Moving Coframes.

I. A Practical Algorithm

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Abstract. This is the first in a series of papers devoted to the development and applications of a new general theory of moving frames. In this paper, we formulate a practical and easy to implement explicit method to compute moving frames, invariant differential forms, differential invariants and invariant differential operators, and solve general equivalence problems for both finite-dimensional Lie group actions and infinite Lie pseudo-groups. A wide variety of applications, ranging from differential equations to differential geometry to computer vision are presented. The theoretical justifications for the moving coframe algorithm will appear in the next paper in this series.

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1. Introduction.

First introduced by Gaston Darboux, and then brought to maturity by Élie Cartan, [**6**, **8**], the theory of moving frames ("repères mobiles") is acknowledged to be a powerful tool for studying the geometric properties of submanifolds under the action of a transformation group. While the basic ideas of moving frames for classical group actions are now ubiquitous in differential geometry, the theory and practice of the moving frame method for more general transformation group actions has remained relatively undeveloped and is as yet not well understood. The famous critical assessment by Weyl in his review, [**47**], of Cartan's seminal book, [**8**], retains its perspicuity to this day:

"I did not quite understand how he [Cartan] does this in general, though in the examples he gives the procedure is clear ... Nevertheless, I must admit I found the book, like most of Cartan's papers, hard reading."

Implementations of the method of moving frames for certain groups having direct geometrical significance — including the Euclidean, affine, and projective groups — can be found in both Cartan's original treatise, [**8**], as well as many standard texts in differential geometry; see, for example, the books of Guggenheimer, [**19**], which gives the method center stage, Sternberg, [**44**], and Willmore, [**50**]. The method continues to attract the attention of modern day researchers and has been successfully extended to some additional examples, including, for instance, holomorphic curves in projective spaces and Grassmannians. The papers of Griffiths, [**18**], Green, [**17**], Chern, [**12**], and the lecture notes of Jensen, [**23**], are particularly noteworthy attempts to place Cartan's intuitive constructions on a firm theoretical and differential geometric foundation. However, none of the proposed modern geometrical formulations of the theory incorporates the full scope or range of applicability of the method as originally envisioned by Cartan. To this day, both the formulation and construction of moving frames for general Lie group actions has remained obscure, particularly for anyone interested in new applications. Although they strive for generality, the range of examples treated remains rather limited, and Weyl's pointed critique of Cartan's original version still, in our opinion, applies to all of these later efforts.

There are two main goals of this series of papers devoted to a study of Cartan's method of moving frames. The first, of utmost importance for applications and the subject of the present work, is to develop a practical algorithm for constructing moving frames that is easy to implement, and can be systematically applied to concrete problems arising in different applications. Our new algorithm, which we call the method of "moving coframes", not only reproduces all of the classical moving frame constructions, often in a simpler and more direct fashion, but can be readily applied to a wide variety of new situations, including infinite-dimensional pseudo-groups, intransitive group actions, restricted reparametrization problems, joint group actions, to name a few. Although one can see the germs of our ideas in the above mentioned references, our approach is different, and, we believe, significantly easier to implement in practical examples. Standard presentations of the method rely on an unusual hybrid of vector fields and differential forms. Our approach is inspired by the powerful Cartan equivalence method, [**11**, **16**, **38**], which has much of the flavor of moving frame-type computations, but relies solely on the use of differential forms, and the operation of exterior differentiation. The moving coframe method we develop does have a complete analogy with the Cartan equivalence method; indeed, we shall see that the method includes not only all moving frame type equivalence problems, under both finite-dimensional Lie transformation groups and infinite Lie pseudo-groups, but also includes the standard Cartan equivalence problems in a very general framework.

Our second goal is to rigorously justify the moving coframe method by proposing a new theoretical foundation for the method of moving frames. This will form the subject of the second paper in the series, [**15**], and will be based on a second algorithm, known as regularization. The key new idea is to avoid the technically complicated normalization procedure during the initial phases of the computation, leading to a fully regularized moving frame. Once a moving frame and coframe, along with the complete system of invariants, are constructed in the regularized framework, one can easily restrict these invariants to particular classes of submanifolds, producing (in nonsingular cases) the standard moving frame. This approach enables us to successfully bypass branching and singularity complications, and enables one to treat both generic and singular submanifolds on the same general footing. Once the regularized solution to the problem has been properly implemented, the *a posteriori* justification for the usual normalization and reduction procedure can be readily provided. Details and further examples appear in part II, [**15**].

