Dirac's theory of constraints in field theory and the canonical form of Hamiltonian differential operators

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A simple algorithm for constructing the canonical form of Hamiltonian systems of evolution equations with constant coefficient Hamiltonian differential operators is given. The result of the construction is equivalent to the canonical system derived using Dirac's theory of constraints from the corresponding degenerate Lagrangian.

I. INTRODUCTION

In the classical theory of Hamiltonian systems, great emphasis is placed on the introduction of canonical coordinates—the positions and conjugate momenta of classical mechanics.[1] Canonical coordinates serve to simplify many of the equations and transformations required in the study of finite-dimensional Hamiltonian systems. Most quantization procedures require that the Hamiltonian system be in canonical form before proceeding. Hamiltonian perturbation theories are much easier to develop in canonical coordinates.[2,3] However, in recent years there has been a renewed interest in Hamiltonian systems in noncanonical coordinates. The principle motivation has been the development of an infinite-dimensional theory of Hamiltonian systems of evolution equations in which the role of the skew-symmetric symplectic matrix J is played by a skew-adjoint Hamiltonian differential operator, and the Hamiltonian function is replaced by a Hamiltonian functional.[4,5] Applications to stability questions in fluid mechanics and plasma physics[6] and also to completely integrable (soliton) equations[7,8] have been just a few of the important consequences of this general theory. A significant open problem in this theory is the *Darboux problem* of whether one can always determine suitable canonical coordinates for such a Hamiltonian system. In this paper, a general result of this type for constant coefficient Hamiltonian differential operators is proved, along with some extensions of the result to more general field-dependent Hamiltonian operators.

In the case of finite-dimensional Hamiltonian systems, Darboux' theorem guarantees that canonical coordinates can always be found, provided that the Poisson bracket has constant rank.[9] For maximal rank (symplectic) Poisson brackets, the proof of Weinstein[10] is especially appealing in that it readily extends to certain infinite-dimensional situations. There are two main steps in Weinstein's proof: first the Hamiltonian operator is reduced to a constant operator by a clever change of variables; second, one shows that any constant-coefficient skew-adjoint operator can be placed into canonical form. In this light, the present paper can be viewed as an implementation of the second part of Weinstein's proof in the case of constant-coefficient skew-adjoint differential operators. The first part of the proof is far more difficult, and, unfortunately, the infinite-dimensional version of Darboux' theorem due to Weinstein does not appear to be applicable to the Hamiltonian differential operators of interest. The problem is that Weinstein requires some form of Banach manifold structure to effect his proof, but for differential operators that depend on the dependent variables it is not at all obvious how to impose such a structure. Even if one could mimic Weinstein's proof, the resulting changes of variable would be horribly nonlocal, and therefore be of limited use. Thus the question of whether Darboux' theorem is valid for Hamiltonian differential operators remains an important open problem. Only in special cases, including first- and third-order scalar operators, and some first-order matrix operators is the answer known.[11,26] (Results of Dubrovin and Novikov[12] indicate that Darboux' theorem may not hold for matrix operators involving more than one independent variable, but they only consider a limited class of changes of variable, so the general Darboux problem remains unanswered.)

The underlying motivation of this paper can be found in the recent applications of Dirac's theory of constraints by Nutku to produce canonical forms of a number of Hamiltonian systems of evolution equations of physical interest, including the equations of shallow water waves and gas dynamics[13] and the Korteweg–de Vries equation.[14] In the finite-dimensional theory of the calculus of variations, for nondegenerate Lagrangians the passage from the Euler–Lagrange equations to the corresponding canonical form of Hamilton's equations is classical.[1] Dirac's theory of constraints was designed to handle degenerate Lagrangians and produce canonical Hamiltonian systems, which, when subjected to the appropriate constraints, reduce to the original Euler–Lagrange equations.[15] In Nutku's applications of this theory, one begins with a Hamiltonian system of evolution equations, whose Poisson bracket is not in canonical form. The next step is to replace the original Hamiltonian system of evolution equations by an equivalent system of Euler–Lagrange equations; this appears to require that the Hamiltonian operator be constant coefficient. The resulting Lagrangian function is inevitably degenerate, so to construct a corresponding canonical Hamiltonian system one is required to invoke the Dirac machinery. The details of the construction can be found in Refs. 13 and 14.

However, given the fact that one begins with a (noncanonical) Hamiltonian system, the entire procedure seems to be a bit roundabout, and it would be useful to have a direct method of constructing canonical Hamiltonian systems from more general Hamiltonian evolution equations. In this paper a simple constructive procedure for effecting this transformation to canonical coordinates is presented. The only restriction is that the original Hamiltonian differential operator does not depend on the field variables or their derivatives; typically the operator will be a constant-coefficient, skew-adjoint differential operator, but explicit dependence on the spatial variables is also allowed. The method is illustrated with a number of examples, including elementary derivations of Nutku's Hamiltonians for gas dynamics and the Korteweg–de Vries equation. More general Hamiltonian operators are less easy to deal with directly. At present, the only recourse is to first determine a transformation that will place the operator in constant-coefficient form, and then apply the method described here.

