{[k-1]}$, $H \in X$.

(H2) $\Phi(\cdot, c^{[k-1]}, J)\colon \Omega \to \bar{\mathbb{R}}$ is measurable for every $(c^{[k-1]}, J) \in Y^{[k-1]} \times \mathbb{R}^{\sigma_r}$, (H3) $\Phi(x, \cdot, \cdot)\colon Y^{[k-1]} \times \mathbb{R}^{\sigma_r} \to \bar{\mathbb{R}}$ is continuous for almost all $x \in \Omega$,

(H4) $\Phi(x, c^{[k-1]}, \cdot)\colon \mathbb{R}^{\sigma_r} \to \bar{\mathbb{R}}$ is convex for almost all $x \in \Omega$ and for all $c^{[k-1]} \in Y^{[k-1]}$,

(H5) $F(x, c^{[k-1]}, H) \geqslant \phi(x) + C_0(\sum_{i=1}^{r-1} |J^i(H)|^{\alpha_i} + \Psi(|J^r(H)|))$

for almost all $x \in \Omega$ and for all $(c^{[k-1]}, H) \in Y$, where $\phi \in L^1(\Omega)$, $C_0 > 0$ is constant, the exponents $\alpha_i \geqslant 1$ satisfy $\alpha_1 \geqslant (r-1)p/(p+1)$ and $1/\alpha_1 + 1/\alpha_i \leqslant 1$ for $1 \leqslant i \leqslant r-1$, and where $\Psi\colon \mathbb{R}^+ \to \mathbb{R}$ is a convex function satisfying $\Psi(t)/t \to \infty$ as $t \to \infty$.

(Note that (H1)–(H4) imply that $F(x, c^{[k-1]}, \cdot)\colon (J^{[r]})^{-1}E(x, c^{[k-1]}) \to \mathbb{R}$ is r-polyconvex for almost all $x \in \Omega$ and all $c^{[k-1]} \in Y^{[k-1]}$, where $E(x, c^{[k-1]})$ is the (possibly empty) convex open set on which $\Phi(x, c^{[k-1]}, \cdot)$ is finite, and that $F(x, c^{[k-1]}, H) = +\infty$ if $H \notin (J^{[r]})^{-1} E(x, c^{[k-1]})$.)

We suppose that Ω is connected and that the boundary $\partial\Omega$ of Ω is strongly Lipschitz; this implies that $\partial\Omega$ forms a measure space with respect to $(p-1)$-dimensional Hausdorff measure μ. We require the admissible functions u to satisfy nonlinear boundary conditions on $\partial\Omega$. Let M be a positive integer, and let $H\colon \partial\Omega \times Y^{[k-1]} \to \mathbb{R}^M$ satisfy

(C1) $H(\cdot, c^{[k-1]})\colon \partial\Omega \to \mathbb{R}^M$ is μ-measurable for every $c^{[k-1]} \in Y^{[k-1]}$,

(C2) $H(x, \cdot)\colon Y^{[k-1]} \to \mathbb{R}^M$ is continuous for μ-almost all $x \in \partial\Omega$,

(C3) There exist measurable subsets $\partial\Omega_i$ of $\partial\Omega$ with $\mu(\partial\Omega_i) > 0$, $1 \leqslant i \leqslant q$, and a constant $K \geqslant 0$, such that if $H(x, c^{[k-1]}) = 0$ for some $x \in \partial\Omega_i$ and $c^{[k-1]} = (c_l^j) \in Y^{[k-1]}$, then $|c_0^i| \leqslant K$.

We define the set $\mathscr{A}$ of admissible functions by

$$\mathscr{A} = \{u \in W^{k,1}(\Omega)\colon I(u) < \infty \text{ and } H(x, \nabla^{[k-1]}u(x)) = 0 \ \mu\text{-almost everywhere in } \partial\Omega\}.$$

In the definition the derivatives $\nabla^{[k-1]}u$ are understood in the sense of trace. We *assume* that $\mathscr{A}$ is nonempty.