Beyond the traditional application to the differential geometry of curves and surfaces in certain homogeneous spaces, there are a host of applications of the method that lend great importance to its proper implementation. Foremost are the equivalence and symmetry theorems of Cartan, that characterize submanifolds up to a group transformation by the functional relationships among their fundamental differential invariants. The method provides an effective means of computing complete systems of differential invariants and associated invariant differential operators, which are used to generate all the higher order invariants. The fundamental differential invariants and their derived invariants, up to an appropriate order, serve to parametrize the "classifying manifold" associated with a given submanifold; the Cartan solution to the equivalence problem states that two submanifolds are (locally) congruent under a group transformation if and only if their classifying manifolds are identical. Moreover, the dimension of the classifying manifold completely determines the dimension of the symmetry subgroup of the submanifold in question. We note that the differential invariants also form the fundamental building blocks of basic physical theories, enabling one to construct suitably invariant differential equations and variational principles, cf. [**38**].

Additional motivation for pursuing this program comes from new applications of moving frames to computer vision promoted by Faugeras, [**13**], with applications to invariant curve and surface evolutions, and the use of the classifying (or "signature") manifolds in the invariant characterization of object boundaries that forms the basis of a fully group-invariant object recognition visual processing system, [**5**]. Although differential invariants have evident direct applications to object recognition in images, the often high order of differentiation makes them difficult to compute in an accurate and stable manner. One alternative approach, [**35**], is to use joint differential invariants, or, as they are known in the computer vision literature, "semi-differential invariants", which are based on several points on the submanifold of interest. Although a few explicit examples of joint differential invariants are known, there is, as far as we know, no systematic classification of them in the literature. We show how the method of moving coframes can be readily used to compute complete systems of joint differential invariants, and illustrate with some examples of direct interest in image processing. The approximation of higher order differential invariants by joint differential invariants and, generally, ordinary joint invariants leads to fully invariant finite difference numerical schemes for their computation, which were first proposed in [**5**]. The moving coframe method should aid in the understanding and extension of such schemes to more complicated situations.

In this paper, we begin with a review of the basic equivalence problems for submanifolds under transformation groups that serve to motivate the method of moving frames. Section 3 provides a brief introduction to one of the basic tools that is used in the moving coframe method — the left-invariant Maurer–Cartan forms on a Lie group. Two practical means of computing the Maurer–Cartan forms, including a novel method based directly on the group transformation rules, are discussed. Section 4 begins the presentation of the moving coframe method for the simplest category of examples — finite-dimensional transitive group actions — and illustrates it with an equivalence problem arising in the calculus of variations and in classical invariant theory. Section 5 extends the basic method to intransitive Lie group actions. The simplest example of an infinite-dimensional pseudo-group, namely the reparametrization pseudo-group for parametrized submanifolds, is discussed in Section 6 and illustrated with a well studied geometrical example — the case of curves in the Euclidean plane. This is followed by a discussion of curves in affine and projective geometry, reproducing classical moving frame computations in a simple direct manner based on the moving coframe approach; in Section 7, the connections between the classical and moving coframe methods are explained in further detail. Section 8 employs the moving coframe method to completely analyze the joint differential invariants in two particular geometrical examples — two-point differential invariants for curves in the Euclidean and affine plane. Section 9 discusses how to analyze more general pseudo-group actions, illustrating the method with two examples arising in classical work of Lie, [**28**], Vessiot, [**46**], and Medolaghi, [**34**]. In addition, we show how to solve the equivalence problem for second order ordinary differential equations under the pseudo-group of fiber-preserving transformations using the moving coframe method, thereby indicating how all Cartan equivalence problems can be treated by this method. Finally, we discuss some open problems that are under current investigation. In all cases, the paper is designed for a reader who is interested in applications, in that only the basic algorithmic steps are discussed in detail. In order not to cloud the present practically-oriented exposition, precise theoretical justifications for the algorithms proposed here will appear in the second paper in this series, [**15**].