II. HAMILTONIAN OPERATORS

For the basic theory of Hamiltonian systems of evolution equations, we refer the reader to the works of Gel'fand and Dorfman,[4] and the author.[5,16] We let $x = (x^1,...,x^p)$ denote the spatial variables, and $u = (u^1,...,u^q)$ the field variables (dependent variables), so each u^α is a function of $x^1,...,x^p$ and the time t. We will be concerned with autonomous systems of evolution equations

$$u_t = K[u] ,$$

in which $K[u] = (K_1[u],...,K_q[u])$ is a q-tuple of *differential functions*, where the square brackets indicate that each K_α is a function of x, u, and finitely many partial derivatives of each u^α with respect to $x^1,...,x^p$. A system of evolution equations is said to be *Hamiltonian* if it can be written in the form

$$u_t = \mathscr{D}\cdot E_u(H) . \tag{1}$$

Here $\mathscr{H}[u] = \int H[u]\, dx$ is the Hamiltonian functional, and the Hamiltonian function $H[u]$ depends on x, u, and the derivatives of the u's with respect to the x's; $E_u = (E_1,...,E_q)$ denotes the Euler operator or variational derivative with respect to u. The Hamiltonian operator $\mathscr{D}$ is a $q\times q$ matrix differential operator, which may depend on both x, u, and derivatives of u (but not on t), and is required to be (formally) skew-adjoint relative to the L^2-inner product $\langle f,g\rangle = \int f\cdot g\, dx = \int \Sigma f^\alpha \cdot g^\alpha\, dx$, so

$$\mathscr{D}^* = -\mathscr{D} ,$$

where * denotes the formal L^2 adjoint of a differential operator.[16] In addition, $\mathscr{D}$ must satisfy a nonlinear "Jacobi condition" that the corresponding Poisson bracket

$$\{\mathscr{P},\mathscr{Q}\} = \int E_u(P)\cdot\mathscr{D}E_u(Q)\, dx ,$$

$$\mathscr{P} = \int P[u]\, dx, \quad \mathscr{Q} = \int Q[u]\, dx ,$$

satisfies the Jacobi identity.[4,5,16] In the special case that $\mathscr{D}$ is a field-independent skew-adjoint differential operator, meaning that the coefficients of $\mathscr{D}$ do not depend on u or its derivatives (but may depend on x), the Jacobi conditions are automatically satisfied; for more general field-dependent operators, there is a nontrivial computation to be effected to determine whether or not it is genuinely Hamiltonian.

Since we will be using changes of variables, it is essential that we determine how they affect objects like Euler operators and Hamiltonian operators. The changes of variables to be considered here are of the form $u = Q[v]$, where $Q[v] = (Q_1[v],...,Q_q[v])$ is a q-tuple of differential functions, depending on the variables x, $v = (v^1,...,v^q)$ and derivatives of v with respect to x. Let D_Q denote the *Fréchet derivative* of Q with repect to v, which is the $q\times q$ matrix differential operator defined by the formula

$$D_Q(w) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} Q[v+\epsilon w], \quad w = (w^1,...,w^q) .$$

Alternatively, note that if $u = Q[v]$, then

$$u_t = D_Q[v_t] . \tag{2}$$

Let D_Q^* denote the (formal) L^2 adjoint of D_Q.

Proposition 1: Let $u = Q[v]$ be a change of variables. Then the variational derivatives with respect to u and v are related by the formula

$$E_v = D_Q^*\cdot E_u . \tag{3}$$

Proposition 2: Let $u_t = \mathscr{D}\cdot E_u(H)$ be a Hamiltonian system with Hamiltonian operator $\mathscr{D}$. Let $u = Q[v]$ be a change of variables. Then the corresponding Hamiltonian operator $\widetilde{\mathscr{D}}$ in the v variables is related to that in the u variables by the formula

$$D_Q\cdot\widetilde{\mathscr{D}}\cdot D_Q^* = \mathscr{D} . \tag{4}$$

The corresponding Hamiltonian system in the v variables is

$$v_t = \widetilde{\mathscr{D}}\cdot E_v(H) ,$$

in which we take the variational derivative of H with respect to v.

These results are special cases of an even more general theorem on how Euler operators and Hamiltonian operators behave under changes in both the independent and dependent variables.[11,17] Note that (4) follows easily from (2) and (3).

Example 3: Suppose $u(x,t)$ is scalar valued, $x\in\mathbb{R}$, and let $\varphi(x,t)$ be a potential function for u, so the change of variables is

$$u = Q[\varphi] = \varphi_x .$$

The corresponding Fréchet derivative is easily seen to be $D_Q = D_x$, with adjoint $D_Q^* = -D_x$. Therefore, by (3),

$$E_\varphi(H) = -D_x E_u(H) , \tag{5}$$

for any differential function H.