THEOREM 5.5. *Under the above hypotheses I attains its minimum on $\mathscr{A}$.*

Proof. Let $u^{(n)}$ be a minimizing sequence for I in $\mathscr{A}$. By (H5)

$$\int_\Omega \left[\sum_{i=1}^{r-1} |J^i(\nabla^k u^{(n)}(x))|^{\alpha_i} + \Psi(|J^r(\nabla^k u^{(n)}(x))|)\right] dx \leqslant C \tag{5.9}$$

for all n, where here and below C denotes a generic constant. In particular,

$$\|\nabla^k u^{(n)}\|_{\alpha_1} \leqslant C. \tag{5.10}$$

The Poincaré inequality (compare (3.19))

$$\int_{\Omega} |\nabla^{[k-1]} u^{(n)}|^{\alpha_1}\, dx \leqslant C \left(\sum_{i=1}^{q} \int_{\partial\Omega_i} |u^{(n)i}|^{\alpha_1}\, d\mu + \int_{\Omega} |\nabla^k u^{(n)}|^{\alpha_1}\, dx \right), \quad (5.11)$$

and (C3) thus imply that

$$\|u^{(n)}\|_{k,\alpha_1} \leqslant C. \quad (5.12)$$

By (5.9), (5.12) and standard results on Sobolev spaces, there exists a subsequence $u^{(\mu)}$ such that as $\mu \to \infty$,

$$\begin{aligned}
&u^{(\mu)} \rightharpoonup u_\infty && \text{in } W^{k,\alpha_1}(\Omega),\\
&J^i(\nabla^k u^{(\mu)}) \rightharpoonup J^i_\infty && \text{in } (L^{\alpha_i}(\Omega))^{N_i}, \quad 2 \leqslant i \leqslant r-1,\\
&J^r(\nabla^k u^{(\mu)}) \rightharpoonup J^r_\infty && \text{in } (L^1(\Omega))^{N_r},\\
&\nabla^{[k-1]} u^{(\mu)} \to \nabla^{[k-1]} u_\infty && \text{almost everywhere in } \Omega \text{ and } \partial\Omega.
\end{aligned}$$

By Theorem 4.11,

$$J^i_\infty = J^i(\nabla^k u_\infty), \qquad 2 \leqslant i \leqslant r,$$

and so

$$J^{[r]}(\nabla^k u^{(\mu)}) \rightharpoonup J^{[r]}(\nabla^k u_\infty) \qquad \text{in } (L^1(\Omega))^{\sigma_r}.$$

Since

$$I(u^{(\mu)}) = \int_{\Omega} \Phi(x, \nabla^{[k-1]} u^{(\mu)}(x), J^{[r]}(\nabla^k u^{(\mu)}(x)))\, dx,$$

we may apply Theorem 5.4 to deduce that

$$I(u_\infty) \leqslant \varliminf_{\mu\to\infty} I(u^{(\mu)}) = \inf_{u \in \mathscr{A}} I(u).$$

By (C2), $H(x, \nabla^{[k-1]} u_\infty(x)) = 0$ μ-almost everywhere in $\partial\Omega$, and thus $u_\infty \in \mathscr{A}$ and $I(u_\infty) = \inf_{u\in\mathscr{A}} I(u)$ as required. ∎

We illustrate some features of Theorem 5.5 by means of a simple example.

