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A Nonlinear Hamiltonian Structure for the Euler Equations

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The Euler equations for inviscid incompressible fluid flow have a Hamiltonian structure in Eulerian coordinates, the Hamiltonian operator, though, depending on the vorticity. Conservation laws arise from two sources. One parameter symmetry groups, which are completely classified, yield the invariance of energy and linear and angular momenta. Degeneracies of the Hamiltonian operator lead in three dimensions to the total helicity invariant and in two dimensions to the area integrals reflecting the point-wise conservation of vorticity. It is conjectured that no further conservation laws exist, indicating that the Euler equations are not completely integrable, in particular, do not have *soliton-like* solutions.

1. INTRODUCTION

The discovery of a number of remarkable model nonlinear-wave equations, the Korteweg–de Vries equation being the prototypical example, has stimulated a resurgence of interest in infinite-dimensional Hamiltonian systems. Only recently, in the work of Gel'fand and Dorfman [12] and the author [18], has the proper characterization of a nonlinear–Hamiltonian structure in this context been formulated. Of particular importance for the present investigation is the consequent establishment of a general Noether relationship between symmetries of the evolutionary system and conservation laws, even when the explicit dependence of the Hamiltonian operator on the dependent variables complicates the direct association of a suitable variational principle.

The primary purpose of this paper is to show that the Euler equations of inviscid, incompressible-fluid flow can, in their natural Eulerian coordinates, be put into Hamiltonian form, and to discuss the consequences from the view point of symmetry group theory. A group-theoretic understanding of known conservation laws is the immediate benefit of this approach; further, more speculative conclusions on the integrability of the equations and the interactive properties of waves can, pending the completion of further investigations, be drawn. This approach differs from the Hamiltonian formulation of Arnol'd [2] which is in Lagrangian (moving) coordinates and

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reduces the Euler equations to the equations for the geodesic flow on an infinite-dimensional group of volume-preserving diffeomorphisms. The precise interrelationship between these two Hamiltonian structures deserves further investigation.

As any treatise on hydrodynamics, e.g., [14], will explain the motion of an inviscid, incompressible-ideal fluid is governed by the system of equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.2}$$

first obtained by Euler. Here $\mathbf{u} = (u, v, w)$ are the components of the three-dimensional velocity field and p the pressure of the fluid at a position $\mathbf{x} = (x, y, z)$. Our considerations will also apply to two-dimensional motions, where $\mathbf{u} = (u, v)$ and $\mathbf{x} = (x, y)$. For simplicity, the case of a fluid of infinite extent is treated here, although typical boundary conditions, e.g., fluid motion in a bounded container, can be incorporated with minimal difficulty into the general theory. Other generalizations to flow on Riemannian manifolds [11] or in higher dimensions offer no additional complications, although the classification of symmetries and conservation laws does depend on the specific geometry and boundary conditions. Suffice it to say that none of the above constraints can possibly enlarge the basic lists of symmetries and conservation laws, and, in most cases, will radically deplete them. Thus the flat two- and three-dimensional problems for an infinite fluid constitute the optimal settings for these kinds of results.

A system of partial differential equations is *Hamiltonian* if it can be written in the form

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \mathcal{D} E(H), \tag{1.3}$$

where $\mathcal{D}$ is a skew-adjoint matrix of (pseudo-) differential operators, $H(\boldsymbol{\omega})$ is the Hamiltonian functional and E denotes the Euler operator or variational derivative with respect to $\boldsymbol{\omega}$ (These considerations will be somewhat formal since the Sobolev subspace of L^2 under consideration will not be precisely specified here. See Ebin and Marsden [11] for questions of existence and uniqueness for the Euler equations.) In addition, if $\mathcal{D}$ explicitly depends on the function $\boldsymbol{\omega}$ and its derivatives, a closure condition on an associated symplectic form must also hold. This condition is the infinite-dimensional analogue of Darboux' theorem giving necessary and sufficient conditions for a variable skew-symmetric matrix to be equivalent via a change of coordinates to the standard sympletic matrix, cf. [25]. In infinite dimensions, however, degeneracies of $\mathcal{D}$ preclude any easily defined change of coordinates to a standard form.

As they stand, the Euler equations (1.1)–(1.2) are not in Hamiltonian form owing to the lack of an equation explicitly governing the time evolution of the pressure. Arnold's strategy to obviate this difficulty was to project each term of the system (1.1) onto a canonically chosen divergence-free representative, thereby eliminating the pressure terms at the expense of introducing a nonlocal operator on the nonlinear terms. Here the more conventional procedure of simply taking the curl of (1.1) performs the same function, leading to the *vorticity equations*

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \boldsymbol{\omega}, \tag{1.4}$$

where $\boldsymbol{\omega} = (\xi, \eta, \zeta) = \nabla \times \mathbf{u}$ is the vorticity associated with fluid at a position $\mathbf{x}$. The vorticity equations can also be written in the suggestive form (similar to a *Lax representation*)

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \{\mathbf{u}, \boldsymbol{\omega}\}, \tag{1.4$'$}$$

where $\{\ ,\ \}$ is a *Poisson bracket* defined so that (1.4′) matches (1.4). The representation (1.4′) though, is, in more than one way, misleading.