Similarly, if $\mathscr{D}$ is any Hamiltonian operator in the u variables, then the corresponding Hamiltonian operator in the v variables $\widetilde{\mathscr{D}}$ is related by the formula

$$D_x\cdot\widetilde{\mathscr{D}}\cdot(-D_x) = \mathscr{D} .$$

For example, consider the Harry Dym equation[7]

$$u_t = D_x^3(u^{-1/2}) , \tag{6}$$

which is in Hamiltonian form (1) with Hamiltonian operator

$$\mathscr{D} = D_x^3 ,$$

and Hamiltonian function

$$H = 2\sqrt{u} .$$

If we introduce a potential function $\varphi_x = u$, then the corresponding potential form of (6) is the equation

$$\varphi_t = D_x^2 (\varphi_x^{-1/2}) . \quad (7)$$

(Here, and elsewhere, we set the integration constants to 0 when introducing a potential function.) The Hamiltonian for (7) is just

$$\tilde{H} = H = 2\sqrt{\varphi_x} ,$$

and the Hamiltonian operator is $\mathscr{D} = -D_x$, since $D_x^3 = D_x \cdot (-D_x) \cdot (-D_x)$. Indeed, (7) is the same as the evolution equation

$$\varphi_t = \mathscr{D} \cdot E_\varphi (\tilde{H}) ,$$

as the reader can check.

III. THE GARDNER HAMILTONIAN OPERATOR

In order to simply illustrate the main ideas of the paper, we begin by discussing the elementary Hamiltonian operator $\mathscr{D} = D_x$, originally found by Gardner in connection with the Korteweg–de Vries equation.[18] Thus we are looking at a single evolution equation of the form

$$u_t = D_x \cdot E_u (H) , \quad (8)$$

in which $\mathscr{H} = \int H[u]\, dx$ is the corresponding Hamiltonian functional. We first show that any such Hamiltonian system can always be derived from a Lagrangian variational problem.[19]

Proposition 4: Let $u_t = D_x \cdot E_u(H)$ be a Hamiltonian evolution equation relative to the Hamiltonian operator D_x. Let $\varphi(x,t)$ be the potential of $u(x,t)$, so $\varphi_x = u$. Then the Hamiltonian evolution equation is equivalent to the Euler–Lagrange equation for the variational problem $\mathscr{L} = \int L[\varphi]\, dx$ with Lagrangian

$$L[\varphi] = \varphi_x \varphi_t - 2H[\varphi_x] . \quad (9)$$

Proof: Formula (5) immediately implies that the Euler–Lagrange equation for $\mathscr{L}$ is

$$E_\varphi (L) = -2\varphi_{xt} - 2E_\varphi (H) = -2\{u_t - D_x E_u (H)\} = 0 ,$$

which coincides with a multiple of the original Hamiltonian system (8).

We now apply Dirac's theory of constraints to the Lagrangian (9) as explained in Nutku.[13,14] The Lagrangian is degenerate, and the first constraint should be determined by

$$c_1 = \pi - \frac{\partial L}{\partial \varphi_t} = \pi - \varphi_x = \pi - u = 0 ,$$

in which π will be the canonical momentum dual to φ. As shown by Nutku, this constraint is second class in the terminology of Dirac, and so to derive the further constraints we need to investigate the canonical Poisson brackets of the constraint with the Hamiltonian.

In the version of the Dirac theory used by Nutku, the Lagrangian is required to only depend on first-order derivatives of the potential φ. This is equivalent to the fact that the Hamiltonian $H = H(x,u)$ depends only on x and u, and not any derivatives of u, so that the Hamiltonian system (8) is a simple nonlinear wave equation

$$u_t = [H_u (x,u)]_x = H_{xu} (x,u) + H_{uu} (x,u) \cdot u_x .$$

The corresponding potential form is the equation

$$\varphi_t = H_u (x,\varphi_x) .$$

The Lagrangian (9) for this equation is

$$L = \varphi_x \varphi_t - 2H(x,\varphi_x) .$$

Therefore, provided there are no further constraints coming from the Poisson brackets of the constraint with the Hamiltonian, the total Hamiltonian has the form

$$H^* = 2H(x,\varphi_x) + \lambda(\pi - \varphi_x) ,$$

where the multiplier λ remains to be determined. [In the notation of Ref. 13, the free part of the Hamiltonian has been determined as

$$H_0 = \varphi_t \pi - L = \varphi_t \pi - \varphi_x \varphi_t + 2H(x,\varphi_x) = 2H(x,\varphi_x).]$$