2. The Basic Equivalence Problems.

We begin our exposition with a discussion of the basic equivalence problems which can be handled by the method of moving frames; see Jensen, [**23**; p. VI], for additional details. Suppose G is a transformation group acting smoothly on an m-dimensional manifold M. In classical applications, G is a finite-dimensional Lie group, but, as we shall see, the method can be extended to infinite-dimensional Lie pseudo-group actions, e.g., the group of conformal transformations on a Riemannian surface, the group of canonical transformations on a symplectic space, or the group of contact transformations on a jet space. In either situation, a basic equivalence problem is to determine whether two given submanifolds are congruent modulo a group transformation. We shall divide the basic problem into two different versions, depending on whether one allows reparametrizations of the submanifolds in question. Formally, these can be stated as follows.

The Fixed Parameter Equivalence Problem: Given two embeddings $\iota: X \to M$ and $\bar{\iota}: X \to M$ of an $n-$dimensional manifold X into M does there exist a group transformation $g \in G$ such that

$$\bar{\iota}(x) = g \cdot \iota(x) \qquad \text{for all} \qquad x \in X. \tag{2.1}$$

The Unparametrized Equivalence Problem: Given two submanifolds $N, \overline{N} \subset M$ of the same dimension n, determine whether there exists a group transformation $g \in G$ such that

$$g \cdot N = \overline{N}. \tag{2.2}$$

Submanifolds satisfying (2.2) are said to be *congruent* under the group action.

In both problems we shall only consider the question in the small, meaning that (2.1) only needs to hold on an open subset of X, or that congruence, (2.2), holds in a suitable neighborhood of given points $z_0 \in N$, $\bar{z}_0 \in \overline{N}$. Global issues require global constructions that lie outside the scope of the Cartan approach to equivalence problems.

Note that the problem of determining the symmetries of a submanifold, meaning the set of all group elements that preserve the submanifold, forms a particular case of the equivalence problem. Indeed, a *symmetry* of a submanifold is merely a self-equivalence. For instance, the unparametrized symmetries of a given submanifold $N \subset M$ are those group elements that (locally) satisfy $g \cdot N = N$. Note that the symmetry group of a given submanifold forms a subgroup $H \subset G$ of the full transformation group.

Example 2.1. A classical example is inspired by the geometry of curves in the Euclidean plane. A curve $C \subset \mathbb{R}^2$ is parametrized by a smooth map $\mathbf{x}(t) = (x(t), y(t))$ defined on (a subinterval of) $\mathbb{R}$. The underlying group for Euclidean planar geometry is the Euclidean group $\mathrm{E}(2) = \mathrm{O}(2) \ltimes \mathbb{R}^2$ consisting of translations, rotations and (in the non-oriented case) reflections.

In the fixed parametrization problem, we are given two parametrized curves $\mathbf{x}(t)$ and $\overline{\mathbf{x}}(t)$, and want to know when there exists a Euclidean motion such that $\overline{\mathbf{x}}(t) = R \cdot \mathbf{x}(t) + a$ for all t, where the rotation $R \in \mathrm{O}(2)$ and translation $a \in \mathbb{R}^2$ are both independent of t. Physically, we are asking when two moving particles differ by a fixed Euclidean motion at all times, a problem that has significant applications to motion detection and recognition of moving objects.

In the unparametrized problem, we are interested in determining when two curves are congruent under a Euclidean motion, meaning $\overline{C} = R \cdot C + a$ for some fixed Euclidean transformation $(R, a) \in \mathrm{E}(2)$. This occurs if and only if there exists a change of parameter $\bar{t} = \tau(t)$ such that $\overline{\mathbf{x}}(\tau(t)) = R \cdot \mathbf{x}(t) + a$ for some fixed Euclidean transformation (R, a).

A Euclidean symmetry of a curve C is a Euclidean transformation (R, a) that preserves the curve: $R \cdot C + a = C$. For instance, the Euclidean symmetries of a circle consist of the rotations around its center. In the fixed parameter version, the circle must be parametrized by a constant multiple of arc length for this to remain valid.