For the vorticity equations, a number of candidates for the skew-adjoint Hamiltonian operator $\mathscr{D}$ are readily apparent, but only one choice, namely,

$$\mathscr{D} = (\boldsymbol{\omega} \cdot \nabla - \nabla \boldsymbol{\omega}) \nabla \times, \tag{1.5}$$

where $\nabla \boldsymbol{\omega}$ denotes the Jacobian matrix of $\boldsymbol{\omega}$, satisfies the closure condition, and is truly Hamiltonian. The Hamiltonian functional is the kinetic energy

$$H = \int \tfrac{1}{2} \mathbf{u}^2 \, dx \qquad (d\mathbf{x} \equiv dx\,dy\,dz), \tag{1.6}$$

the integration taking place over all space. In Section 3, it is proved that $\mathscr{D}$ is Hamiltonian and system (1.3) with definitions (1.5)–(1.6) is equivalent to the vorticity equations.

Once a system is known to be Hamiltonian, the generalization of Noether's theorem presented in [18] provides conservation laws associated with many of the symmetry groups of the system. Buchnev [7] and, in a less comprehensive fashion, Strampp [24] have classified the one-parameter symmetry groups of the Euler equations. The basic Lie–Ovsjannikov infinitesimal techniques, as explained in [6, 17, 21], reduce this problem to a more or less straightforward computational exercise. In both two and three dimensions the symmetry group consists of changes to arbitrarily moving coordinate frames, spatial rotations, time translations, two groups of scaling transformations, and the addition of arbitrary functions of t to the pressure. No further Lie symmetries continuously deformable to the identity are possible.

The Hamiltonian version of Noether's theorem allows two different mechanisms for the appearance of conservation laws. Any one parameter group of time-independent "canonical" symmetries yields a conservation law. For the Euler equations, the usual invariants of energy and linear and angular momenta, known to Helmholtz and Kelvin, arise in this fashion. A second source of conservation laws is the appearance of the inverse of the Hamiltonian operator $\mathscr{D}$ in the Noether formulas. When $\mathscr{D}$ is a matrix of differential operators, in general, there is a nontrivial kernel, and this in turn provides new candidates for conservation laws. In three dimensions, the incompressibility of the fluid implies that all but one of these are trivial, the only exception being the curious conserved quantity

$$\int \mathbf{u} \cdot \boldsymbol{\omega} \, d\mathbf{x}.$$

Moffatt [16] names this quantity *total helicity* and relates it to the invariance of the degree of knottedness of tangled vortex filaments. In two dimensions only a single component ζ of the vorticity is nonvanishing, and the correspondingly higher degeneracy of $\mathscr{D}$ leads to the invariance of the *area integrals*

$$\int A(\zeta) \, d\mathbf{x},$$

where A is an arbitrary function of the vorticity. All known local conservation laws in Eulerian coordinates are thus ascribed a group-theoretic interpretation.

The final question is whether the above techniques succeed in classifying all local conservation laws of the Euler equations. The elegant Hamiltonian structure of Arnol'd, exploited to great effect by Ebin and Marsden, [11], and also the suggestive *Lax representation* (1.3′) have led to recent speculation that the Euler equations might actually be completely integrable in the sense that the Korteweg–de Vries equation is. The hallmarks of complete integrability include an infinity of independent conservation laws, solution by inverse scattering methods and the clean interaction of solitary wave (soliton) solutions, cf., [22]. In two dimensions, the appearance of infinitely many area integrals further adds weight to this conjecture. In the case of finite-area vortex regions (patches of uniform vorticity), the n-fold rotationally symmetric solutions ("V-states") found by Deem and Zabusky, [9, 10], are tempting candidates for the role of solitons, but the precise significance of these special solutions is at present unclear.

Since the one-parameter groups of geometrical (Lie) symmetries of the Euler equations have been classified, Noether's theorem assures us that we have completely classified all nontrivial conservation laws quadratic in the first-order derivatives of the velocity field. Alternative classifications by direct methods are done in Howard [13] and Serre [23]. Conservation laws of higher degree in the first-order derivatives of $\mathbf{u}$ or depending essentially on higher order derivatives of $\mathbf{u}$ will not arise from symmetries or from degeneracies of the Hamiltonian operator, unless one generalizes the notion of symmetry to include nonlocal transformations governed by evolution equations [19]. (These are also (mis-) named *Lie–Bäcklund transformations* [1].) A complete classification of the conservation laws of the Euler equations is thereby equivalent to a classification of all generalized symmetries, this latter task being amenable to straightforward, albeit intricate, computational methods. Although the completion of this program awaits a completely rigorous analysis of the quadratic terms in the defining equations of the symmetry group, which is deferred to a subsequent exposition, preliminary evidence points to the conclusion that no generalized symmetries, and hence, no further conservation laws, exist. This assertion, if indeed true, would strongly suggest that the Euler equations are not completely integrable. From this point of view, the area integrals in two dimensions are not indications of integrability, but merely a fortuitous accident arising from the higher degeneracy of the Hamiltonian operator. Moreover, the existence of an inverse scattering formulation or the clean interaction of solitary wave solutions for the Euler equations thereby becomes less likely. In a sense, this latter conclusion is supported by numerical experiments of Zabusky *et al.* [27], in which vortex filaments and breaking phenomena routinely occur in the interactions of separate patches of uniform vorticity. The situation here can also be compared with results of Benjamin and Olver [5], [20], where the water wave problem is proved to have only finitely many conservation laws, and also with the famous theorems of Bruns and Poincaré, cf., [26], which demonstrated that the n-body problem, the most interesting physically-exact, finite-dimensional Hamiltonian system, is not completely integrable for $n \geqslant 3$. The general conclusion seems to be that exact physical systems are usually not completely integrable, whereas some model equations used for approximation often are integrable. A cautious attitude to the indiscriminant interpretation of special results for the model equations is therefore highly desirable.