Using the canonical Poisson bracket relations[14]

$$\{\varphi(x),\pi(x')\} = \delta(x - x') ,$$

δ being the Dirac delta function, we find

$$\{c_1(x),c_1(x')\} = -2\delta'(x - x') .$$

Therefore

$$\{c_1(x),H^*(x')\} = 2[H_u(x',\varphi_x(x')) - \lambda] \cdot \delta'(x - x') ,$$

from which we see that $\lambda = H_u(x,\varphi_x)$ is required in order to make the Poisson bracket vanish. Thus the total Hamiltonian is

$$H^*[\varphi,\pi] = \pi \cdot H_u(x,\varphi_x) + 2H(x,\varphi_x) - H_u(x,\varphi_x) \cdot \varphi_x . \quad (10)$$

The canonical equations corresponding to H^*, which are

$$\varphi_t = E_\pi [H^*] = H_u (x,\varphi_x) ,$$
$$\pi_t = -E_\varphi [H^*]$$
$$= D_x \{(\pi - \varphi_x) H_{uu} (x,\varphi_x) + H_u (x,\varphi_x)\} ,$$

are easily seen to reduce to the original wave equation when subjected to the constraint $\pi = u$.

The goal now is to generalize this construction to Hamiltonian functions which depend on higher-order derivatives of the field variable u. Rather than try to follow through the complete derivation using the Dirac theory, as in Ref. 14, we proceed directly to the general result. In order to state it, we need to introduce the multiplication operator

$$N = u \frac{\partial}{\partial u} + u_x \frac{\partial}{\partial u_x} + u_{xx} \frac{\partial}{\partial u_{xx}} + \cdots ,$$

whose action on differential functions is to multiply each term by its algebraic degree in u and its derivatives. For example,

$$N(u_{xx} + xu^2 u_x + u^5) = u_{xx} + 3xu^2 u_x + 5u^5 .$$

Theorem 5: Let $u_t = D_x \cdot E_u (H)$ be a Hamiltonian system with Hamiltonian operator D_x. Then the corresponding canonical Hamiltonian system has total Hamiltonian

$$H^*[\pi,\varphi] = \pi \cdot E_u (H) + (2 - N) H , \quad (11)$$

in which φ is the potential for u, π the corresponding momentum, and u is to be replaced by φ_x on the right-hand side of (11). The corresponding canonical Hamiltonian system for H^* takes the form

$$\varphi_t = E_\pi (H^*), \quad \pi_t = -E_\varphi (H^*) , \quad (12)$$

and, when subjected to the constraint $\pi = u = \varphi_x$, is equivalent to the original Hamiltonian system.

For example, in the case that $H = H(x,u)$ just depends on u, then (11) reduces to the formula (10) derived using the Dirac theory.

Proof: It suffices to check that when $\pi = u$, the pair of evolution equations in (12) reduce to the original evolution equation (8). The first one is easy, since $E_\pi(H^*) = E_u(H)$, and so we just derive the potential form $\varphi_t = E_u(H)$ of the original equation. For the second, we require a lemma of Olver and Shakiban.[20]

Lemma 6: Let $u(x)$ be real valued, and let $L[u]$ be any differential function. Then

$$E_u(u \cdot E_u(L)) = (N+1)E_u(L) . \tag{13}$$

[Indeed, if P is a differential polynomial, then the condition $E_u(u \cdot P) = (N+1)P$ is both necessary and sufficient that $P = E_u(L)$ be the Euler–Lagrange expression for some Lagrangian L.]

Corollary 7: Let $u(x)$ and $\pi(x)$ be real-valued functions, and $L[u]$ any differential function depending only on u and its derivatives. Then

$$E_u(\pi \cdot E_u(L))|_{\pi = u} = N[E_u(L)] = E_u((N-1)L) . \tag{14}$$

Proof: The second equality is clear since the Euler operator E_u reduces the algebraic degree of a differential function by 1. To prove the first, we use the well-known formula for the Euler operator

$$E_u = \sum_{n=0}^{\infty} (-D_x)^n \cdot \frac{\partial}{\partial u_n} ,$$

where $u_n = \partial^n u / \partial x^n$. Therefore

$$E_u(u \cdot E_u(L)) = E_u(L) + \sum_n (-D_x)^n \left\{ u \cdot \frac{\partial E_u(L)}{\partial u_n} \right\} .$$

On the other hand, since the restriction to $\pi = u$ commutes with the operation of total differentiation D_x (but not with the partial derivatives $\partial / \partial u_n$), the left-hand side of (14) equals

$$\sum_n (-D_x)^n \left\{ \pi \cdot \frac{\partial E_u(L)}{\partial u_n} \right\} \bigg|_{\pi = u} = \sum_n (-D_x)^n \left\{ u \cdot \frac{\partial E_u(L)}{\partial u_n} \right\} .$$

The equivalence of (14) and (13) is now clear.