Example 2.2. Consider the action

$$A\colon \quad (x,u) \longmapsto \left(\frac{\alpha x+\beta}{\gamma x+\delta}, \frac{u}{\gamma x+\delta} \right), \qquad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2), \tag{2.3}$$

of the general linear group GL(2) on $\mathbb{R}^2$. This forms a multiplier representation of GL(2), cf. [**14, 38**], which lies at the heart of classical invariant theory. We restrict our attention to curves given by the graphs of functions $u = f(x)$, thereby avoiding issues of reparametrization. Two such curves are equivalent if and only if their defining functions f and $\bar f$ are related by the formula

$$f(x) = (\gamma x+\delta)\, \bar f \left(\frac{\alpha x+\beta}{\gamma x+\delta} \right) = (\gamma x+\delta) \bar f(\bar x), \tag{2.4}$$

for some nonsingular matrix A. Equation (2.4) is the fundamental equivalence condition for first order Lagrangians that depend only on a derivative coordinate in the calculus of variations, cf. [**36**]. Moreover, if $f(x) = \sqrt[n]{P(x)}$, and $\bar f(\bar x) = \sqrt[n]{\overline{P}(\bar x)}$, then (2.4) becomes†

$$P(x) = (\gamma x+\delta)^n\, \overline{P} \left(\frac{\alpha x+\beta}{\gamma x+\delta} \right) = (\gamma x+\delta)^n\, \overline{P}(\bar x). \tag{2.5}$$

In the case when P and $\overline{P}$ are polynomials of degree n, (2.5) indicates their equivalence under projective transformations, and so forms the fundamental equivalence problem of classical invariant theory.

In the general unparametrized equivalence problem, typically, the submanifolds N and $\overline{N}$ are formulated via explicit parametrizations $\iota\colon X \to M$, with image $N = \iota(X)$ and $\bar\iota\colon X \to M$, with $\overline{N} = \bar\iota(X)$, where, for simplicity, the parameter spaces are taken to be the same. (Indeed, since our considerations are always local, we shall not lose any generality by assuming that $X \subset \mathbb{R}^n$ is an open subset of Euclidean space.) In such cases, we can easily reformulate the unparametrized equivalence problem in the following form.

The Reparametrization Equivalence Problem: Given two embeddings $\iota\colon X \to M$ and $\bar\iota\colon X \to M$ of an $n-$dimensional manifold X into M does there exist a local diffeomorphism $\Phi\colon X \to X$, i.e., a change of parameter, and a group transformation $g \in G$ such that

$$\bar\iota(\Phi(x)) = g \cdot \iota(x), \qquad \text{for all} \qquad x \in X. \tag{2.6}$$

We shall see that by solving the fixed parametrization problem, first in the case of G being a finite dimensional Lie transformation group, then extending this to the case of G being an infinite Lie pseudo-group of transformations, that we will then be able to solve the reparametrization problem. For instance, we can reformulate the unparametrized equivalence problem for curves in the Euclidean plane as a fixed parametrization problem for curves in the extended space $E = \mathbb{R} \times \mathbb{R}^2$, which has coordinates $(t, \mathbf{x}) = (t, x, y)$. The

† We are ignoring the branching of the n^{th} root here. See [**36, 38**] for a more precise version of this construction.

extended curve is given as the graph $\{(t, \mathbf{x}(t))\}$ of the original parametrized curve, and the pseudo-group $\mathcal{G} = \mathcal{D}\mathit{iff}(1) \times \mathrm{E}(2)$ acting on E consists of a finite-dimensional group, the Euclidean group $\mathrm{E}(2)$ acting on $\mathbb{R}^2$, together with the infinite-dimensional pseudo-group $\mathcal{D}\mathit{iff}(1)$ consisting of all smooth (local) diffeomorphisms $\bar{t} = \tau(t)$ of the parameter space $\mathbb{R}$.

The formulation of the reparametrization problem in the form (2.6) indicates an intermediate extension of the two cases, in which one only allows a subclass of all possible reparametrizations.

The Restricted Reparametrization Equivalence Problem: Given two embeddings $\iota: X \to M$ and $\bar{\iota}: X \to M$ of an $n-$dimensional manifold X into M and a Lie pseudo-group of transformations $\mathcal{H}$ acting on X, determine whether there exists a group transformation $g \in G$ such that (2.6) holds for some reparametrization $\Phi \in \mathcal{H}$ in the prescribed pseudo-group.

For example, one might consider the problem of equivalence of surfaces in Euclidean space, in which one is only allowed conformal, or area preserving, or Euclidean reparametrizations. The general reparametrization equivalence problem is, of course, a special case when the pseudo-group $\mathcal{H} = \mathcal{D}\mathit{iff}(X)$ is the entire local diffeomorphism group.

