Non-Abelian Integrable Systems of the Derivative Nonlinear Schrödinger Type

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In the classification of integrable systems, the symmetry method has proven to be the most systematic and successful. Many commutative integrable systems have now been completely classified using this direct, computational approach. Attention has now shifted to various noncommutative generalizations.

This note is a continuation of the systematic classifications of non-abelian integrable systems by the authors, [11], in which the dependent variables are matrix-valued, or, more generally, take values in a general associative algebra. The equations found to be integrable via the symmetry approach fall into two categories: those which, like Burgers' equation, can be linearized by differential substitutions, and those which, like the Korteweg–deVries equation can be solved by inverse scattering. One can construct a Lax pair for the system based on the recursion operator, [10], but the associated scattering problem is nonstandard, and must be reduced to a standard form amenable to direct analysis, typically by using the squared eigenfunction construction, [1].

The so-called derivative nonlinear Schrödinger equation has the form

$$\varphi_t = i\varphi_{xx} + |\varphi|^2\varphi_x, \tag{1}$$

and is known to be integrable (see [7]). The complex equation (1) is equivalent, under a complex change of variables, to the pair of real equations

$$\begin{cases} u_t = u_{xx} + 4uvu_x + 2u^2v_x, \\ v_t = -v_{xx} + 4uvv_x + 2v^2u_x. \end{cases} \tag{2}$$

Note that the system (2) is homogeneous if we assign weightings of 1 to the dependent variables u and v, while x and t-differentiation have weights 2 and 4, respectively. The general form of such a homogeneous system is given by

$$\begin{cases} u_t = u_{xx} + a_1u^2u_x + a_2uvu_x + a_3v^2u_x + a_4u^2v_x + a_5uvv_x + a_6v^2v_x + \\ \qquad b_1u^5 + b_2u^4v + b_3u^3v^2 + b_4u^2v^3 + b_5uv^4 + b_6v^5 \\ v_t = -v_{xx} - \bar{a}_1v^2v_x - \bar{a}_2vuv_x - \bar{a}_3u^2v_x - \bar{a}_4v^2u_x - \bar{a}_5vuv_x - \bar{a}_6u^2u_x - \\ \qquad \bar{b}_1v^5 - \bar{b}_2v^4u - \bar{b}_3v^3u^2 - \bar{b}_4v^2u^3 - \bar{b}_5vu^4 - \bar{b}_6u^5. \end{cases} \tag{3}$$

Our choice of notation for the coefficients stresses the existence of a transformation $u \leftrightarrow v,\ t \leftrightarrow -t$, which preserve the class of equations (3). Under this involution a_1 is replaced by $\bar{a}_1$ and so on.

The class (3) contains some integrable cases. Namely, for any constants α and β, the following systems are completely integrable (see [9])

$$\begin{cases} u_t = u_{xx} + 2\alpha u^2 v_x + 2\beta uvu_x + \alpha(\beta - 2\alpha)u^3v^2, \\ v_t = -v_{xx} + 2\alpha v^2 u_x + 2\beta uvv_x - \alpha(\beta - 2\alpha)u^2v^3, \end{cases} \tag{4}$$

$$\begin{cases} u_t = u_{xx} + 2\alpha uvu_x + 2\alpha u^2 v_x - \alpha\beta u^3v^2, \\ v_t = -v_{xx} + 2\beta v^2 u_x + 2\beta uvv_x + \alpha\beta u^2v^3. \end{cases} \tag{5}$$

If $\alpha = 1$ and $\beta = 2$, then (4) coincides with (2). Two more well known systems described by (4) correspond to $\alpha = 1$ and $\beta = 0$, [1], and $\alpha = 0$ and $\beta = 1$, [4]. All these systems were integrated with the help of the inverse scattering method. The system (5) with $\alpha = \beta = 1$ have been considered in [5], and turns ouit to be linearizable.

It is easy to see that rescaling t, x, u and v in (4) and (5), we can make α equal to 1, unless it is zero, so the essential parameter is the ratio of α and β. Both systems (4) and (5) have a higher symmetry of the form

$$\begin{cases} u_\tau = u_{xxxx} + f(u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xxx}, v_{xxx}), \\ v_\tau = -v_{xxxx} + g(u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xxx}, v_{xxx}), \end{cases} \tag{6}$$

where f and g are homogeneous polynomials of weight 9.

It is known that the majority of integrable equations possess matrix generalizations (see [13, 11]). In particular, a matrix counterpart of the derivative nonlinear Schrödinger system (2) takes the following form (see [6])

$$\begin{cases} u_t = u_{xx} + 2uvu_x + 2u_xvu + 2uv_xu, \\ v_t = -v_{xx} + 2vuv_x + 2v_xuv + 2vu_xv. \end{cases} \tag{7}$$

We will call (7) the *non-abelian derivative NLS-equation*. The dependent variables u and v can be square matrices of arbitrary size, or, more generally functions with values in an arbitrary (perhaps infinite dimensional) associative algebra. As in the scalar case, this equation has a higher symmetry of the form (6), where f and g are homogeneous *non-commutative* polynomials of weight 9.

