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BiHamiltonian systems

1. Introduction

Roughly speaking, a biHamiltonian system is a system of differential equations (either first order ordinary differential equations or evolution equations) which can be written in Hamiltonian form in two distinct ways. In 1978 Magri, [7], proved a remarkable theorem that, subject to certain technical hypotheses, any such biHamiltonian system is completely integrable in the sense that it possesses an infinite number of conservation laws in involution. Examples include the Korteweg-deVries equation, the Boussinesq equation and most of the other "soliton" equations, as well as the equations of gas dynamics and shallow water theory. Magri's result has been refined by Gel'fand and Dorfman, [2], and Fuchssteiner and Fokas, [1]; an alternative proof appears in Olver, [11; Chapter 7]. A good survey of recent work on biHamiltonian systems can be found in Kosmann-Schwarzbach, [3].

The present paper begins with a brief summary of the basic features of Magri's theorem, illustrated by a couple of examples of physical interest. Connections with Hamiltonian perturbation theory and the construction of model equations for complicated physical systems is discussed. In the second half of the paper, it is shown how the exactness of a certain differential complex known as the Poisson complex provides a simple proof of a refined version of Magri's theorem. The conditions under which the Poisson complex is known to be exact are discussed, along with several conjectures.

The paper will assume a certain familiarity with Poisson structures and Hamiltonian systems, both in the finite dimensional case of ordinary differential equations as well as the infinite dimensional generalization to systems of evolution equations, as presented in [11; Chapters 6,7].

2. Magri's theorem

We begin with the basic definition of a biHamiltonian system and the standard version of the theorem of Magri.

DEFINITION: The operators $\mathcal{D}$ and $\mathcal{E}$ are said to form a *Hamiltonian pair* if all three operators $\mathcal{D},\mathcal{E}$ and $\mathcal{D}+\mathcal{E}$ are Hamiltonian. A system of differential equations is said to be *biHamiltonian* with respect to the *Hamiltonian pair* $\mathcal{D},\mathcal{E}$ if there exist Hamiltonian function(al)s H_0 and H_1 such that the system can be written in the two Hamiltonian forms

$$u_t = \mathcal{D}\cdot\delta H_1 = \mathcal{E}\cdot\delta H_0 .$$

Here u denotes the dependent variables in the equation and δ the gradient operator or variational derivative.

Corresponding to any Hamiltonian pair $\mathcal{D},\mathcal{E}$ is a *recursion operator*

$$R = \mathcal{E}\cdot\mathcal{D}^{-1},$$

where we are only viewing the inverse $\mathcal{D}^{-1}$ in a formal manner, i.e. if $P = \mathcal{D}Q$, then we set $Q = \mathcal{D}^{-1}P$. (Note that this may not uniquely determine Q.) We need to place one restriction on one of the differential operators, say $\mathcal{D}$, which appears in the Hamiltonian pair. We say a differential operator $\mathcal{D}$ is *degenerate* if there exists a nonzero differential operator $\tilde{\mathcal{D}}$ such that the product $\tilde{\mathcal{D}}\cdot\mathcal{D}$ is the zero operator; the condition is then that $\mathcal{D}$ be *nondegenerate*, meaning that it is not degenerate according to the above definition. Degeneracy is strictly a matrix phenomenon - any (non-zero) scalar differential operator is automatically nondegenerate.

THEOREM: Let

$$u_t = K_1[u] = \mathcal{D}\cdot\delta H_1 = \mathcal{E}\cdot\delta H_0 \tag{1}$$

be a biHamiltonian system of evolution equations. Assume that the differential operator $\mathcal{D}$ is nondegenerate. Let $R = \mathcal{E}\cdot\mathcal{D}^{-1}$ be the corresponding recursion operator. Assume inductively that for $n = 1,2,3,\ldots, K_n[u]$ lies in the image of $\mathcal{D}$, and so we can define the differential function

$$K_{n+1}[u] = RK_n[u].$$

Then

(i) There exist a sequence of functionals H_0, H_1, H_2,... such that each of the corresponding evolution equations

$$u_t = K_n[u] = \mathcal{D}\cdot\delta H_n = E\cdot\delta H_{n-1}$$

is also a biHamiltonian system.