Returning to the proof of the theorem, we only need compute

$$E_\varphi(H^*) = E_\varphi\{\pi \cdot E_u(H) + (2-N)H\} = -D_x \cdot E_u\{\pi \cdot E_u(H) + (2-N)H\} ,$$

cf. (5), and restrict to $\pi = u$. According to (14), this equals

$$E_\varphi(H^*)|_{\pi = u} = -D_x \cdot E_u\{(N-1)H + (2-N)H\} = -D_x \cdot E_u(H) ,$$

which explains the factor $(2-N)$ in the formula (11) for the total Hamiltonian. Therefore, when restricted to the constraint $\pi = u$, the second evolution equation in (12) becomes

$$u_t = -E_\varphi(H^*)|_{\pi = u} = D_x \cdot E_u(H) ,$$

which is the same as the original Hamiltonian system! This completes the proof.

Example 8: Consider the evolution equation

$$u_t = u_x + u_{xxx} + u u_x + u^2 u_x , \tag{15}$$

which is a combination of the Korteweg–de Vries and modified Korteweg–de Vries equations. This is in Hamiltonian form (8), with Hamiltonian

$$H = \tfrac{1}{2} u^2 - \tfrac{1}{2} u_x^2 + \tfrac{1}{6} u^3 + \tfrac{1}{12} u^4 .$$

Note that

$$E_u(H) = u + u_{xx} + \tfrac{1}{2} u^2 + \tfrac{1}{3} u^3 ,$$

while

$$(N-2)H = \tfrac{1}{6} u^3 + \tfrac{1}{6} u^4 .$$

Therefore the total Hamiltonian (11) is

$$H^* = \pi(u + u_{xx} + \tfrac{1}{2} u^2 + \tfrac{1}{3} u^3) - \tfrac{1}{6} u^3 - \tfrac{1}{6} u^4 = \pi(\varphi_x + \varphi_{xxx} + \tfrac{1}{2}\varphi_x^2 + \tfrac{1}{3}\varphi_x^3) - \tfrac{1}{6}\varphi_x^3 - \tfrac{1}{6}\varphi_x^4 .$$

The corresponding canonical Hamiltonian system is

$$\varphi_t = E_\pi(H^*) = \varphi_x + \varphi_{xxx} + \tfrac{1}{2}\varphi_x^2 + \tfrac{1}{3}\varphi_x^3 ,$$
$$\pi_t = -E_\varphi(H^*) = \pi_x + \pi_{xxx} + \pi_x \varphi_x + \pi \varphi_{xx} + \pi_x \varphi_x^2 + 2\pi\varphi_x\varphi_{xx} - \varphi_x\varphi_{xx} - 2\varphi_x^2\varphi_{xx} . \tag{16}$$

The first is just the potential form of the original equation (15); restricting to $\pi = \varphi_x = u$, the second reduces to (15) identically. Thus we are justified in labeling (16) as the canonical form of the modified Korteweg–de Vries equation (15). If the last term in (15) does not appear, we are back to the Korteweg–de Vries equation as treated by Nutku.[14]

IV. CANONICAL FORMS AND FACTORIZATIONS OF HAMILTONIAN OPERATORS

Theorem 5 readily generalizes to systems of evolution equations which are in field-independent Hamiltonian form

$$u_t = \mathscr{D} \cdot E(H) , \tag{17}$$

in which the Hamiltonian H depends on $x = (x^1,...,x^p)$, $u = (u^1,...,u^q)$, and the derivatives of the u's with respect to the x's. The corresponding Lagrangian form of such a system is written in terms of the "potential" $\psi = (\psi^1,...,\psi^q)$, satisfying $\mathscr{D}\psi = u$. The Lagrangian function is

$$L[\psi] = (\mathscr{D}\psi) \cdot \psi_t - 2H ,$$

in which $\mathscr{D}\psi$ is to be substituted for u in H. Using the change of variables formula (3), which is

$$E_\psi(H) = \mathscr{D}^* \cdot E_u(H) = -\mathscr{D} \cdot E_u(H)$$

(the second equality following from the skew-adjointness of $\mathscr{D}$), we easily check that the Euler–Lagrange equations $E_\psi(L) = 0$ for L are the same as the Hamiltonian system (17).

As it turns out, for each possible *factorization*,

$$\mathscr{D} = \mathscr{D}_1 \cdot \mathscr{D}_2 , \tag{18}$$

of the differential operator $\mathscr{D}$ into the product of two differential operators $\mathscr{D}_1$ and $\mathscr{D}_2$, there is a corresponding canonical Hamiltonian system that reduces to (17). Either $\mathscr{D}_1$ or $\mathscr{D}_2$ can be the identity operator, in which case the other one coincides with $\mathscr{D}$, but this is not the only possible choice in (18). Once a factorization has been chosen, we define canonically conjugate "positions" $\varphi = (\varphi^1,...,\varphi^q)$ and momenta $\pi = (\pi^1,...,\pi^q)$ by the equations

$$\mathscr{D}_1\varphi = u, \quad \mathscr{D}_2^*\pi = u\,,$$

where $\mathscr{D}_2^*$ is the adjoint of $\mathscr{D}_2$. Thus, the choice of $\mathscr{D}_1$ and $\mathscr{D}_2$ might be determined on physical grounds as to which variables might reasonably be labeled "position" or "momentum"; however, from a mathematical point of view, any choice of $\mathscr{D}_1$ and $\mathscr{D}_2$ satisfying (18) is allowable.