EXAMPLE 5.6. Let $k=2$, $p=2$, $q=1$. Let $\Omega \subset \mathbb{R}^2$ have strongly Lipschitz boundary $\partial\Omega$, and define

$$I(u) = \int_{\Omega} [(1+u^2)(\Delta u)^4 + (u_{xx}u_{yy} - u_{xy}^2)^{2m}]\, dx\, dy,$$

where $m \geqslant 2$ is an integer. Let $\mathscr{A} = \{u \in W^{2,1}(\Omega): I(u) < \infty$ and $u|_{\partial\Omega} = f$ a.e. in $\partial\Omega\}$, where f is a sufficiently smooth function. Then I attains its minimum at some $\bar{u} \in \mathscr{A}$. This follows from Theorem 5.5 if we set

$$\Phi(a; v, w, z; \delta) = (1 + a^2)(v + z)^4 + \delta^{2m},$$

and let $r = 2$, $\alpha_1 = 4$, $\Psi(t) = t^{2m}$. The coercivity condition (H5) follows from the estimate

$$\begin{aligned} F(u, u_{xx}, u_{xy}, u_{yy}) &= \Phi(u; u_{xx}, u_{xy}, u_{yy}; u_{xx}u_{yy} - u_{xy}^2) \\ &\geqslant (u_{xx}^2 + u_{yy}^2 + 2u_{xy}^2 + 2\delta)^2 + \delta^{2m} \\ &\geqslant \tfrac{1}{2}(u_{xx}^4 + u_{yy}^4 + 4u_{xy}^4) + \delta^{2m} - 4\delta^2, \end{aligned}$$

where $\delta = u_{xx}u_{yy} - u_{xy}^2$. This example clearly shows the advantage of making the coercivity assumption on F rather than on Φ, as was done in [3–5]. The results of Meyers [27] imply that $I(u)$ is sequentially weakly lower semicontinuous on $W^{2,4m}(\Omega)$, but are insufficient to establish the existence of $\bar{u}$ since it is not clear that a minimizing sequence for I exists which is bounded in $W^{2,4m}(\Omega)$. Note also that in the case $m = 2$ one can compute $(d/dt)\, I(\bar{u} + t\phi)|_{t=0}$ for $\phi \in \mathscr{D}(\Omega)$ using Hölder's inequality and the dominated convergence theorem, but that for $m > 2$ this method fails, so that it is not obvious that $\bar{u}$ satisfies the Euler-Lagrange equation in the sense of distributions.

We now briefly mention some ways in which Theorem 5.5 may be extended.

1. *Addition of Surface Integrals*

Let $\partial\Omega$ be sufficiently smooth, let $l \geqslant k$, and define

$$\hat{I}(u) = \int_\Omega F(x, \nabla^{[k]}u(x))\, dx + \int_{\partial\Omega} G(x, \nabla^{[l-1]}u(x), J^{[m]}(\nabla_s^l u(x)))\, d\mu,$$

where $J^{[m]}(\nabla_s^l u(x))$ denotes a complete set of Jacobians of lth-order tangential (surface) derivatives of u, up to and including those of order m. If $G(x, c^{[l-1]}, \cdot)$ is convex, and if G satisfies suitable continuity and growth conditions, then $\hat{I}$ attains its minimum subject to appropriate boundary conditions on $\partial\Omega$. Further integrals over lower-dimensional manifolds can also be added and the existence of minimizers established using our methods. The details of these results are left to the reader.

2. *Weakened Growth Conditions*

For certain integrands F satisfying coercivity conditions which are not of polynomial type, a version of Theorem 5.5 can be proved in which the rôle of the spaces $L^{\gamma}(\Omega)$ is played by Orlicz spaces $L_A(\Omega)$, where A denotes a suitable convex function. For the case of nonlinear elasticity, this was carried out in [3]. The main step is to extend the sequential weak continuity results for the J_K^α to Orlicz–Sobolev spaces, using the imbedding theorems for these spaces.