(ii) The functionals H_n are in involution with respect to either Poisson bracket:

$$\{H_n, H_m\}_{\mathcal{D}} = 0 = \{H_n, H_m\}_E,$$

and hence provide an infinite hierarchy of conservation laws for each of the biHamiltonian systems

(iii) The flows of the biHamiltonian systems all mutually commute.

Although quite powerful, the theorem's main defect is the extra assumption that each $K_n[u]$ lies in the image of $\mathcal{D}$, an assumption which always seems to hold in practice, but which can be quite complicated to rigorously verify. One of the advantages of the Poisson complex approach to the proof of the theorem to be discussed below is that, under certain conditions, it allows one to eliminate this extraneous assumption entirely.

EXAMPLE 1: The archetypical example is the Korteweg-deVries equation

$$u_t = u_{xxx} + uu_x.$$

This can be written in Hamiltonian form in two distinct ways, the first being

$$u_t = D_x(u_{xx} + \frac{1}{2}u^2) = \mathcal{D}\cdot\delta H_1,$$

where the Hamiltonian operator is $\mathcal{D} = D_x$, and $H_1[u] = \int(-\frac{1}{2}u_x^2 + \frac{1}{6}u^3)dx$ is one of the classical conservation laws. The second is a bit less obvious. We find that the Korteweg-deVries equation can be written in the form

$$u_t = (D_x^3 + \frac{2}{3}uD_x + \frac{1}{3}u_x)(u) = E\cdot\delta H_0,$$

with second Hamiltonian operator $E = D_x^3 + \frac{2}{3} uD_x + \frac{1}{3} u_x$ and conserved functional $H_0[u] = \int \frac{1}{2} u^2 dx$. The proof that the operators D and E form a Hamiltonian pair is most easily effected using the calculus of functional multi-vectors discussed in [11; Chapter 7]. The recursion operator R in this case is the well-known Lenard operator

$$R = D_x^2 + \frac{2}{3} u + \frac{1}{3} u_x D_x^{-1},$$

and the corresponding hierarchy of commuting Hamiltonian evolution equations is the standard hierarchy of higher order Korteweg-deVries equations. It is known that the soliton equations associated with higher order Lax equations also admit a biHamiltonian structure, [5].

<u>EXAMPLE 2:</u> The equations of gas dynamics

$$u_t + uu_x + v^{\gamma-2} v_x = 0,$$

$$v_t + (uv)_x = 0,$$

were recently shown by Nutku, [8], to have not just a biHamiltonian but actually a triHamiltonian structure! The first Hamiltonian form is obvious; we have

$$\mathcal{D}_1 = \begin{pmatrix} 0 & -D_x \\ -D_x & 0 \end{pmatrix}, \quad H_1[u,v] = \int \left(\frac{1}{2} u^2 v + \gamma^{-1}(\gamma-1)^{-1} v^\gamma \right) dx.$$

(If $\gamma = 0$, or if $\gamma = 1$, which corresponds to Poisson's equation of nonlinear acoustics, then the Hamiltonian functional is slightly different. From now on, for simplicity, we leave these two special cases aside.)

The second Hamiltonian structure has

$$\mathcal{D}_2 = -\gamma^{-1} \begin{pmatrix} v^{\gamma-3}(2vD_x + (\gamma-2)v_x) & (\gamma-1)uD_x + u_x \\ (\gamma-1)uD_x + (\gamma-2)u_x & 2vD_x + v_x \end{pmatrix}, \quad H_0[u,v] = \int uvdx.$$

Finally, we have a third Hamiltonian form, with

$$\mathcal{D}_3 = -\begin{pmatrix} v^{\gamma-3}(2uvD_x+(\gamma-2)uv_x+vu_x) & [\frac{1}{2}(\gamma-1)u^2+\frac{2}{\gamma-1}\ v^{\gamma-1}]D_x+uu_x+v^{\gamma-2}v_x \\ [\frac{1}{2}(\gamma-1)u^2+\frac{2}{\gamma-1}\ v^{\gamma-1}]D_x+(\gamma-2)uu_x+v^{\gamma-2}v_x & 2uvD_x+(uv)_x \end{pmatrix},$$

and

$$H_{-1}[u,v] = (\gamma+1)^{-1}\int v\,dx.$$

In the latter two cases, the operators $\mathcal{D}_2$ and $\mathcal{D}_3$ are clearly skew-adjoint, but there is a nontrivial computation required to verify the Jacobi identity condition.