In general, the solution to any equivalence problem is governed by a complete system of invariants. In the present context, the invariants are the fundamental differential invariants for the transformation group action in question. Thus, any solution method must, as a consequence, produce the differential invariants in question.

Example 2.3. In the case of curves in Euclidean geometry, the ordinary curvature† function $\kappa = |\,\mathbf{x}_t\,|^{-3}(\mathbf{x}_t \wedge \mathbf{x}_{tt})$ is the fundamental differential invariant. For the fixed parametrization problem, there is a second fundamental differential invariant — the speed $v = |\,\mathbf{x}_t\,|$. Furthermore, all higher order differential invariants are obtained by successively differentiating the curvature (and speed) with respect to arc length $ds = v\,dt = |\,\mathbf{x}_t\,|\,dt$, which is the fundamental Euclidean invariant one-form. (In the fixed problem, one can replace s derivatives by t derivatives since dt is also invariant if we disallow any changes in parameter.) A similar result holds for general transformation groups — one can obtain all higher order differential invariants by successively applying certain invariant differential operators to the fundamental differential invariants, cf. [**38**].

The functional relationships between the fundamental differential invariants will solve the equivalence problem. Roughly speaking, one uses the differential invariants to parametrize a "classifying" or "signature" manifold associated with the given submanifold, and the result is that, under suitable regularity hypotheses, two submanifolds will be congruent under a group transformation if and only if their classifying manifolds are *identical*. For example, in the unparametrized Euclidean curve problem, the classifying curve is parametrized by the two curvature invariants (κ, κ_s), whereas in the fixed problem, one uses all four invariants $(v, \kappa, v_s, \kappa_s)$ to parametrize the classifying curve. See [**5**] for applications of

† Here $|\,\mathbf{a}\,|$ is the usual Euclidean norm and $\mathbf{a} \wedge \mathbf{b}$ is the scalar-valued cross product between vectors in the plane.

the classifying curve to the problem of object recognition in computer vision. Of course, this "solution" reduces one to another potentially difficult *identification problem* — when do two parametrized submanifolds coincide? One approach to the latter problem is to use the Implicit Function Theorem to realize the classifying submanifold as the graph of a function, which eliminates the reparametrization ambiguity. Alternatively, in an algebraic context, a solution can be provided by Gröbner basis techniques, cf. [**4**]. Neither approach completely resolves the general identification problem, but particular cases can often be handled effectively.

Remark: A more standard solution to the equivalence problem depends on the choice of a base point $\mathbf{x}_0 = \mathbf{x}(t_0)$ on the curve. Then the curvature $\kappa(s)$ as a function of arc length $s = \int_{\mathbf{x}_0}^{\mathbf{x}} ds$ uniquely characterizes the curve up to Euclidean congruence, [**19**; p. 24]. The classifying curve approach has two distinct advantages: first, there is no choice of base point required, which eliminates the translational ambiguity inherent in the curvature function $\kappa(s)$; second, the classifying curve is completely local, whereas the arc length s is a nonlocal function of the curve. Note that the classifying curve can be computed directly, without appealing to the arc length parametrization.

The differential invariants can also be used to determine the structure of the symmetry group. In the case of an effectively acting Lie group G, the codimension of the symmetry subgroup H of the submanifold N, i.e., $\dim G - \dim H$, is the same as the number of functionally independent differential invariants on the submanifold. In particular, the maximally symmetric submanifolds occur when all differential invariants are constant; if G acts transitively, then these can be identified with the homogeneous submanifolds of M, i.e., the orbits of suitable closed subgroups of G, cf. [**23**]. For instance, in the Euclidean case, the maximally symmetric curves are where the curvature is constant, which are the circles and straight lines, since these are the orbits of the one-parameter subgroups of E(2). (Technically, these retain the infinite-dimensional reparametrization group $\mathcal{D}\mathit{iff}(1)$ as an additional symmetry group.) In the fixed parameter version, the circles and straight lines must be parametrized by a constant multiple of their arc length in order to retain their distinguished symmetry status.