Another way to weaken the growth conditions on F is to define $J_K^\alpha(\nabla^k u)$ using its expression as a divergence (cf. (4.8)). We denote this distribution by $\tilde{J}_K^\alpha(\nabla^k u)$; it is defined inductively as follows. Let $\alpha = (\nu_1, I_1; \ldots; \nu_r, I_r)$, $K = (k_1, \ldots, k_r)$. If $r = 1$, then $\tilde{J}_K^\alpha(\nabla^k u) \stackrel{\text{def}}{=} u^{\nu_1}_{I_1, k_1}$. If the Jacobians $\tilde{J}$ of order $r-1$ have been defined, then

$$\tilde{J}_K^\alpha(\nabla^k u) \stackrel{\text{def}}{=} \sum_{s=1}^{r} (-1)^{s+1} \frac{\partial}{\partial x^{k_s}} \left(u^{\nu_1}_{I_1} \tilde{J}^{\alpha'}_{K(s)}(\nabla^k u)\right),$$

where $\alpha' = (\nu_2, I_2; \ldots; \nu_r, I_r)$, $K(s) = (k_1, \ldots, \hat{k}_s, \ldots, k_r)$, provided that each product $u^{\nu_1}_{I_1} \tilde{J}^{\alpha'}_{K(s)}(\nabla^k u) \in L^1_{\text{loc}}(\Omega)$. It is easily proved that $\tilde{J}_K^\alpha(\nabla^k u)$ if $u \in W^{k,r}(\Omega)$. Since $W^{k,m}(\Omega) \subset W^{k-1,\bar{m}}(\Omega)$, where $1/\bar{m} = 1/m - 1/p$, it is clear that $J_K^\alpha(\nabla^k u)$ is well defined if $u \in W^{k,m}(\Omega)$ and each $\tilde{J}^{\alpha'}_{K(s)}(\nabla^k u) \in L^\delta(\Omega)$, where $1/\delta + 1/m - 1/p \leqslant 1$. Since $u \in W^{k,m}(\Omega)$ implies $\tilde{J}^{\alpha'}_{K(s)}(\nabla^k u) \in L^{m/(r-1)}(\Omega)$, it follows that $\tilde{J}_K^\alpha(\nabla^k u)$ is defined if $m \geqslant rp/(p+1)$. However, if $r > m > rp/(p+1)$ and $u \in W^{k,m}(\Omega)$ it can happen $\tilde{J}_K^\alpha(\nabla^k u) \neq J_K^\alpha(\nabla^k u)$; an example is given in [3]. The arguments used to prove Theorems 4.11, 5.5 show that

$$\tilde{I}(u) = \int_\Omega \Phi(x, \nabla^{[k-1]} u(x), \tilde{J}^{[r]}(\nabla^k u(x)))\, dx$$

attains its minimum on $\tilde{\mathscr{A}}$, where $\tilde{J}^{(r)}$, $\tilde{\mathscr{A}}$ are defined in the obvious way, provided Φ satisfies (H2)–(H4), and provided the estimate

$$\Phi(x, c^{[k-1]}, \tilde{J}^{[r]}) \geqslant \phi(x) + C_0 \left(\sum_{i=1}^{r-1} |\tilde{J}^i|^{\tilde{\alpha}_i} + \Psi(|\tilde{J}^r|) \right)$$

holds, where ϕ, C_0, Ψ are as in (H5), but where the $\tilde{\alpha}_i$ need satisfy only

$$\frac{1}{\alpha_1} - \frac{1}{p} + \frac{1}{\alpha_i} < 1, \qquad 1 \leqslant i < r.$$

Unfortunately, it is known whether $\tilde{J}_K^\alpha(\nabla^k u) = J_K^\alpha(\nabla^k u)$ a.e. if the former is a function, so that the meaning of the result is not clear.

Certain J_K^α possess "better" identities than (4.8) which enable them to be defined as distributions under correspondingly weaker conditions on u. For example, if $p = 2$, $q = 1$,

$$u_{xx} u_{yy} - u_{xy}^2 = (u_x u_y)_{xy} - (\tfrac{1}{2} u_y^2)_{xx} - (\tfrac{1}{2} u_x^2)_{yy} \tag{5.13}$$

has meaning as a distribution if $u \in W^{1,2}(\Omega)$! This distribution is sequentially weakly continuous from $W^{2,s}(\Omega) \to \mathscr{D}'(\Omega)$ for any $s > 1$. Each such identity can be used to weaken the growth conditions further, in the way described above. A complete solution to the problem of determining all those null Lagrangians possessing better identities can be found in [46].