There are thus three recursion operators,

$$R_1 = \mathcal{D}_2\cdot\mathcal{D}_1^{-1},\quad R_2 = \mathcal{D}_3\cdot\mathcal{D}_1^{-1},\quad R_3 = \mathcal{D}_3\cdot\mathcal{D}_2^{-1},$$

but only two independent ones since

$$R_2 = R_3\cdot R_1.$$

However, interestingly, both R_1 and R_3 lead to the same hierarchy of flows and Hamiltonians, with R_3 mapping K_n in the R_1-hierarchy to K_{n+2}, so it only gives every other Hamiltonian. There is also a second hierarchy of Hamiltonian functionals and commuting flows, which starts with the functional

$$\tilde{H}_{-1}[u,v] = (\gamma+1)^{-1}\int u\,dx.$$

All the Hamiltonian functionals in both hierarchies depend only on u and v and do not involve derivatives of them. However, Verosky, [12], has found conservation laws which do depend on higher order derivatives, and corresponding generalized symmetries. It is unclear precisely how these fit into the given triHamiltonian framework, or whether there are additional Hamiltonian structures as yet undetermined.

Surely one of the reasons for the wealth of structure of the gas dynamics equation is that they can be linearized by the hodograph transformation. Recently Verosky has shown how certain three dimensional hyperbolic conservation laws, modelling gas dynamics with variable entropy, have similar higher order conservation laws. The Hamiltonian structure of these higher order systems is currently under investigation.

A second direction for generalization is given by Kupershmidt, [4], who provided a tri-Hamiltonian structure for a system of Boussinesq equations modelling shallow water wave motion; see also [11; Exercise 7.11].

3. Hamiltonian perturbation theory

BiHamiltonian systems appear to be very special, but surprisingly, they occur in numerous situations as model equations for more complicated physical systems. For example, the Korteweg-deVries equation is a model for shallow water waves, plasmas, elastic media, and many other physical situations where both nonlinear and dispersive effects are important. A possible explanation for this phenomenon was proposed in [10] using a form of non-canonical Hamiltonian perturbation theory. In outline, the method begins with a Hamiltonian system

$$u_t = \mathcal{D}\cdot\delta H,$$

which represents the original physical system, e.g. the water wave problem. The relevant models are derived by first introducing a small parameter ε into the problem (possibly by rescaling), and then substituting a (formal) perturbation expansion

$$u = v + \varepsilon\varphi(v) + \varepsilon^2\psi(v) + \ldots.$$

Truncating at a specified order in ε produces the desired model equation. If the perturbation expansion is not canonical, i.e. does not preserve the Poisson bracket induced by $\mathcal{D}$, then not only the Hamiltonian functional will have an expansion in powers of ε,

$$H = H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \ldots,$$

but also the Hamiltonian operator itself will have an expansion

$$\mathcal{D} \mapsto \mathcal{D}_0 + \varepsilon\mathcal{D}_1 + \varepsilon^2\mathcal{D}_2 + \ldots.$$

(It should be remarked that, owing to the nonlinearity of the Jacobi condition, it is not necessarily the case that the operators $\mathcal{D}_1$, $\mathcal{D}_2$, etc. are Hamiltonian, but this often happens.) Thus, to first order in ε, the model system takes the form

$$v_t = \mathcal{D}_0 \delta H_0 + \varepsilon\{\mathcal{D}_1 \delta H_0 + \mathcal{D}_0 \delta H_1\}, \tag{2}$$

which is not naturally Hamiltonian with respect to either $\mathcal{D}_0$ or $\mathcal{D}_1$.

