Corrections and additions to

Kogan, I.A., and Olver, P.J., Invariants of objects and their images under surjective maps, *Lobachevskii J. Math.* **36** (2015), 260–285.

Last updated: September 21, 2015.

⋆⋆ *On page 268, the right hand side of the formula appearing at the end of the first paragraph is missing a factor* $1/z$*:*

$$\psi(x,y,z) = \varphi(x/z, y/z, z)/z.$$

⋆⋆ *On page 277, in the equation appearing after* (3.21)*, all* w*'s should be replaced with* y*'s*:

$$x=0, \quad y=0, \quad z=1, \quad y_1=0, \quad z_1=0, \quad y_2=1, \quad z_2=0, \quad y_3=0, \quad z_3=3.$$

⋆⋆ *On page 278, equation* (3.25) *is missing an overall actor of* $3^{-2/3}$:

$$\begin{aligned}\widehat{\eta} &= 3^{-2/3}\left(\widehat{\alpha}^{-5/3}\,\widehat{\alpha}_{\sigma\sigma} - \tfrac{7}{6}\,\widehat{\alpha}^{-8/3}\,\widehat{\alpha}_{\sigma}^2 + \tfrac{3}{2}\,\widehat{\alpha}^{-2/3}\left(\widehat{\kappa}_\sigma + \tfrac{1}{4}\widehat{\kappa}^2 - 1\right)\right) \\ &= 3^{1/3}\left(-2\,\widehat{\alpha}^{-1/2} D_\sigma^2(\widehat{\alpha}^{-1/6}) + \tfrac{1}{2}\,\widehat{\alpha}^{-2/3}\left(\widehat{\kappa}_\sigma + \tfrac{1}{4}\widehat{\kappa}^2 - 1\right)\right).\end{aligned} \tag{3.25}$$

⋆⋆ *On page 278, the Remark starting before equation* (3.27) *is significantly changed. Several formulae are corrected and new formulae are inserted. To preserve the numbering in subsequent sections, we have added a* $*$ *to the additional formula tags.*

Remark: In Example 12, we introduced another invariant differential form, the pull-back of the projective arc length element (2.31). We find that

$$d\widehat{\xi} = {\Pi_0}^{(5)\,*}\,d\xi \equiv (3\,\widehat{\alpha})^{1/3}\,d\sigma = (3\,\alpha)^{1/3}\,ds, \tag{3.27}$$

where, as before, $\widehat{\alpha} = \widehat{\kappa} + 2\widehat{\tau}$, and we set

$$\alpha = \widehat{\alpha}\,\kappa^{3/2} = \kappa_s + 2\,\tau,$$

while $d\sigma$ and ds are given by (3.18) and (3.12), respectively. We showed in Example 12 that $\widehat{\eta}$ and $\widehat{\zeta}_1 = \widehat{\iota}(Z_1)$, given by (2.27) and (2.35), respectively, provide another generating set of centro-affine invariants under the invariant differentiation

$$D_{\widehat{\xi}} = (3\,\alpha)^{-1/3} D_s = (3\,\widehat{\alpha})^{-1/3}\,D_\sigma,$$

and, therefore, can be expressed in terms of $\widehat{\kappa}$ and $\widehat{\tau}$. We of course, already have such an expression for $\widehat{\eta}$, given by (3.25), and can rewrite it in the alternative form using $D_{\widehat{\xi}}$:

$$\widehat{\eta} = \frac{\widehat{\alpha}\,\widehat{\alpha}_{\widehat{\xi}\widehat{\xi}} - \frac{5}{6}\,\widehat{\alpha}_{\widehat{\xi}}^2}{\widehat{\alpha}^2} \;+\; \frac{3}{2}\,\frac{(3\,\widehat{\alpha})^{1/3}\,\widehat{\kappa}_{\widehat{\xi}} + \frac{1}{4}\widehat{\kappa}^2 - 1}{(3\,\widehat{\alpha})^{2/3}}. \tag{3.28}$$

We further find that

$$\widehat{\zeta}_1 = -\frac{\widehat{\alpha}_\sigma + \frac{3}{2}\widehat{\kappa}\,\widehat{\alpha}}{3^{4/3}\,\widehat{\alpha}^{4/3}} = -\frac{1}{3}\,\frac{\widehat{\alpha}_{\widehat{\xi}}}{\widehat{\alpha}} - \frac{1}{6}\,\frac{3^{2/3}\,\widehat{\kappa}}{\widehat{\alpha}^{1/3}}. \tag{3.29}$$