We also need the general multiplication operator

$$N = \sum u_J^\alpha \frac{\partial}{\partial u_J^\alpha}, \quad u_J^\alpha = \frac{\partial^n u^\alpha}{\partial x^{j_1} \cdots \partial x^{j_n}}\,,$$

the sum being over all $\alpha = 1,...,q$ and all multi-indices $J = (j_1,...,j_n)$, $n \geqslant 0$, $1 \leqslant j_\nu \leqslant p$, corresponding to all possible derivatives of the u's. The effect of N is, as before, to multiply a monomial by its algebraic degree in the u's and their derivatives. With this definition, Lemma 6 has an immediate generalization due to Shakiban.[21] In this case, formula (13) still holds, with $u \cdot E_u(H) \equiv \Sigma u^\alpha \cdot E_\alpha(H)$.

Theorem 9: Consider a Hamiltonian system of evolution equations $u_t = \mathscr{D} \cdot E(H)$, in which the Hamiltonian operator $\mathscr{D}$ is a skew-adjoint $q \times q$ matrix differential operator, whose coefficients do not depend on u or their derivatives. Let $\mathscr{D} = \mathscr{D}_1 \cdot \mathscr{D}_2$ be any factorization of $\mathscr{D}$ as a product of two differential operators. Define canonically conjugate variables φ and π by the equations $\mathscr{D}_1\varphi = u$, $\mathscr{D}_2^*\pi = u$. Define the total Hamiltonian

$$H^*[\varphi,\pi] = (\mathscr{D}_2^*\pi) \cdot E_u(H) + (2-N)H\,, \tag{19}$$

in which one substitutes $\mathscr{D}_1\varphi$ for u wherever it occurs on the right-hand side of (19). Then the original Hamiltonian system is equivalent to the canonical Hamiltonian system

$$\varphi_t = E_\pi(H^*), \quad \pi_t = -E_\varphi(H^*)\,, \tag{20}$$

when subjected to the constraints

$$\mathscr{D}_1\varphi = u = \mathscr{D}_2^*\pi\,. \tag{21}$$

Proof: The first canonical equation is easy; we find it has the form

$$\varphi_t = E_\pi(H^*) = \mathscr{D}_2 E_u(H)\,,$$

evaluated at $u = \mathscr{D}_1\varphi$. Applying the operator $\mathscr{D}_1$ to both sides of this equation, we recover the original Hamiltonian system since $\mathscr{D} = \mathscr{D}_1 \cdot \mathscr{D}_2$. For the second canonical system, we require the identity

$$E_u\{(\mathscr{D}_2^*\pi) \cdot E_u(H)\}|_{u=\mathscr{D}_2^*\pi} = E_u[(N-1)H]\,,$$

which follows from formula (13) (in the general case) just as (14) did before. Therefore, evaluating the canonical equation

$$\pi_t = -E_\varphi(H^*) = -E_\varphi\{(\mathscr{D}_2^*\pi) \cdot E_u(H) + (2-N)H\}$$

on the constraint $u = \mathscr{D}_2^*\pi$, we find, using (3),

$$\pi_t = -\mathscr{D}_1^* \cdot E_u[(N-1)H + (2-N)H]$$
$$= -\mathscr{D}_1^* \cdot E_u(H)\,.$$

Finally applying $\mathscr{D}_2^*$ to this system, we recover

$$u_t = \mathscr{D}_2^* \cdot \mathscr{D}_1^* \cdot E_u(H) = -\mathscr{D}^* \cdot E_u(H) = \mathscr{D} \cdot E_u(H)\,,$$

since $\mathscr{D}$ is skew-adjoint. Thus the second canonical equation, when evaluated on the constraints, is equivalent to the original Hamiltonian system, and Theorem 9 is proven.

Example 10: Consider the equations of gas dynamics for a polytropic gas

$$u_t + uu_x + v^{\gamma-2}v_x = 0\,,$$
$$v_t + uv_x + vu_x = 0\,,$$

the case $\gamma = 2$ also covering the equations of shallow water wave motion.[13] These are in Hamiltonian form