However, in certain cases the order ε terms in (2) are proportional. When this happens, we have

$$\mathcal{D}_1 \delta H_0 = \lambda \mathcal{D}_0 \delta H_1 \tag{3}$$

for some constant λ, and so (2) is Hamiltonian with respect to $\mathcal{D}_0$. But, except for the inessential constant λ, this condition is precisely the biHamiltonian condition (1), and so (2) is a linear combination of the first two equations in the biHamiltonian hierarchy induced from (3). Thus, when the first order model equation arising from a noncanonical perturbation expansion of a Hamiltonian system happens also (in the above sense) to be Hamiltonian, then it is actually biHamiltonian, and hence completely integrable. Indeed, as shown in [10], this is precisely what happens for the Korteweg-deVries equation when derived from the water wave problem via the Boussinesq expansion for shallow water. Although this provides a possible mechanism for the appearance of soliton models for many physical systems, it does not explain all of them. For example, Kupershmidt's tri-Hamiltonian Boussinesq system, [4], arises from a canonical perturbation expansion of the water wave problem.

4. The Poisson complex

Associated with any Hamiltonian structure is a certain differential complex, called the Poisson complex, which plays as fundamental a role as the deRham complex does in the theory of differential forms. The Poisson complex, though, involves the dual objects to differential forms, which are known as multi-vectors, or, in the infinite-dimensional case, functional multi-vectors, cf. [11; Chapter 7]. Specifically, the space Λ_k of k-vectors consists of all smoothly varying k-linear alternating maps on the space T^*

of differential one-forms.

The most important operation on multi-vectors is the bracket introduced by Schouten, which generalizes the Lie bracket between vector fields. See [6] for the finite-dimensional version of the Schouten bracket and [10] for the generalization to functional multi-vectors. If Φ is a k-vector and Ψ an ℓ-vector, then their Schouten bracket $[\Phi,\Psi]$, is a $(k+\ell-1)$-vector. The Schouten bracket satisfies the following properties:

(i) Bilinearity: $[\Phi,\Psi]$ is a bilinear function of Φ and Ψ (with respect to multiplication by real constants).

(ii) Super-symmetry: $[\Phi,\Psi] = (-1)^{k\ell}\cdot[\Psi,\Phi]$, for Φ a k-vector and Ψ an ℓ-vector.

(iii) Super-Jacobi Identity: If Φ is a k-vector, Ψ an ℓ-vector and Θ an m-vector, then

$$(-1)^{km}\cdot[[\Phi,\Psi],\Theta] + (-1)^{\ell m}\cdot[[\Theta,\Phi],\Psi] + (-1)^{k\ell}\cdot[[\Psi,\Theta],\Phi] = 0.$$

(iv) Lie Derivative: If $\mathbf{v}$ is a vector field, i.e. a "uni-vector" ($k = 1$), and Ψ a k-vector, then the Schouten bracket $[\mathbf{v},\Psi]$ is also a k-vector, and coincides with the Lie derivative of Ψ with respect to $\mathbf{v}$. In particular, the Schouten bracket of two vector fields is the same as their Lie bracket.

Each bi-vector Θ determines an alternating, bilinear map on the space of one-forms, and hence a bilinear, skew-symmetric "bracket" on the space of real-valued function(al)s:

$$\{F,H\} \equiv \langle\Theta;\ dF,\ dH\rangle.$$

This bracket automatically satisfies the Leibniz rule, and hence to be a true Poisson bracket must only satisfy the additional restriction imposed by the Jacobi identity. This can be easily expressed in invariant form using the tri-vector $[\Theta,\Theta]$ obtained by bracketing Θ with itself:

$$\{\{F,H\},P\} + \{\{P,F\},H\} + \{\{H,P\},F\} = \tfrac{2}{3}\langle[\Theta,\Theta];\ dF,\ dH,\ dP\rangle.$$