On the other hand, $\widehat{\eta}$ and $\widetilde{\zeta} = \widetilde{\iota}(Z)$, given by (2.33), provide an alternative generating set of centro-equi-affine invariants under the invariant differentiation $D_{\widehat{\xi}}$. We can express these invariants in terms of κ and τ (or, rather κ and α) and their derivatives with respect to D_s. We find that

$$\widetilde{\zeta} = \frac{1}{(3\,\alpha)^{1/3}} \tag{3.30*}$$

and comparing with (2.33), we observe that the expression $3z^3\alpha$, evaluated at a point on a space curve $\widehat{C}$, equals μ_χ, the derivative of the equi-affine curvature with respect to equi-affine arc-length evaluated at the corresponding point of its projection. The formula for $\widehat{\eta}$ becomes rather simple:

$$\widehat{\eta} = \frac{\alpha_{ss}\,\alpha - \frac{7}{6}\,\alpha_s^2 - \frac{3}{2}\,\kappa\,\alpha^2}{3^{2/3}\,\alpha^{8/3}} \tag{3.31*}$$

and can be compared with formula (2.27) for the projective curvature in terms of the planar equi-affine invariants. If we replace α by μ_χ and κ by μ in the above formula, we obtain a very similar formula to (2.27) — the difference is in the overall factor and also in the coefficient of the last term in the numerator. In part this may be explained by the fact that $\mu_\chi = 3z^3\alpha$, as observed above. The centro-affine invariant (3.29) has a particular simple expression in terms of centro-equi-affine invariant $\widetilde{\zeta}$, or, equivalently, α:

$$\widehat{\zeta}_1 = \widetilde{\zeta}_s = -\frac{\alpha_s}{(3\,\alpha)^{4/3}} \tag{3.32*}$$

We finally note that we can also write

$$\widehat{\eta} = -3\widetilde{\zeta}\,\widetilde{\zeta}_{ss} + \tfrac{3}{2}\widetilde{\zeta}_s^2 - \tfrac{3}{2}\widetilde{\zeta}^2\kappa = -6\,\widetilde{\zeta}^{3/2}(\widetilde{\zeta}^{1/2})_{ss} - \tfrac{3}{2}\widetilde{\zeta}^2\kappa = -6\,\widetilde{\zeta}^{3/2}(D_s^2 + \tfrac{1}{4}\,\kappa)\widetilde{\zeta}^{1/2}. \tag{3.33*}$$

Alternatively, since

$$D_s = \frac{1}{\widetilde{\zeta}}\,D_{\widehat{\xi}}, \tag{3.34*}$$

we can rewrite (3.33*) as

$$\widehat{\eta} = \tfrac{3}{2}\,\widetilde{\zeta}^{-2}\,(3\,\widetilde{\zeta}_{\widehat{\xi}}^2 - 2\,\widetilde{\zeta}\,\widetilde{\zeta}_{\widehat{\xi}\widehat{\xi}} - \widetilde{\zeta}^4\,\kappa) \tag{3.35*}$$

and then solve for κ:

$$\kappa = -\frac{1}{3}\,\frac{2\,\widehat{\eta}\,\widetilde{\zeta}^2 + 6\,\widetilde{\zeta}\,\widetilde{\zeta}_{\widehat{\xi}\widehat{\xi}} - 9\,\widetilde{\zeta}_{\widehat{\xi}}^2}{\widetilde{\zeta}^4} \tag{3.36*}$$

Formulae (3.30*), (3.31*), (3.34*), and (3.36*) give strikingly simple relationships between two natural generating sets of the differential algebra of centro-equi-affine invariants:

(a) κ and α under D_s;   (b) $\widehat{\eta}$ and $\widetilde{\zeta}$ under $D_{\widehat{\xi}}$.

The first set is naturally expressed in terms of the position vector of a curve and its derivatives, while the second set has a natural relationship with the invariants of the image of the curve under projective and equi-affine actions on the plane. Indeed, recall that $\widehat{\eta}$ and $d\widehat{\xi}$ are the projective curvature and arc length element, respectively, of the image curve, while, from (2.33), (2.19), $\widetilde{\zeta} = z\,\mu_\chi^{-1/3}$, where μ_χ is the derivative of the equi-affine curvature with respect to the equi-affine arc length.

⋆⋆ *On page 280, add the following sentence after equation* (3.34):

Taking into account that translations leave jet variables of the first order and higher invariant (i.e. $y_k \circ T_{\widehat{\mathbf{c}}}^{-1} = y_k$ and $z_k \circ T_{\widehat{\mathbf{c}}}^{-1} = z_k$, for $k \geq 1$) we observe that $I_{\widehat{\mathbf{c}},k}$ and $J_{\widehat{\mathbf{c}},k}$ are, in fact, normalized invariants.

⋆⋆ *On page 282, add after the last set of displayed equations*:

Combining (3.5) with the Replacement Theorem, we can compute explicit relations between normalized invariants for invariantizations $\widehat{\iota}_{\mathbf{b}}$ and $\widehat{\iota}$. For example,

$$\widehat{\iota}\,(y_5) = \widehat{\iota}_{\mathbf{b}}(y_5 + 5\,b_1\,z_4 - b_2\,z_5) \circ T_{\mathbf{b}}{}^{(5)}, \tag{3.56}$$

while

$$\widehat{\iota}_{\mathbf{b}}(y_5) = \widehat{\iota}\,(y_5 - 5\,z_4\,b_1 + z_5\,b_2 - 10\,b_1\,b_2) \circ (T_{\mathbf{b}}^{-1})^{(5)}. \tag{3.57}$$