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \mathscr{D} \begin{pmatrix} E_u(H) \\ E_v(H) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix} \begin{pmatrix} -uv \\ -\frac{1}{2}u^2 - (\gamma-1)^{-1}v^{\gamma-1} \end{pmatrix},$$

with Hamiltonian function

$$H[u,v] = -\tfrac{1}{2}u^2v - \{\gamma(\gamma-1)\}^{-1}v^\gamma\,.$$

Let $\varphi_x = u$, $\psi_x = v$ be the corresponding potentials, with $\pi = v, \rho = u$, the canonically conjugate momenta. The reader can see that this corresponds to the factorization (18) in which $\mathscr{D}_1 = D_x$, and $\mathscr{D}_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Note that

$$(2-N)H = \frac{1}{2}u^2v + \frac{\gamma-2}{\gamma(\gamma-1)}v^\gamma\,.$$

Therefore, according to (19), the corresponding canonical total Hamiltonian is

$$H^*[\varphi,\psi,\pi,\rho] = -\left\{\frac{1}{2}u^2 + \frac{1}{(\gamma-1)}v^{\gamma-1}\right\}\pi - uv\rho$$
$$+ \frac{1}{2}u^2v + \frac{\gamma-2}{\gamma(\gamma-1)}v^\gamma$$
$$= -\left\{\frac{1}{2}\varphi_x^2 + \frac{1}{(\gamma-1)}\psi_x^{\gamma-1}\right\}\pi - \varphi_x\psi_x\rho$$
$$+ \frac{1}{2}\varphi_x^2\psi_x + \frac{\gamma-2}{\gamma(\gamma-1)}\psi_x^\gamma\,.$$

This is the same as that derived by Nutku,[13] but the derivation here is far more straightforward. The Hamiltonian system

$$u_t = E_\pi(H^*), \quad v_t = E_\rho(H^*)\,,$$
$$\pi_t = -E_u(H^*), \quad \rho_t = -E_v(H^*)\,,$$

when subjected to the constraints

$$\varphi_x = \rho = u, \quad \psi_x = \pi = v\,,$$

is easily seen to be equivalent to the original system.

There are, of course, other possible factorizations of the Hamiltonian operator $\mathscr{D}$, and these lead to different canonical total Hamiltonians. For example, if we choose $\mathscr{D}_1$ to be the identity operator, while $\mathscr{D}_2 = \mathscr{D}$, then the velocities u,v are the canonical "positions," while the conjugate momenta π,ρ are related by $\pi_x = -v$, $\rho_x = -u$. In this case formula (19) gives the total Hamiltonian as

$$H^*[\varphi,\psi,\pi,\rho] = -uv\pi - \left\{\frac{1}{2}u^2 + \frac{1}{(\gamma-1)}v^{\gamma-1}\right\}\rho$$
$$+ \frac{1}{2}u^2v + \frac{\gamma-2}{\gamma(\gamma-1)}v^\gamma\,,$$

and the gas dynamic equations are equivalent to the canonical system

$$u_t = E_\pi(H^*), \quad v_t = E_\rho(H^*),$$
$$\pi_t = -E_u(H^*), \quad \rho_t = -E_v(H^*),$$

when subject to the constraints $\pi_x = -v$, $\rho_x = -u$, as can be easily checked.

This last remark indicates that there are other possible canonical formulations of the Korteweg–de Vries example (15) above. The procedure of example 8 amounts to choosing the factorization (18) with $\mathscr{D}_2$ the identity operator. If, on the other hand, we were to choose $\mathscr{D}_1$ to be the identity, then we would have canonically conjugate variables u and π, with $\pi_x = -u$, and total Hamiltonian

$$H^* = \pi(u + u_{xx} + \tfrac{1}{2}u^2 + \tfrac{1}{3}u^3) - \tfrac{1}{6}u^3 - \tfrac{1}{6}u^4.$$

While simpler than the Hamiltonian found above, this is not the version prescribed by the Dirac theory. It is, however, related to the Dirac Hamiltonian by a canonical transformation.

Example 11: For a higher-order example, consider the Harry Dym equation

$$u_t = D_x^3(u^{-1/2}), \tag{22}$$

which is in Hamiltonian form (17) with

$$\mathscr{D} = D_x^3, \quad H = 2\sqrt{u}.$$

If we choose $\mathscr{D}_1 = D_x^2$, $\mathscr{D}_2 = D_x$, so that $\varphi_{xx} = u$, $\pi_x = -u$ are conjugate variables, then the total Hamiltonian is

$$H^* = -\pi_x \varphi_{xx}^{-1/2} + 3\varphi_{xx}^{1/2},$$

with the canonical system (20) equivalent to the Harry Dym equation when subjected to the constraints $\varphi_{xx} = u$, $\pi_x = -u$.

Alternatively, we can choose $\mathscr{D}_1$ to be the identity, so u and π are conjugate, where $\pi_{xxx} = -u$, in which case

$$H^* = -\pi_{xxx}\cdot u^{-1/2} + 3u^{1/2}$$

is the total Hamiltonian. Other factorizations are also possible.