2. Symmetries of the Euler Equations

The Lie symmetry group of a system of differential equations is the largest group of transformations on the space of independent and dependent variables which transforms solutions of the system to other solutions. The Lie–Ovsjannikov theory provides a systematic, computational procedure for finding the connected component of the symmetry group. The basic method is presented in Ovsjannikov, [21], Bluman and Cole, [6], and Olver, [17]. The notation and procedure developed in this latter reference, especially for the computation of the symmetry group of the Navier–Stokes equations,[1] will be used here; the reader should consult [17] for background information.

For the three-dimensional Euler equations, the symmetry group will consist of (local) diffeomorphism of the eight-dimensional space with coordinates **x**, t, **u**, p. The key to the Lie–Ovsjannikov theory is the reliance on infinitesimal techniques. The vector field

$$\mathbf{v} = \alpha\partial_x + \beta\partial_y + \gamma\partial_z + \delta\partial_t + \lambda\partial_u + \mu\partial_v + \nu\partial_w + \pi\partial_p \tag{2.1}$$

is the infinitesimal generator of a one parameter group; here $\alpha, \ldots, \pi$ are functions of $\mathbf{x}, t, \mathbf{u}, p$, and $\partial_x \equiv \partial/\partial x$. The infinitesimal criterion of invariance of the differential equations (1.1)–(1.2) depends on the prolongation of the vector field **v** to the space of derivatives of **u**, p, and can be written as

$$\lambda^t + u\lambda^x + v\lambda^y + w\lambda^z + u_x\lambda + u_y\mu + u_z\nu = -\pi^x, \tag{2.2a}$$

$$\mu^t + u\mu^x + v\mu^y + w\mu^z + v_x\lambda + v_y\mu + v_z\nu = -\pi^y, \tag{2.2b}$$

$$\nu^t + u\nu^x + v\nu^y + w\nu^z + w_x\lambda + w_y\mu + w_z\nu = -\pi^z, \tag{2.2c}$$

$$\lambda^x + \mu^y + \nu^z = 0. \tag{2.2d}$$

The functions λ^t, μ^x, etc., are the coefficients of the prolongation of **v** corresponding to u_t, v_x, etc. Typical expression for these functions are

$$\begin{aligned} \lambda^t &= D_t\lambda - u_x D_t\alpha - u_y D_t\beta - u_z D_t\gamma - u_t D_t\delta, \\ \mu^x &= D_x\mu - v_x D_x\alpha - v_y D_x\beta - v_z D_x\gamma - v_t D_x\delta, \end{aligned} \tag{2.3}$$

and so on. Here D_t denotes the total derivative with respect to t, etc. The symmetry equations (2.2) must be satisfied whenever the Euler equations are. We may therefore substitute for p_x, p_y, p_z, and w_z whenever they occur by the expressions from (1.1)–(1.2).

Since the solution of the symmetry equations (2.2) is a fairly routine, although tedious, computational exercise, we shall content ourselves with just

[1] It should be remarked that there is an error in the computation of the symmetry group of the Navier–Stokes equations in [17], resulting in the omission of the changes to arbitrarily moving coordinate systems as in the classification of Theorem 2.1 for the Euler equations. This mistake has been corrected in a recent paper of Lloyd [28].

stating the result of such a computation. Details can be found in Buchnev [7] and Strampp [24], although the latter reference makes an unnecessary assumption on the form of the solutions before actually solving the equations.

THEOREM 2.1. *The Lie symmetry group of the Euler equations in three dimensions is generated by the vector fields*

$$\begin{aligned}
&\mathbf{v}_a = a\partial_x + a'\partial_u - a''x\partial_p, && \mathbf{v}_b = b\partial_y + b'\partial_v - b''y\partial_p,\\
&\mathbf{v}_c = c\partial_z + c'\partial_w - c''z\partial_p, && \mathbf{v}_0 = \partial_t,\\
&\mathbf{s}_1 = x\partial_x + y\partial_y + z\partial_z + t\partial_t, && \mathbf{s}_2 = t\partial_t - u\partial_u - v\partial_v - w\partial_w - 2p\partial_p,\\
&\mathbf{r}_z = y\partial_x - x\partial_y + v\partial_u - u\partial_v, && \mathbf{r}_y = z\partial_x - x\partial_z + w\partial_u - u\partial_w,\\
&\mathbf{r}_x = z\partial_y - y\partial_z + w\partial_v - v\partial_w, && \mathbf{v}_q = q\partial_p,
\end{aligned} \tag{2.4}$$

where a, b, c, q are arbitrary functions of t.