Therefore, a bivector Θ determines a Poisson bracket if and only if

$$[\Theta,\Theta] = 0. \tag{4}$$

Let $\delta = \delta_\Theta$ be the map taking k-vectors to (k+1)-vectors defined by bracketing with the Poisson bivector Θ:

$$\delta(\Psi) = [\Theta,\Psi].$$

The determining property (4) along with the Jacobi identity for the Schouten bracket immediately implies that

$$\delta(\delta(\Psi)) = [\Theta,[\Theta,\Psi]] = 0$$

for any multi-vector Ψ. Therefore the maps δ determine a complex, called the <u>Poisson complex</u> corresponding to the Poisson bivector Θ:

$$0 \to \mathbb{R} \to \Lambda_0 \xrightarrow{\delta} \Lambda_1 \xrightarrow{\delta} \Lambda_2 \xrightarrow{\delta} \Lambda_3 \xrightarrow{\delta} \ldots,$$

meaning that the composition of two successive maps, $\delta\circ\delta = 0$, is always 0. This complex, in the finite-dimensional case, is due to Lichnerowicz, [6], and was generalized to infinite dimensions in Olver, [10].

The first stage of the Poisson complex, $\delta : \Lambda_0 \to \Lambda_1$, takes a function(al) H to the corresponding Hamiltonian vector field $\hat{\mathbf{v}}_H = \delta(H)$ determined by H (relative to the Poisson structure determined by Θ). Therefore, if a vector field $\mathbf{v}$ is a Hamiltonian vector field, then closure of the Poisson complex at the Λ_1-stage implies that

$$\delta(\mathbf{v}) = [\Theta,\mathbf{v}] = \mathbf{v}(\Theta) = 0. \tag{5}$$

If the Poisson complex is <u>exact</u> at the Λ_1-stage, then (5) is both necessary and sufficient for $\mathbf{v}$ to be a Hamiltonian vector field. In this case, (5) is a simple and readily verifiable condition that will tell whether or not a given vector field is Hamiltonian with respect to the given Poisson bracket.

Let us investigate in detail the infinite dimensional case of functional multi-vectors. We begin by determining the explicit form of the conditions (5).

<u>PROPOSITION:</u> Let

$$\Theta = \frac{1}{2}\int \{\vartheta \wedge \mathcal{D}\vartheta\}dx$$

be the functional bi-vector corresponding to the skew-adjoint (matrix) differential operator $\mathcal{D}$, cf. [11; page 431], and let $\mathbf{v}_K$ be an evolutionary vector field. Then

$$[\Theta, \mathbf{v}_K] = \mathbf{v}_K(\Theta) = \frac{1}{2}\int \{\vartheta \wedge E\vartheta\}dx,$$

where

$$E = \mathbf{v}_K(\mathcal{D}) - D_K\cdot\mathcal{D} - \mathcal{D}\cdot D_K^*.$$

(Here D_K denotes the Fréchet derivative of K, and D_K^* its adjoint.)

<u>PROOF:</u> We have,

$$\mathbf{v}_K(\Theta) = \frac{1}{2}\int \{\mathbf{v}_K(\vartheta) \wedge \mathcal{D}\vartheta + \vartheta \wedge \mathbf{v}_K(\mathcal{D})\vartheta + \vartheta \wedge \mathcal{D}[\mathbf{v}_K(\vartheta)]\}dx.$$

We need the formula

$$\mathbf{v}_K(\vartheta) = -D_K^*(\vartheta)$$

for the action of an evolutionary vector field on the basic uni-vectors ϑ, cf. [10]. Therefore

$$\mathbf{v}_K(\Theta) = \frac{1}{2}\int \{-D_K^*(\vartheta) \wedge \mathcal{D}\vartheta + \vartheta \wedge \mathbf{v}_K(\mathcal{D})\vartheta - \vartheta \wedge \mathcal{D}[D_K^*(\vartheta)]\}dx.$$

Integrating the first term by parts completes the demonstration.