V. FIELD-DEPENDENT HAMILTONIAN OPERATORS

If the Hamiltonian operator depends explicitly on the dependent variables u, or their derivatives, then the above theory does not appear to be directly applicable. Indeed, a significant outstanding problem in the subject is whether some version of Darboux' theorem is true for all Hamiltonian differential operators, i.e., given a Hamiltonian differential operator, is it always possible to find canonical coordinates? The only case that has been completely answered to date is the case of first-order scalar differential operators in one independent variable.[11] In this case, provided one admits differential substitutions,[22] which change both the independent and dependent variables in the problem, one can always reduce such an operator to constant coefficient form, and hence, using the methods of this paper, to canonical form. The proof, however, is constructive, and does not appear to easily generalize to either higher-order or matrix operators, so the general "Darboux problem" for differential operators remains open. See, also, Ref. 26.

If one *can* reduce a Hamiltonian operator to constant coefficient form using some change of variables, then the methods discussed above are applicable, and canonical coordinates can always be found. In the case of bi-Hamiltonian systems,[4,7] or even multi-Hamiltonian systems,[23,24] this opens up the possibility of several different systems of canonical variables, which are *not* related to each other by canonical transformations. The implications of this phenomenon for quantization theory or perturbation theory remain to be developed. Here we just present a few examples to illustrate the main ideas.

Example 12: The Harry Dym equation (22) has a second Hamiltonian structure,[7] with first-order Hamiltonian operator

$$\widetilde{\mathscr{D}} = 2uD_x + u_x,$$

and Hamiltonian function

$$\widetilde{H} = \tfrac{1}{8}u^{-5/2}\cdot u_x^2.$$

Using the results in Ref. 11, or by direct inspection, we see that the transformation

$$u = \tfrac{1}{2}v^2$$

transforms $\widetilde{\mathscr{D}}$ into the constant-coefficient operator D_x; indeed

$$D_Q\cdot D_x\cdot D_Q^* = v\cdot D_x\cdot v = v^2 D_x + vv_x = 2uD_x + u_x = \widetilde{\mathscr{D}}.$$

In terms of v,

$$\widetilde{H} = 2^{-1/2}v^{-3}v_x^2,$$

and

$$E_v(\widetilde{H}) = \sqrt{2}(-v^{-3}v_{xx} + \tfrac{3}{2}v^{-4}v_x^2).$$

Therefore, using Theorem 9, the canonical total Hamiltonian is

$$\widetilde{H}^*[\varphi,\pi] = \sqrt{2}\,\{\pi(-\varphi_x^{-3}\varphi_{xxx} + \tfrac{3}{2}\varphi_x^{-4}\varphi_{xx}^2) + \tfrac{1}{2}\varphi_x^{-3}\varphi_{xx}^2\},$$

where $\varphi_x = v$, $\pi = v$ are the canonically conjugate variables. The reader can check that the canonical Hamiltonian system (20) for $\widetilde{H}^*$, when subjected to the constraints $u = \frac{1}{2}\varphi_x^2 = \frac{1}{2}\pi^2$, coincides with the Harry Dym equation (22). Thus we have constructed a second, inequivalent, canonical form for this equation.

Example 13: As a final example, consider the Korteweg–de Vries equation

$$u_t = u_{xxx} + uu_x.$$

The first Hamiltonian structure was considered in example 8. There is also a second Hamiltonian structure,[7] with Hamiltonian operator

$$\widetilde{\mathscr{D}} = D_x^3 + \tfrac{2}{3}uD_x + \tfrac{1}{3}u_x,$$

and Hamiltonian function

$$\widetilde{H} = \tfrac{1}{2}u^2.$$

According to Kupershmidt and Wilson,[8] the second Hamiltonian operator for the Korteweg–de Vries equation can be put into constant-coefficient form D_x by the Miura transformation[25]

$$u = v_x - \tfrac{1}{6} v^2 ,$$

which has the effect of transforming the Korteweg–de Vries equation into the modified Korteweg–de Vries equation

$$v_t = v_{xxx} - \tfrac{1}{6} v^2 v_x .$$

Indeed,

$$\begin{aligned} D_Q \cdot D_x \cdot D_Q^* &= (D_x - \tfrac{1}{3} v) \cdot D_x \cdot (- D_x - \tfrac{1}{3} v) \\ &= - D_x^3 - (\tfrac{2}{3} v_x - \tfrac{1}{9} v^2) D_x - (\tfrac{2}{3} v_{xx} - \tfrac{1}{9} v v_x) \\ &= - D_x^3 - \tfrac{2}{3} u D_x - \tfrac{1}{3} u_x = - \mathscr{D} . \end{aligned}$$

Thus, using (19), we obtain the canonical total Hamiltonian

$$\widetilde{H}^*[\varphi,\pi] = \pi(\varphi_{xxx} - \tfrac{1}{18} \varphi_x^3) + \tfrac{1}{36} \varphi_x^4 ,$$

where $\varphi_x = \pi = v$. In this case, we obtain a second canonical representation of the Korteweg–de Vries equation corresponding to the canonical Hamiltonian system for $\widetilde{H}^*$ subject to the constraints[26]

$$u = \varphi_{xx} - \tfrac{1}{6} \varphi_x^2 = \pi_x - \tfrac{1}{6} \pi^2 .$$

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