These vector fields exponentiate to familiar one-parameter symmetry groups of the Euler equations. For instance, a linear combination of the first three fields, $\mathbf{v}_a + \mathbf{v}_b + \mathbf{v}_c$ generates the group transformations

$$(\mathbf{x}, t, \mathbf{u}, p) \to (\mathbf{x} + \varepsilon\mathbf{a}, t, \mathbf{u} + \varepsilon\mathbf{a}', p - \varepsilon\mathbf{x}\cdot\mathbf{a}'' + \tfrac{1}{2}\varepsilon^2\mathbf{a}\cdot\mathbf{a}''),$$

where ε is the group parameter, and $\mathbf{a} = (a, b, c)$. These represent changes to arbitrarily moving coordinate systems, and have the interesting consequence that for a fluid with no free surfaces, the only essential effect of changing to a moving coordinate frame is to add an extra component, namely, $-\varepsilon\mathbf{x}\cdot\mathbf{a} + \frac{1}{2}\varepsilon^2\mathbf{a}\cdot\mathbf{a}''$, to the resulting pressure.

The group generated by $\mathbf{v}_0$ is that of time translations, reflecting the time-independence of the system. The next two groups are scaling transformations:

$$\mathbf{s}_1 : (\mathbf{x}, t, \mathbf{u}, p) \to (\varepsilon\mathbf{x}, \varepsilon t, \mathbf{u}, p)$$

$$\mathbf{s}_2 : (\mathbf{x}, t, \mathbf{u}, p) \to (\mathbf{x}, \varepsilon t, \varepsilon^{-1}\mathbf{u}, \varepsilon^{-2}p).$$

The vector fields $\mathbf{r}_x, \mathbf{r}_y, \mathbf{r}_z$ generate the orthogonal group $SO(3)$ of simultaneous rotations of space and associated velocity field; e.g., $\mathbf{r}_x$ is just an infinitesimal rotation around the x axis. The final group indicates that arbitrary functions of t can be added to the pressure.

If the calculations were carried through in two spatial dimensions, no new symmetries would result. (This should be somewhat surprising, as the conformal group in two dimensions is much larger than in three dimensions.) The other words, the list of infinitesimal symmetries of the Euler equations in two dimensions is obtained from the list (2.4) for three dimensions simply by setting $z = w = 0 = \partial_z = \partial_w$. No further comment is required here.

Beyond the geometrical–Lie symmetries of a system, another important class of symmetries are the generalized symmetries, as introduced in [1, 19]. The basic step is to allow the coefficient functions $\alpha, \ldots, \pi$ of the vector field (2.1) to depend also on derivatives of the dependent variables $\mathbf{u}, p$. The corresponding one-parameter group now acts on some space of functions of the form $\mathbf{u} = \mathbf{f}(x, t)$, $p = g(x, t)$. The corresponding symmetry equations are exactly the same, i.e., (2.2), but naturally, are more difficult to solve explicitly.

One trivial source of generalized symmetries is when the coefficient functions vanish when the Euler equations are satisfied; these we always ignore. If $\mathbf{v}$ as given by (2.1) is a symmetry, the *standard* representative of $\mathbf{v}$, which is $\hat{\mathbf{v}} = \hat{\lambda}\partial_u + \hat{\mu}\partial_v + \hat{\nu}\partial_w + \hat{\pi}\partial_p$, where

$$\begin{aligned} \hat{\lambda} &= \lambda - u_x\alpha - u_y\beta - u_z\gamma - u_t\delta, \\ \hat{\mu} &= \mu - v_x\alpha - v_y\beta - v_z\gamma - v_t\delta, \\ \hat{\nu} &= \nu - w_x\alpha - w_y\beta - w_z\gamma - w_t\delta, \\ \hat{\pi} &= \pi - p_x\alpha - p_y\beta - p_z\gamma - p_t\delta, \end{aligned} \tag{2.5}$$

also is a symmetry. It is conjectured that no nontrivial standard symmetries exist save the standard representatives of the Lie symmetries found in the previous theorem. As discussed in the introduction, if this conjecture were true, it would strongly indicate the nonintegrability of the Euler equations.

3. The Hamiltonian Operator

Once the system of Euler equations (1.1)–(1.2) have been replaced by the vorticity equations (1.4) it is possible to introduce a Hamiltonian structure. Since the requisite skew-adjoint operator has to depend on the vorticity $\boldsymbol{\omega}$, however, it must be carefully checked that the operator is truly Hamiltonian. The lack of understanding as to the correct condition for a differential operator to be Hamiltonian has caused some confusion as to exactly which skew-adjoint operator should be chosen. The appropriate general theory has only recently been established, inspired by developments in the study of the Korteweg–de Vries equation. The appropriate Hamiltonian condition was first mentioned by Manin [15], subsequently being simplified and geometrically motivated by Gel'fand and Dorfman [12] and the author [18].