<u>COROLLARY:</u> Let $\mathcal{D}$ be a Hamiltonian differential operator. If the evolutionary vector field $\mathbf{v}_K$ is Hamiltonian, meaning that

$$K = \mathcal{D}\cdot\delta H \tag{6}$$

for some Hamiltonian functional $H[u]$, then

$$D_K\cdot\mathcal{D} + \mathcal{D}\cdot D_K^* = \mathbf{v}_K(\mathcal{D}). \tag{7}$$

Conversely, if the Poisson complex corresponding to $\Theta = \frac{1}{2}\int \{\vartheta \wedge \mathcal{D}\vartheta\}dx$ is exact, then (7) is both necessary and sufficient for K to be of the Hamiltonian form (6).

The condition (7) has been isolated by Fokas and Fuchssteiner, [1], and called the condition for $\mathcal{D}$ to be a "Noether operator" for the evolution equation $u_t = K$. The connection with the Poisson complex is new.

5. A proof of Magri's theorem

Before embarking on a discussion of the exactness of the Poisson complex, let us show how it can provide a simple proof of Magri's theorem, without the extra hypothesis that each K_n be in the image of $\mathcal{D}$. Let $\Theta_{\mathcal{D}}$ and Θ_E be the bivectors corresponding to the two Hamiltonian operators $\mathcal{D}$ and E in the Hamiltonian pair. Since the three operators $\mathcal{D}$, E, and $\mathcal{D}+E$ are all required to be Hamiltonian, not only do we have the conditions

$$[\Theta_{\mathcal{D}},\Theta_{\mathcal{D}}] = 0, \quad \text{and} \quad [\Theta_E,\Theta_E] = 0,$$

on the individual bivectors, but also the compatibility condition

$$[\Theta_{\mathcal{D}},\Theta_E] = 0. \tag{8}$$

Assume exactness of the Poisson complex for the bivector $\Theta_{\mathcal{D}}$. The condition that the evolution equation (1) be biHamiltonian is equivalent to the fact that the corresponding evolutionary vector field $\mathbf{v}_1 \equiv \mathbf{v}_{K_1}$ can be written in the two forms

$$\mathbf{v}_1 = [\Theta_{\mathcal{D}},H_1] = [\Theta_E,H_0],$$

using the Schouten bracket. Let

$$\mathbf{v}_2 = [\Theta_E,H_1]$$

be the next evolutionary vector field in the presumed hierarchy. The main task is to prove that $\mathbf{v}_2$ is a Hamiltonian vector field for the operator $\mathcal{D}$, i.e.

$$\mathbf{v}_2 = [\Theta_{\mathcal{D}},H_2]$$

for some functional H_2. By exactness of the $\mathcal{D}$-Poisson complex, a necessary and sufficient condition is that

$$[\Theta_{\mathcal{D}}, \mathbf{v}_2] = 0.$$

To verify this latter condition, we find

$$[\Theta_{\mathcal{D}}, \mathbf{v}_2] = [\Theta_{\mathcal{D}}, [\Theta_{E}, H_1]].$$

By the super-Jacobi identity and the compatibility condition (8) for the Hamiltonian pair, we find

$$\begin{aligned}[\Theta_{\mathcal{D}}, [\Theta_{E}, H_1]] &= -[\Theta_{E}, [\Theta_{\mathcal{D}}, H_1]] \\ &= -[\Theta_{E}, [\Theta_{E}, H_o]] \\ &= 0,\end{aligned}$$

the last equality being a consequence of the closure of the E-Poisson complex. (Note that we do not require exactness of the E-Poisson complex.)

Thus, we have shown that if

$$\mathbf{v}_n = [\Theta_{\mathcal{D}}, H_n] = [\Theta_{E}, H_{n-1}]$$

is a biHamiltonian vector field, the same is true of

$$\mathbf{v}_{n+1} = [\Theta_{E}, H_n] = [\Theta_{\mathcal{D}}, H_{n+1}].$$

The inductive step is clear. This completes the proof of the first statement in Magri's theorem. The other two statements are proved in the usual manner, cf. [11; Theorem 7.24].