3. *Constraints*

A wide variety of constrained minimization problems can be handled by our methods. Of particular interest are pointwise constraints on the derivatives $\nabla^k u$. Suppose, for example, that L is a given null Lagrangian of order $\leqslant r$. Let θ be a given measurable function, and let the hypotheses of Theorem 5.5 hold, with the exception that the inequality in (H5) is required to be satisfied only for x, $c^{[k-1]}$, H such that $L(H) = \theta(x)$. Then I attains its minimum on

$$\mathscr{A} = \{u \in W^{k,1}(\Omega) : I(u) < \infty, H(x, \nabla^{[k-1]}u(x)) = 0 \ \mu\text{-a.e.}, \\ L(\nabla^k(u(x))) = \theta(x) \text{ a.e.}\},$$

provided that $\mathscr{A}$ is nonempty. This follows using the sequential weak continuity of L. Arbitrary continuous constraints on lower order derivatives of u can be treated using compactness.

By Theorems 3.3 and 5.3, the hypotheses of Theorem 5.5 imply that $F(x, c^{[k-1]}, \cdot)$ is Λ-convex on the set $(J^{[r]})^{-1}E(x, c^{[k-1]})$. The stronger condition that $F(x, c^{[k-1]}, \cdot)$ be *strictly* Λ-convex has a connection with regularity of weak solutions to the Euler-Lagrange equations for I that we now describe. We first give a geometrical interpretation of elements of Λ.

LEMMA 5.7. *Let $\mu \in \mathbb{R}^p$ be nonzero, let $a \in \mathbb{R}$, and let π denote the hyperplane $\{x : \langle x, \mu \rangle = a\}$. A function $u \in C^{k-1}(\mathbb{R}^p)$ exists satisfying*

$$\begin{aligned} \nabla^k u(x) &= G \quad \text{if} \quad \langle x, \mu \rangle > a, \\ &= H \quad \text{if} \quad \langle x, \mu \rangle < a, \end{aligned}$$

where G, $H \in X$, if and only if

$$G - H = \lambda \otimes^k \mu \tag{5.14}$$

for some $\lambda \in \mathbb{R}^q$.

Proof. Without loss of generality we take μ to be a unit vector in the x^1-direction and $a = 0$. If u exists with the properties stated then

$$u^i = \sum_{|K|=k} \frac{1}{K!} G^i_K x^K + P^i(x) \qquad \text{if} \quad x^1 > 0,$$

$$= \sum_{|K|=k} \frac{1}{K!} H^i_K x^K + P^i(x) \qquad \text{if} \quad x^1 < 0,$$

where P^i is a polynomial of degree $k-1$. Therefore $G^i_K = H^i_K$ unless $K = (k, 0,..., 0)$, so that

$$G - H = \lambda^i \otimes^k e_1$$

as required. The converse is obvious. ∎

Remark. The jump condition (5.14) still holds if π is replaced by a smooth surface with normal μ at some point x_0, and if u is C^{k-1} in a neighborhood of x_0 with $\nabla^k u$ continuous on either side of the surface and tending to limits G, H at x_0.

For simplicity we consider a C^1 integrand $F: X \to \mathbb{R}$, so that

$$I(u) = \int_\Omega F(\nabla^k u(x))\, dx.$$

The corresponding Euler–Lagrange equations are

$$\int_\Omega \frac{\partial F}{\partial H^i_I} (\nabla^k u(x))\, \phi_I(x)\, dx = 0 \qquad \text{for all} \quad \phi \in \mathscr{D}(\Omega). \tag{5.15}$$

It is easily shown that if Ω intersects π then a function u as in Lemma 5.7 satisfies (5.15) if and only if

$$\sum_{|I|=q} \left(\frac{\partial F}{\partial H^i_I} (H + \lambda \otimes^k \mu) - \frac{\partial F}{\partial H^i_I} (H) \right) \mu_I = 0. \tag{5.16}$$

By multiplying (5.16) by λ^i it follows that if F is strictly Λ-convex then there are no nontrivial weak solutions u of the above type. Conversely, if every such weak solution is trivial and if, for example, F attains a local minimum at some $H_0 \in X$, then the method of [6] shows that F is strictly Λ-convex.