In direct analogy with the finite-dimensional case of ordinary differential equations, a system of evolution equations involving the dependent variables

$\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ is called *Hamiltonian* if it can be written in the special form

$$\boldsymbol{\omega}_t = \mathscr{D}E(H), \tag{3.1}$$

where H is the Hamiltonian functional, E denotes the Euler operator, variational derivative, or gradient of H with respect to $\boldsymbol{\omega}$, and $\mathscr{D}$ is a skew-adjoint matrix of differential or pseudo-differential operators. If $\mathscr{D}$ actually depends on $\boldsymbol{\omega}$ and its derivatives, a further condition must be satisfied. This is best expressed in the exterior algebra of differential forms involving the dependent variables and their spatial derivatives, as developed in [18] to which we refer for the details. Any identity involving these differential forms always holds modulo the image of the total divergence, the total derivatives acting as Lie derivatives. We are thus allowed the luxury of integrating by parts (and discarding the boundary terms) in any computation. In this context, the Hamiltonian condition is that the fundamental symplectic two form

$$\Omega = -\tfrac{1}{2}\, d\boldsymbol{\omega}^T \wedge \mathscr{D}^{-1}\, d\boldsymbol{\omega} \tag{3.2}$$

is closed, i.e.,

$$d\Omega = 0 \tag{3.3}$$

modulo total divergence. (Here $\boldsymbol{\omega}$ and $d\boldsymbol{\omega}$ are viewed as column vectors, and $\mathscr{D}^{-1}$ is the *formal inverse* of the matrix of operators $\mathscr{D}$.) The important consequences of an equation being in Hamiltonian form are contained in the following theorem, proved in [18].

THEOREM 3.1. *Consider the Hamiltonian system* (3.1), *so the fundamental two-form* (3.2) *is closed.*

(a) *The Hamiltonian functional* $\int H\, d\mathbf{x}$ *is a constant on solutions.*

(b) *The two form* Ω *is an absolute-integral invariant in the sense of Cartan.*

(c) *If* $\mathbf{v} = \boldsymbol{\lambda} \cdot \partial_{\boldsymbol{\omega}} = \sum \lambda_j \partial_{\omega_j}$ *is the standard representative of a time-independent infinitesimal symmetry of* (2.1) (*either Lie or generalized*), *then the one form*

$$\theta = \mathbf{v} \,\lrcorner\, \Omega = \mathscr{D}^{-1} \boldsymbol{\lambda} \cdot d\mathbf{u} \tag{3.4}$$

(*assuming* $\boldsymbol{\lambda}$ *lies in the image of* $\mathscr{D}$) *is an absolute integral invariant of the evolution equation. If, moreover,*

$$\theta = dT$$

for some function T, then T is the conserved density of a conservation law of (4.1). *In other words*

$$\int T\,d\mathbf{x} = \text{const.}$$

for solutions $\mathbf{u}$ *decaying sufficiently rapidly at large distances.*

(*A one form* $\theta = \mathbf{f}\cdot d\boldsymbol{\omega} = \sum f_j d\omega_j$ *is an absolute integral invariant, means that for any one parameter family of solutions* $\boldsymbol{\omega}(\mathbf{x}, t, \varepsilon)$, $\varepsilon \in \mathbb{R}$, *decaying rapidly for large* $|\mathbf{x}|$, *the integral*

$$\iint \mathbf{f}(\mathbf{u}) \cdot \frac{\partial \boldsymbol{\omega}}{\partial \varepsilon}\, d\varepsilon\, d\mathbf{x} \tag{3.5}$$

is a constant.)

(d) *The formula*

$$\{P, Q\} = E(P)^T \mathscr{D}^{-1} E(P)$$

defines a Poisson bracket on the space of functionals. This means that $\{\ ,\ \}$ *is skew symmetric (modulo divergences), and satisfies the Jacobi identity. Moreover, if P is associated with the vector field* $\mathbf{v}_P = \mathscr{D}E(P)\cdot\partial_{\omega}$, *then the Poisson bracket* $\{P, Q\}$ *is associated with the Lie bracket of the vector fields* $\mathbf{v}_P$ *and* $\mathbf{v}_Q$ (*defined using prolongation*).

If $\mathscr{D}$ is a matrix of differential operators, the closure condition (3.2) on the fundamental two form is somewhat difficult to verify in practice, and can be replaced with an equivalent condition on the associated cosymplectic form

$$\tilde{\Omega} = \tfrac{1}{2}\, d\boldsymbol{\omega}^T \wedge \mathscr{D}\, d\boldsymbol{\omega}. \tag{3.6}$$

This condition is that $\tilde{\Omega}$ be closed under the exterior derivation $d_{\mathscr{D}}$ based on the operator $\mathscr{D}$ and defined by the properties

$$\begin{aligned} d_{\mathscr{D}}\omega_i &= \sum_j \mathscr{D}_{ij}\omega_j, \\ d_{\mathscr{D}} d &= 0 = [d_{\mathscr{D}}, D_{x_j}], \end{aligned} \tag{3.7}$$

as well as linearity and the derivation property on forms. Thus $\mathscr{D}$ is Hamiltonian if and only if

$$d_{\mathscr{D}}\tilde{\Omega} = 0; \tag{3.8}$$

see [18] for a proof.

Here we shall follow the development of Hamiltonian operators set forth in [18]. To be technically correct, the cosymplectic form should be written as a two tensor, i.e., sum of wedge products of pairs of vector fields. The derivation $d_{\mathscr{D}}$ will then act on the spaces of k-tensors and is then closely related to (but not identical with) the Schouten–Nijenhuis bracket $[\tilde{\Omega}, \cdot\,]$ between k-tensors. However, the closure condition (3.8) *is* identical to the condition of the vanishing of the Schouten–Nijenhuis bracket

$$[\tilde{\Omega}, \tilde{\Omega}] = 0.$$

For a complete discussion of this point of view, see Gel'fand and Dorfman [12, Sect. 5].