In this paper we classify integrable non-abelian generalizations of the integrable systems (3). Namely, we consider the systems of the form

$$\begin{cases} u_t = u_{xx} + F(u, v, u_x, v_x), \\ v_t = -v_{xx} + G(u, v, u_x, v_x), \end{cases} \tag{8}$$

where F and G are non-commutative polynomials of weight 5. Each of them contains 56 arbitrary constants.

From the methodological point of view, this work is very similar to the classification of non-abelian equations of the nonlinear Schrödinger type that was previously done in [11]. We refer the reader to this paper for motivations and references. The history of development of the symmetry approach to classification of scalar equations was described in the surveys [12, 9, 8].

The main result of the paper is a list of integrable non-abelian systems (8). These systems can be interesting for the following reasons. First, they can lead to integrable quantum models of the same type. Second, limiting procedures such as Whitham averaging give us new integrable systems of the hydrodynamical type. Third, symmetry reductions of these systems allow us to produce non-abelian ordinary differential equations having the Painlevé property, [11].

For the actual computation we used a computer program implemented a MATHEMATICA package. It available at the web site

`http://www.math.umn.edu/~olver`.

For non-abelian systems to compactify the answer it is convenient to add to the transformation

$$u \leftrightarrow v, \qquad t \leftrightarrow -t \tag{9}$$

an involution $*$, satisfying the conditions

$$(P^*)^* = P, \qquad (P\ Q)^* = Q^*\ P^*.$$

For the matrix equations one can identify the involution $*$ with the transpose.

Theorem. Up to scallings of t, x, u, v, the transformation (9) and the involution $*$, there exist, besides (7), the following homogeneous non-abelian systems (8) of weight 5, having a homogeneous symmerty (6) of weight 7:

$$\begin{cases} u_t = u_{xx} + 2u_x vu, \\ v_t = -v_{xx} + 2vuv_x \end{cases} \tag{10}$$

$$\begin{cases} u_t = u_{xx}, \\ v_t = -v_{xx} + 2vuv_x + 2vu_x v, \end{cases} \tag{11}$$

$$\begin{cases} u_t = u_{xx} + 2uv_x u - 2uvuvu, \\ v_t = -v_{xx} + 2vu_x v + 2vuvuv, \end{cases} \tag{12}$$

$$\begin{cases} u_t = u_{xx} + 2u^2v_x + 2u_x uv - 2uv_x u - 2uvu_x - 2uvu^2v + 2u^2v^2u - 2u^3v^2 + 2u^2vuv, \\ v_t = -v_{xx} + 2u_x v^2 + 2uvv_x + 2u^2v^3 - 2uvuv^2, \end{cases} \tag{13}$$

$$\begin{cases} u_t = u_{xx} + 2u^2v_x + 2u_x uv - 2uvu_x - 2uvu^2v + 2u^2vuv - 2u^3v^2, \\ v_t = -v_{xx} + 2u_x v^2 + 2uvv_x - 2v_x uv + 2uv^2uv - 2uvuv^2 + 2u^2v^3, \end{cases} \tag{14}$$

$$\begin{cases} u_t = u_{xx} - 2u^2v_x - 2u_x uv + 2uv_x u + 2uvu_x + 2u_x vu - 2uvu^2v + 4u^2vuv - 2u^3v^2, \\ v_t = -v_{xx} - 2u_x v^2 - 2uvv_x + 2vu_x v + 2v_x uv + 2vuv_x + 2uv^2uv - 4uvuv^2 + 2u^2v^3, \end{cases} \tag{15}$$

$$\begin{cases} u_t = u_{xx} + 2u^2v_x + 2u_x uv + 2u_x vu + 2u^2v^2u - 2u^3v^2, \\ v_t = -v_{xx} + 2u_x v^2 + 2uvv_x + 2vuv_x - 2vu^2u^2 + 2u^2v^3, \end{cases} \tag{16}$$

$$\begin{cases} u_t = u_{xx} + 2u^2 v_x + 2u_x uv - 2uv_x u - 2uvuvu + 2u^2 v^2 u - 2u^3 v^2 + 2u^2 vuv, \\ v_t = -v_{xx} + 2u_x v^2 + 2uvv_x - 2vu_x v + 2vuvuv - 2vu^2 u^2 + 2u^2 v^3 - 2uvuv^2, \end{cases} \quad (17)$$

This list contains a number of new non-abelian systems. We do not as yet know how to integrate them. The empirical observation claiming that any equation having one higher symmetry is completely integrable has been remarkably efficient during last 15 years. Recently, Beukers, Sanders and Wang, [3], rigorously proved that a fourth order system due to Bakirov, [2], has a sixth order symmetry, but no higher order symmetries. Nevertheless, we do not expect this pathology to enter into the present classification, and are absolutely sure that all systems from our list are integrable.

It is interesting to compare the list with the abelian integrable systems (4) and (5). The systems (7) and (16) are different non-abelian analogues of (4) with $\alpha = 1,\ \beta = 2$. The systems (10), (15) and (17) in the commutative case coincide with (4) with $\alpha = 1,\ \beta = 0$. At last, the systems (11) and (13) correspond to (5) with $\alpha = 0,\ \beta = 1$.

Thus, among the one-parameter abelian families (4) and (5) there are only 4 equations that have non-abelian generalizations. On the other hand, each of these four has several such generalizations.

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