6. Applications to Theories of Elastic Materials

Consider a material body which in a reference configuration occupies the bounded connected open set $\Omega \subset \mathbb{R}^3$. We assume that $\partial\Omega$ is strongly

Lipschitz. In a deformed configuration the particle with position vector $x \in \Omega$ moves to the point with position vector $u(x) \in \mathbb{R}^3$. We are interested in those orientation-preserving deformations u that are invertible, so that interpenetration of matter does not occur. To ensure invertibility of u it would in general be necessary to consider self-contact effects, so that we content ourselves with the less stringent local invertibility condition

$$\det \nabla u(x) > 0 \qquad \text{a.e. in } \Omega. \tag{6.1}$$

The connection between (6.1) and global invertibility has been studied in the case of pure displacement boundary conditions in [7].

We begin by considering the classical case when the material is hyperelastic, so that there exists a stored-energy function $W(x, \nabla u)$. We consider a mixed displacement pure traction equilibrium boundary-value problem in which the external body force possesses a potential $\psi(x, u)$. We are required to minimize the total energy functional

$$I(u) \stackrel{\text{def}}{=} \int_\Omega [W(x, \nabla u(x)) + \psi(x, u(x))] \, dx$$

subject to the boundary condition

$$u(x) = g(x) \qquad \text{for almost all} \quad x \in \partial\Omega_1,$$

where g is a given measurable function and where $\partial\Omega_1 \subset \partial\Omega$ with $\mu(\partial\Omega_1) > 0$. Let $M^{3\times 3}$ denote the space of real 3×3 matrices, and let $M_+^{3\times 3} = \{H \in M^{3\times 3}: \det H > 0\}$. We assume that $W: \Omega \times M_+^{3\times 3} \to \mathbb{R}$ and that

(A1) there exists a function $\Phi: \Omega \times \mathbb{R}^{19} \to \overline{\mathbb{R}}$ such that for almost all $x \in \Omega$

$$\begin{aligned} \Phi(x, H, \operatorname{adj} H, \det H) &= W(x, H) &&\text{if } \det H > 0, \\ &= +\infty &&\text{otherwise.} \end{aligned}$$

(A2) $\Phi(\cdot, J): \Omega \to \overline{\mathbb{R}}$ is measurable for every $J \in \mathbb{R}^{19}$, $\Phi(x, \cdot): \mathbb{R}^{19} \to \overline{\mathbb{R}}$ is convex and continuous for almost all $x \in \Omega$,

(A3) $W(x, H) \geqslant \phi(x) + C_0(|H|^{\alpha_1} + |\operatorname{adj} H|^{\alpha_2} + \Psi(\det H))$

for almost all $x \in \Omega$ and all $H \in M_+^{3\times 3}$, where $\phi \in L^1(\Omega)$, $C_0 > 0$ is constant, $\alpha_1 \geqslant 2$, $\alpha_2 \geqslant \alpha_1/(\alpha_1 + 1)$, and where $\Psi: (0, \infty) \to \mathbb{R}$ is a convex function satisfying $\Psi(t)/t \to \infty$ as $t \to \infty$,

(A4) $\Psi: \Omega \times \mathbb{R}^3 \to \mathbb{R}^+$ is measurable in x for all $u \in \mathbb{R}^3$, and continuous in u for almost all $x \in \Omega$.