To place the vorticity equations in Hamiltonian form, we introduce the vector stream function $\boldsymbol{\psi}$, chosen via the Hodge decomposition theorem so that

$$\nabla \times \boldsymbol{\psi} = \mathbf{u}, \qquad \nabla \cdot \boldsymbol{\psi} = 0.$$

Note that this implies that

$$\boldsymbol{\omega} = -\Delta \boldsymbol{\psi},$$

where Δ denotes the Laplacian. We can rewrite the vorticity equations in the form

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = (\boldsymbol{\omega} \cdot \nabla - \nabla \boldsymbol{\omega})\, \nabla \times \boldsymbol{\psi} = \mathscr{D} E(H), \tag{3.9}$$

where

$$\mathscr{D} = (\boldsymbol{\omega} \cdot \nabla - \nabla \boldsymbol{\omega})\, \nabla \times . \tag{3.10}$$

Here

$$H = \int \tfrac{1}{2} \mathbf{u}^2 \, d\mathbf{x}$$

is the total energy of the system. Note that the variational derivative of H in (3.9) is with respect to $\boldsymbol{\omega}$. If $\delta\boldsymbol{\omega}$ an infinitesimal variation in $\boldsymbol{\omega}$, with corresponding variation $\delta\mathbf{u}$ in $\mathbf{u}$, then both $\delta\mathbf{u}$ and $\delta\boldsymbol{\omega}$ are divergence free. Therefore, for variations with compact support,

$$\delta H = \int \mathbf{u} \cdot \delta\mathbf{u} \, d\mathbf{x} = \int (\nabla \times \boldsymbol{\psi}) \cdot \delta\mathbf{u} \, d\mathbf{x} = \int \boldsymbol{\psi} \cdot (\nabla \times \delta\mathbf{u}) \, d\mathbf{x} = \int \boldsymbol{\psi} \cdot \delta\boldsymbol{\omega} \, d\mathbf{k},$$

hence $E(H) = \boldsymbol{\psi}$ and (3.9) is indeed the vorticity equations. The main result of this section is that the system (3.9) with operator (3.10) is genuinely Hamiltonian.

THEOREM 3.2. *The skew-adjoint operator* $\mathscr{D} = (\boldsymbol{\omega} \cdot \nabla - \nabla\boldsymbol{\omega})\nabla \times$ *is a Hamiltonian operator.*

Proof. The proof that $\mathscr{D}$ is skew adjoint is left to the reader. To prove that $\mathscr{D}$ is Hamiltonian, the first step is to show that the associated two form (3.6) has the form

$$\tilde{\Omega} = \xi d\Delta v \wedge d\Delta w + \eta d\Delta w \wedge d\Delta u + \zeta d\Delta u \wedge d\Delta v. \tag{3.11}$$

Notationally, the easiest way of demonstrating this is to index the components of position $\mathbf{x} = (x_1, x_2, x_3)$, velocity $\mathbf{u} = (u_1, u_2, u_3)$, and vorticity $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$. Since

$$\nabla \times \boldsymbol{\omega} = -\Delta \mathbf{u} \tag{3.12}$$

we have

$$\mathscr{D}\boldsymbol{\omega} = (\nabla\boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla)\Delta\mathbf{u},$$

or, in coordinates,

$$\mathscr{D}_{ij}\omega_j = D_j\omega_i\Delta u_j - \omega_j D_j \Delta u_i. \tag{3.13}$$

The indices i, j run from 1–3, the summation convention on repeated indices is employed, and D_j denotes the total derivative with respect to x_j. Therefore, by (3.6),

$$\begin{aligned}\tilde{\Omega} &= \tfrac{1}{2} d\omega_i \wedge \mathscr{D}_{ij} d\omega_j \\ &= \tfrac{1}{2}[D_j\omega_i d\omega_i \wedge d\Delta u_j - \omega_j d\omega_i \wedge dD_j\Delta u_i] \\ &= -\tfrac{1}{2}\omega_i d(D_j\omega_i - D_i\omega_j) \wedge d\Delta u_j,\end{aligned}$$

where we have integrated by parts using the fact that both $\mathbf{u}$ and $\boldsymbol{\omega}$ are divergence free. Further use of (3.12), and identification of the components of $\boldsymbol{\omega}$ and $\mathbf{u}$ completes the proof of (3.11). Now we can use criterion (3.8) to prove that $\mathscr{D}$ is Hamiltonian. Thus, by (3.11)–(3.13),

$$\begin{aligned}d_{\mathscr{D}}\tilde{\Omega} = {} & (\xi_x d\Delta u - \omega \cdot \nabla d\Delta u) \wedge d\Delta v \wedge d\Delta\omega \\ & + (\eta_y d\Delta v - \omega \cdot \nabla d\Delta v) \wedge d\Delta w \wedge d\Delta u \\ & + (\zeta_z d\Delta w - \omega \cdot \nabla d\Delta w) \wedge d\Delta u \wedge d\Delta v.\end{aligned}$$

Integrating the third set of terms by parts, we find

$$d_{\mathscr{D}}\tilde{\Omega} = 2(\xi_x + \eta_y + \zeta_z)\, d\Delta u \wedge d\Delta v \wedge d\Delta w = 0,$$

since $\boldsymbol{\omega}$ is divergence free. This completes the proof of the theorem.