6. Exactness of the Poisson complex

In the finite dimensional case, the cohomology of the Poisson complex has been investigated by Lichnerowicz, [6]. The local cohomology is of particular interest, and is known when Θ has constant rank:

THEOREM: Let Θ be a Poisson bivector of constant rank, and let Ψ be a k-vector, $k \geq 1$. Then $\delta_\Theta(\Psi) = [\Theta,\Psi] = 0$ vanishes if and only if in any flat coordinate chart $\Psi = [\Theta,\Phi] + \Psi_0$, where Φ is a (k-1)-vector, and Ψ_0 is a k-vector which is constant on the leaves of the symplectic foliation of M induced by Θ. In particular, if Θ is of maximal rank, then the Poisson complex is locally exact, and the global cohomology equals the deRham cohomology of the manifold M.

In the maximal rank case, the interior product $\iota_\Theta = \iota: \omega \mapsto \omega \lrcorner \Theta$ sets up a linear isomorphism $\iota: T^*M|_x \xrightarrow{\sim} TM|_x$ between the cotangent and tangent spaces and thus, by wedging, between their exterior powers: $\Lambda_K\iota: \Lambda_K T^*M|_x \xrightarrow{\sim} \Lambda_K TM|_x$. This isomorphism maps the deRham complex on M to the Poisson complex, providing an immediate proof of the last statement of the theorem.

In the infinite dimensional case of functional multi-vectors, a similar proof of exactness works in the case of nondegenerate constant coefficient Hamiltonian operators $\mathcal{D}$. The passage to functional forms is effected by the introduction of a "potential function" w satisfying

$$u = \mathcal{D}[w].$$

Using the associated "substitution map", cf. [9], any differential function of u can be rewritten as a differential function depending on w (but of course, many differential functions of w do not correspond to local functions depending only on u and its derivatives).

The interior product $\iota: \omega \mapsto \omega \lrcorner \Theta$ is essentially implied by the basic substitution formula

$$du \mapsto \mathcal{D}\vartheta,$$

which converts a functional form into a functional multi-vector. Under this map, dw goes into ϑ itself, and we can invert the identification between functional multi-vectors and forms by mapping ϑ back to dw, and the coefficient differential functions of u to the corresponding differential functions of w. In this way, just as in finite dimensions, the Poisson complex gets mapped back to the variational (deRham) complex over M, cf. [11; Section 5.4]. Exactness follows immediately, but with one caveat; the expressions are all in terms of the potential w and not in terms of u, so it is unclear how we return to the multi-vector picture.

However, using the methods in [9], it is possible to show that the only obstructions to exactness are certain special linear differential functions of u.

<u>EXAMPLE:</u> For the Hamiltonian operator D_x, exactness of the Poisson complex at the Λ_1-stage is equivalent to the statement that

$$K[u] = D_x E_u(H) \tag{9}$$

for some differential function H (where E_u denotes the Euler operator or variational derivative with respect u) if and only if

$$D_K \cdot D_x + D_x \cdot D_K^* = 0,$$

where D_K is the Fréchet derivative of K with respect to u, cf. (7).

If we replace u by the potential w, where $w_x = u$, then, by standard change of variables formulae, these conditions become

$$K[w_x] = E_w(H) \tag{10}$$

for some differential function H if and only if

$$D_x \cdot \tilde{D}_K \cdot D_x - D_x \cdot \tilde{D}_K^* \cdot D_x = 0, \tag{11}$$

where $\tilde{D}_K$ is the Fréchet derivative of K with respect to w, related to D_K by the formula $D_K = \tilde{D}_K \cdot D_x$. Clearly (11) holds if and only if $\tilde{D}_K$ is self-adjoint, which, by Helmholtz' version of the inverse problem of the calculus of variations, [11; Theorem 5.68], implies that K satisfies (10). Moreover, since K only depends on w_x and higher order derivatives of w, it is easy to see, cf. [9], that the same is true of H, and hence (9) holds.