Note that by (A1), (A2), for almost all $x \in \Omega$

$$W(x, H) \to \infty \qquad \text{as} \quad \det H \to 0+.$$

Let $\mathscr{A} = \{u \in W^{1,1}(\Omega)$: $\det \nabla u(x) > 0$ a.e., $I(u) < \infty$, $u(x) = g(x)$ for almost all $x \in \partial\Omega_1\}$. Applying Theorem 5.5 we immediately obtain

THEOREM 6.1. *Let $\mathscr{A}$ be nonempty. Then I attains its minimum on $\mathscr{A}$.*

This theorem is a slight variant of results in [3, 4]; the reader is referred to these papers for existence theorems under different boundary conditions and for other refinements, including the case of incompressible elasticity.

We next give a version of Theorem 6.1 whose hypotheses are slightly easier to verify. Let $E = M^{3\times 3} \times M^{3\times 3} \times (0, \infty)$. We introduce hypotheses

(A1)′ there exists a function $\Phi: \Omega \times E \to \mathbb{R}$ such that for almost all $x \in \Omega$

$$\Phi(x, H, \operatorname{adj} H, \det H) = W(x, H) \qquad \text{if} \quad \det H > 0.$$

(A2)′ $\Phi(\cdot, J): \Omega \to \mathbb{R}$ is measurable for every $J \in E$, $\Phi(x, \cdot): E \to \mathbb{R}$ is convex for almost all $x \in \Omega$.

THEOREM 6.2. *Let* (A1)′, (A2)′, (A3) *and* (A4) *hold, and suppose further that the function Ψ in* (A3) *satisfies $\Psi(t) \to \infty$ as $t \to 0+$. Let $\mathscr{A}$ be nonempty. Then I attains its minimum on $\mathscr{A}$.*

Proof. Let $u^{(\mu)}$ be the minimizing subsequence constructed in the proof of Theorem 5.5. In particular we have $\det \nabla u^{(\mu)} \rightharpoonup \det \nabla u$ in $L^1(\Omega)$. We claim that $\det \nabla u(x) > 0$ a.e. If not then $\det \nabla u(x) > 0$ for $x \in U$, $m(U) > 0$. Since $\det \nabla u^{(\mu)} > 0$ a.e. it follows that for a further subsequence $\det \nabla u^{(\mu)}(x) \to 0$ a.e. on U. By (A3) and Fatou's lemma, $\int_\Omega W(x, \nabla u^{(\mu)}(x))\, dx \to \infty$, a contradiction. The result now follows by observing that in the proofs of Theorems 5.4, 5.5 the continuity of $\Phi(x, H, A, \delta)$ is needed only for $\delta > 0$, and this follows by convexity. ∎

We now consider more general models of elastic behaviour in which couple-stresses occur; as a general reference see Toupin [43]. These models are relevant for certain materials with microstructure (e.g., liquid crystals), and have been discussed in connection with crack problems (e.g., Atkinson and Leppington [2]) and with surface effects in crystals (cf. Toupin and Gazis [44]). We begin by discussing the case of a *Cosserat continuum*, for which the stored-energy function $W(x, d; \nabla u, \nabla d)$ depends additionally on a vector $d = (d_1, ..., d_N) \in \mathbb{R}^{3N}$ and its gradient. The *directors* $d_i \in \mathbb{R}^3$ model the microstructure of the material. Frame-indifference imposes restrictions on the dependence of W on ∇u, d and ∇d, but these restrictions will not concern us here. The total energy function is

$$I(u, d) = \int_\Omega [W(x, d(x), \nabla u(x), \nabla d(x)) + \psi(x, u(x), d(x))]\, dx,$$

where the body force potential can now depend on d. We seek a minimum for I subject to the boundary conditions

$$u(x) = g(x) \qquad \text{for almost all } x \in \partial\Omega_1,$$
$$d(x) = h(x) \qquad \text{for almost all } x \in \partial\Omega_2,$$