Consider the case of two dimensions. If $\mathbf{u} = (u, v)$ is the velocity, then there is a single component $\zeta = v_x - u_y$ of vorticity. The divergence-free condition on $\mathbf{u}$ implies the existence of a stream function ψ with $\psi_x = u$, $\psi_y = -v$. The vorticity equations take the form, noticed by Christiansen and Zabusky [8] and Benjamin [4],

$$\zeta_t = \partial(\zeta, \psi), \qquad \zeta = -\Delta\psi. \tag{3.14}$$

Here $\partial(\zeta, \psi)$ denotes the Jacobian determinant $\zeta_x \psi_y - \zeta_y \psi_x$. The Hamiltonian operator for this system is the skew-adjoint operator

$$\mathscr{D} = \partial(\zeta, \cdot\,). \tag{3.15}$$

The proof that $\mathscr{D}$ is Hamiltonian follows from (3.8) as in three dimensions. The operator $\partial(\psi, \cdot\,)$, as proposed by Christiansen and Zabusky [8], is *not* Hamiltonian, since the closure condition on the associated two form is not satisfied.

4. Conservation Laws

The Hamiltonian structure of the vorticity equations is now exploited to derive conservation laws from the infinitesimal symmetry group. The basic device is the version of Noether's theorem stated in part (d) of Theorem 3.1. We shall specialize this to the vorticity equations. To begin with, we shall work exclusively in three dimensions.

THEOREM 4.1. *Let* $\mathscr{D} = (\boldsymbol{\omega} \cdot \nabla - \nabla\boldsymbol{\omega})\, \nabla \times$. *For* $\mathbf{v} = \hat{\lambda}\partial_u + \hat{\mu}\partial_v + \hat{\nu}\partial_w + \hat{\pi}\partial_p$ *the standard infinitesimal generator of a time-independent one-parameter group of symmetries of the Euler equations (for instance* $\mathbf{v}$ *might be the standard representative of a Lie symmetry group as in* (2.7)), *let* $\boldsymbol{\lambda} = (\hat{\lambda}, \hat{\mu}, \hat{\nu})$. *If* $\nabla \times \boldsymbol{\lambda}$ *lies in the image of the Hamiltonian operator* $\mathscr{D}$, *then the one-form*

$$\theta = \mathscr{D}^{-1}(\nabla \times \boldsymbol{\lambda}) \cdot d\boldsymbol{\omega} \tag{4.1}$$

is an absolute integral invariant of the Euler equations. Moreover, if $\theta = dT$ *for some function* T, *then* T *is the conserved density of a conservation law of the form* $\int T d\mathbf{x}$.

The only remark needed in the proof of this theorem is that if $\mathbf{v}$ is a standard symmetry of the Euler equations, then its prolongation $\mathrm{pr}\,\mathbf{v} = \mathbf{v} + \rho\partial_\xi + \sigma\partial_\eta + \tau\partial_\zeta$, with $(\rho, \sigma, \tau) = \nabla \times \boldsymbol{\lambda}$, is a symmetry of the vorticity equations.

As mentioned in the introduction, the first interesting class of conservation laws comes from the degeneracy of the Hamiltonian operator. Choosing the trivial symmetry $\mathbf{v} = 0$, Theorem 4.1 says that if $\mathbf{F}$ is any time-independent function with $\mathscr{D}\mathbf{F} = 0$, then the one form $\mathbf{F} \cdot d\boldsymbol{\omega}$ is automatically conserved. We therefore need to characterize the kernel of the differential operator $\mathscr{D}$.

LEMMA 4.2. *The time-independent function* $\mathbf{F}$ *satisfies* $\mathscr{D}\mathbf{F} = 0$ *if and only if*

$$\mathbf{F} = \nabla P + c\mathbf{u},$$

where P is an arbitrary function of $\mathbf{x}, \mathbf{u}, \boldsymbol{\omega}$, *and their derivatives, and c is a constant.*

Proof. It is routine to check that in three spatial dimensions

$$(\boldsymbol{\omega} \cdot \nabla - \nabla\boldsymbol{\omega})\,\mathbf{G} = 0$$

if and only if $\mathbf{G} = c\boldsymbol{\omega}$ for some constant c. Therefore, $\mathbf{F}$ is in the kernel of $\mathscr{D}$ if and only if $\nabla \times \mathbf{F} = c\omega$, from which the lemma follows.