For the Hamiltonian operator D_x^3, the same argument applies, with potential $w_{xxx} = u$. We find that the equation corresponding to (10) is

$$K[w_{xxx}] = E_w(H).$$

In this case, however, it is not true that H can be taken to only depend on third and higher order derivatives of w. Using the methods of [9], it is possible to show that the only counterexample is that of $K_0[u] = (\alpha x+\beta)u_x + \alpha u$ for constants α, β, so $K_0[w_{xxx}] = (\alpha x+\beta)w_{xxxx} + \alpha w_{xxx}$, which is an Euler-Lagrange expression in w, but no Hamiltonian depending only on u and its derivatives can be found. Except for K_0, the D_x^3 Poisson complex is exact at the Λ_1-stage.

In general, one can prove the following theorem on exactness of the Poisson complex at the initial stage:

THEOREM: Let $\mathcal{D}$ be a nondegenerate, skew-adjoint, constant coefficient differential operator. Then, except for a finite dimensional space of linear differential functions, the Poisson complex for $\mathcal{D}$ is exact at the Λ_1-stage. More specifically, there exist linear differential functions $K_1[u],\ldots,K_n[u]$ such that an evolutionary vector field $\mathbf{v}_K$ satisfies the condition

$$D_K\cdot\mathcal{D} + \mathcal{D}\cdot D_K^* = 0$$

if and only if it is "almost" Hamiltonian, meaning that

$$K = \mathcal{D}\cdot E_u(H) + \alpha_1 K_1 + \ldots + \alpha_N K_N$$

for some Hamiltonian H[u], and some constants $\alpha_1,\ldots,\alpha_N$. (In fact, even the linear functions K_j are Hamiltonian provided one admits nonlocal Hamiltonian functionals depending on the potential w.)

A similar exactness result for the later Λ_k-stages, $k \geq 1$, of the Poisson complex should hold, but I have not tried to construct a proof. The extraneous linear terms cause no problem in the implementation of Magri's theorem for such operators.

In the more general case of field-dependent Hamiltonian operators $\mathcal{D}$, the potential method is no longer applicable, and much less is known. I conjecture that a similar theorem holds, but have not been able to prove it in general. However, in certain cases, one can directly prove the equivalence of the two conditions (6) and (7), and we close with one such example.

EXAMPLE: Consider the second Korteweg-deVries Hamiltonian operator

$$\mathcal{D} = D_x^3 + 2uD_x + u_x.$$

(For convenience, we have rescaled u.) Condition (7) reads

$$D_Q \cdot \mathcal{D} + \mathcal{D} \cdot D_Q^* = 2QD_x + Q_x, \qquad (12)$$

where $Q_x \equiv D_x Q$. The goal is to show that this implies that

$$Q = \mathcal{D}E_u(H)$$

for some H. Write

$$D_Q = \mathcal{D} \cdot L + M, \qquad (13)$$

where L and M are differential operators, with M of order ≤ 2. (This determines L and M uniquely.) Substituting into (12), we find the condition

$$\mathcal{D} \cdot (L - L^*) \cdot \mathcal{D} + M \cdot \mathcal{D} + \mathcal{D} \cdot M^* = 2QD_x + Q_x.$$

Considering the form of $\mathcal{D}$ and the fact that M is of order 2 or less, it is not hard to see that this identity can hold if and only if L is self adjoint, $L^* = L$, and

$$M \cdot \mathcal{D} + \mathcal{D} \cdot M^* = 2QD_x + Q_x. \qquad (14)$$

This latter equation immediately implies that M is of order 1, so

$$M = RD_x + S$$

for certain differential functions R and S. Substituting into (14), we find that

$$S = 2D_x R,$$

and

$$\mathcal{D}R = D_x^3 R + 2uR + u_x R = Q,$$

so Q is in the image of $\mathcal{D}$! Finally, recalling (13), we find that $L = D_R$ is just the Fréchet derivative of R, so its self-adjointness implies that (locally)

$$R = E_u(H)\cdot$$

for some H. This completes the proof of (12), and hence the exactness of the $\mathcal{D}$ Poisson complex at the Λ_1-stage.

The same technique works in many other special cases, but so far I have been unable to extend it to general Hamiltonian differential operators.

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