where g, h are given and $\mu(\partial\Omega_i) > 0$, $i = 1, 2$. Obviously, more complicated boundary conditions can be treated. Theorem 5.5 applies with $k = 1$, $p = 3$, $q = 3(N+1)$, and $r = 3$. We leave it to the reader to write down the detailed hypotheses for this example: note that the polyconvexity hypothesis (H4) says that $W(x, d, \cdot, \cdot)$ can be written as a convex function of the minors of the $3(N+1) \times 3$ matrix $\nabla\binom{u}{d}$. The existence of a minimizer for I subject to pointwise constraints on these minors can also be established (cf. the remarks after Theorem 5.5).

The equilibrium equations for liquid crystals (cf. Ericksen [18]) are a special case of those for a Cosserat continuum, but the hypotheses above cannot hold due to the special form of W. We can still apply Theorem 5.5, however, if we formulate the minimization problem in spatial coordinates as is customary in liquid crystal theory. The functional to be minimized is then

$$I(d) = \int_\Omega [W(d(x), \nabla d(x)) + \psi(x, d(x))]\, dx,$$

where Ω is a spatial region ocupied by the fluid, $d \in \mathbb{R}^3$ is a vector describing the shape and orientation of the liquid crystal molecules, W is the free energy, and ψ is the body force potential. We have assumed that the fluid has constant density. It is usually assumed also that d is a unit vector, so that

$$|d(x)| = 1 \qquad \text{a.e. in } \Omega. \tag{6.2}$$

With or without this constraint, Theorem 5.5 and its subsequent remarks apply (with similar hypotheses to the case of classical hyperelasticity) and establish the existence of minimizers for I under various boundary conditions. For remarks on equilibrium solutions of 'infinite energy' in liquid crystal theory see Dafermos [12].

We now turn to the case of "*elastic materials of grade N*." Here the functional to be minimized is

$$I(u) = \int_\Omega [W(x, \nabla u(x), \dots, \nabla^N u(x)) + \psi(x, u(x), \nabla u(x), \dots, \nabla^{N-1} u(x))]\, dx,$$

where W is the stored energy function and ψ the potential of the body forces, couples etc. In this case the full strength of Theorem 5.5 is used (with $k = N$,

$p = q = 3$, $1 \leqslant r \leqslant 3$), the conclusion being the existence of minimizers for I under a wide variety of nonlinear boundary condition on $\nabla^{[n-1]}u$. (For an idea of the kinds of boundary condition that might be interesting see Toupin [43] and Antman [1, p. 675].) Note that there is no convexity assumption on the behaviour of W with respect to the derivatives $\nabla^j u$ with $1 \leqslant j \leqslant N-1$. Thus materials of grade $N > 1$ provide a useful regularization for classical hyperelastic materials which do not satisfy the Legendre–Hadamard condition (e.g., elastic crystals, see Ericksen [20]); by adding to $W(x, \nabla u(x))$ a suitable polyconvex term $\varepsilon E(\nabla^N u(x))$, $\varepsilon > 0$ small, and choosing N as large as we wish, we obtain the existence of minimizers of the desired smoothness for a nearby material of grade N. Note also that if $N = 2$ the hypotheses of Theorem 5.5 are consistent with W blowing up both as $\det \nabla u(x)$ and as various curvatures tend to limiting values, since, for example, the Gaussian curvature of a surface parallel to the (x^1, x^2)-plane in the reference configuration depends on $\nabla^2 u$ only through the second order Jacobians $\partial(u^i_{,1}, u^j_{,2})/\partial(x^1, x^2)$ and these Jacobians appear linearly.

Surface energy terms can be added to $I(u)$ in all the cases discused above, and the existence of minimizers proved as explained in the remarks after Theorem 5.5. For the relevant continuum mechanics and other remarks see Gurtin and Murdoch [22, 23]. With an appropriate identification of variables Theorem 5.5 can also be applied to various nonlinear rod and shell theories.

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