Therefore, Theorem 4.1 implies that every one form of the form

$$(\nabla P + c\mathbf{u}) \cdot d\boldsymbol{\omega}$$

is conserved. Integration by parts shows, however, that

$$\nabla P \cdot d\boldsymbol{\omega} = \nabla P \cdot (\nabla \times d\mathbf{u}) = (\nabla \times \nabla P) \cdot d\mathbf{u} = 0,$$

so only the form $\mathbf{u} \cdot d\boldsymbol{\omega}$ is nontrivial. Further note that

$$\begin{aligned} d(\mathbf{u} \cdot \boldsymbol{\omega}) &= \mathbf{u} \cdot d\boldsymbol{\omega} + \boldsymbol{\omega} \cdot d\mathbf{u} = \mathbf{u} \cdot d\boldsymbol{\omega} + (\nabla \times \mathbf{u}) \cdot d\mathbf{u} \\ &= \mathbf{u} \cdot d\boldsymbol{\omega} + \mathbf{u} \cdot \nabla \times d\mathbf{u} = 2\mathbf{u} \cdot d\boldsymbol{\omega}. \end{aligned}$$

Therefore the quantity

$$I_0 = \int \mathbf{u} \cdot \boldsymbol{\omega}\, d\mathbf{x}$$

is conserved for all solutions $\mathbf{u}$ of the three-dimensional Euler equations. This quantity was named *total helicity* by Moffatt [16] and, in the case where the vorticity is confined to a narrow filament, serves to characterize the degree of knottedness of the filament.

We shall now turn to consider the conservation laws associated with actual symmetries of the Euler equations as computed in Section 2. We shall recover the fundamental integrals representing linear and angular momentum, and energy.

THEOREM 4.3. *The three-dimensional Euler equations admit the following seven conserved quantities*:

(a) *Linear momenta*:

$$\int u\,d\mathbf{x}, \qquad \int v\,d\mathbf{x}, \qquad \int w\,d\mathbf{x}.$$

(b) *Angular momenta*:

$$\int (yu - xv)\,d\mathbf{x}, \quad \int (zu - xw)\,d\mathbf{x}, \quad \int (zv - yw)\,d\mathbf{x}.$$

(c) *Energy*:

$$\tfrac{1}{2}\int \mathbf{u}^2\,d\mathbf{x}.$$

That each of the above quantities is conserved can, of course, be proved directly. In order to derive them from the symmetry groups of Theorem 2.1, first replace each time-independent infinitesimal generator (2.4) by its standard representative as in (2.5). Theorem 4.1 then provides a corresponding conserved one form as in (4.1). It remains to determine which of these one forms are exact, and thereby yield a geniune conservation law. We illustrate this process with the first vector field $\mathbf{v}_a$ for $a(t) \equiv 1$. Its standard representative is the generalized vector field

$$-u_x\partial_u - v_x\partial_v - w_x\partial_w - p_x\partial_p.$$

Next note that if $\mathbf{F}$ is the column vector $(0, 0, y)$, then

$$\mathscr{D}\mathbf{F} = -(\xi_x, \eta_x, \zeta_x) = -\nabla \times (u_x, v_x, w_x).$$

Therefore, by (4.1), the one form

$$y\,d\zeta = d(y\zeta)$$

is conserved, hence the quantity

$$\int y\zeta\,d\mathbf{x}$$

is conserved. Integration by parts shows that this quantity is equivalent to the first component of linear momentum $\int u\,d\mathbf{x}$.

In a similar fashion, the vector fields $\mathbf{v}_b, \mathbf{v}_c$, $(b, c \equiv 1)$, $\mathbf{v}_0, \mathbf{r}_z, \mathbf{r}_y, \mathbf{r}_x$ yield, respectively, the other two components of linear momentum, the energy and the components of angular momentum. Neither of the two scale groups provide conservation laws, or even conserved one-forms, since the vector $\nabla \times \boldsymbol{\lambda}$ obtained from their standard representatives does not lie in the image of the operator $\mathscr{D}$, as can easily be verified. Therefore, Theorem 4.3 completes the list of conservation laws obtainable from Lie symmetries of Euler equations. If the conjecture that there are no generalized symmetries is correct, then this would provide all nontrivial conservation laws of the Euler equations whose densities are local functions depending on the Eulerian coordinates and their derivatives. (Other laws can be written down in Lagrangian coordinates using the method of Arnol'd, cf., [3], but these are not relevant here.)

Except for the higher degree of degeneracy of the Hamiltonian operator, the situation in two spatial dimensions is fairly similar since the symmetry groups are essentially the same.

THEOREM 4.4. *The two-dimensional Euler equations admit the following conservation laws*:

(a) *Linear momenta*:

$$\int u\,d\mathbf{x}, \qquad \int v\,d\mathbf{x}.$$

(b) *Angular momentum*:

$$\int (yu - xv)\,d\mathbf{x}.$$

(c) *Energy*:

$$\tfrac{1}{2}\int \mathbf{u}^2\,d\mathbf{x}.$$

(d) *Area integrals*:

$$\int A(\zeta)\,d\mathbf{x},$$

where A is an arbitrary function of the vorticity $\zeta = u_y - v_x$.

The derivation of these quantities proceeds along the same lines as in the three-dimensional case. The only laws that require comment are the area integrals, and these depend on the following characterization of the kernel of the corresponding Hamiltonian operator $\mathscr{D} = \partial(\zeta, \cdot)$, cf., (3.15).

LEMMA 4.5. *A function f satisfies $\mathscr{D}f = \partial(\zeta, f) = 0$ if and only if $f = f(\zeta)$ is an arbitrary function of ζ.*

Note that we no longer have a quantity corresponding to total helicity (indeed, two-dimensional vortex filaments cannot be tangled!) but the total amount of vorticity, as expressed by the area integrals, is conserved. This completes the group-theoretic characterization of conservation laws for Euler equations.

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