Finally, we remark that differential invariants can be used to construct general invariant differential equations admitting the given transformation group. Specifically, suppose $J_1, \ldots, J_N$ form a complete system of functionally independent k^{th} order differential invariants, defined on an open subset $\mathcal{V}^k \subset \mathrm{J}^k$ of the jet space where the prolonged group action is regular. Then, on $\mathcal{V}^k$, any k^{th} order system of differential equations admitting G as a symmetry group can be written in terms of the differential invariants: $H_\nu(J_1, \ldots, J_N) = 0$. For example, the most general Euclidean-invariant third order differential equation has the form $d\kappa/ds = H(\kappa)$, equating the derivative of curvature with respect to arc length to a function of curvature. Similar comments apply to invariant variational problems, and we refer the reader to [**37, 38**], for details. These results form the foundations of modern physical field theories, in which one bases the differential equations, or variational principle, on its invariance with respect to the theory's underlying symmetry group. The groups in question range from basic Poincaré and conformal invariance, to the exceptional simple Lie groups lying at the foundations of string theory, as well as infinite-dimensional gauge groups and groups of Kac–Moody type. Remarkably, complete systems of differential invariants are known for only a small handful of transformation groups arising in physical applications — a collection that includes *none* of the above mentioned groups! Our moving coframe algorithm provides an direct and effective means for providing such classifications.

3. The Maurer–Cartan Forms.

In our approach to the theory and practical implementation of the method of moving frames, the left-invariant Maurer–Cartan forms on a finite-dimensional Lie group play an essential role. We therefore begin by reviewing the basic definition, and then present two computationally effective methods for finding the explicit formulae for the Maurer–Cartan forms. The theoretical justification for the second method will appear in part II, [**15**].

Throughout this section, G will be an r-dimensional Lie group. We let $L_g \colon h \mapsto g \cdot h$ denote the standard left multiplication map.

Definition 3.1. A one-form μ on G is called a (left-invariant) *Maurer–Cartan form* if it satisfies

$$(L_g)^* \mu = \mu \qquad \text{for all} \qquad g \in G. \tag{3.1}$$

Remark: If one uses the right-invariant Maurer–Cartan forms instead, one is led to an alternative theory of right moving frames. Although the left versions appear almost exclusively in the literature, their right counterparts will play an important role in the theoretical justifications and the regularized method introduced in part II. In this paper, though, we shall exclusively use the left-invariant Maurer–Cartan forms and moving frames; see [**15**] for details.

The space of Maurer–Cartan forms on G is an r-dimensional vector space, which can naturally be identified with the dual to the Lie algebra $\mathfrak{g}$ of left-invariant vector fields on G. If we choose a basis $\mathbf{v}_1, \ldots, \mathbf{v}_r$ of $\mathfrak{g}$, then there is a dual basis $\mu^1, \ldots, \mu^r$ of the space of Maurer–Cartan forms, satisfying $\langle\, \mu^i \,; \mathbf{v}_j \,\rangle = \delta^i_j$, where δ^i_j is the usual Kronecker delta. The basis Maurer–Cartan forms satisfy the fundamental *structure equations*

$$d\mu^i = -\sum_{j<k} C^i_{jk}\, \mu^j \wedge \mu^k, \tag{3.2}$$

where the coefficients C^i_{jk} are the structure constants corresponding to our choice of basis of the Lie algebra $\mathfrak{g}$. The Maurer–Cartan forms are a *coframe* on the Lie group G, meaning that they form a pointwise basis for the cotangent space T^*G, or, equivalently, that we can write any one-form ω on G as a linear combination $\omega = \sum f_i\, \mu^i$ thereof, where the f_i are suitable smooth functions.

The most common method for explicitly determining the Maurer–Cartan forms on a given Lie group is to realize $G \subset \mathrm{GL}(n)$ as a matrix Lie group. The independent entries of the $n \times n$ matrix of one-forms

$$\boldsymbol{\mu} = A^{-1}\, dA \tag{3.3}$$

form a basis for the left-invariant Maurer–Cartan forms on G. Here $A = A(g^1, \ldots, g^r) \in G$ represents the general matrix in G, which we have parametrized by local coordinates

$(g^1, \ldots, g^r)$ near the identity, and $dA = \sum (\partial A / \partial g^i)\, dg^i$ is its differential, which is an $n \times n$ matrix of one-forms.

For example, in the case $G = \mathrm{GL}(2)$, the four independent Maurer–Cartan forms are the components of the matrix

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix} = A^{-1}\, dA = \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} \delta\, d\alpha - \beta\, d\gamma & \delta\, d\beta - \beta\, d\delta \\ \alpha\, d\gamma - \gamma\, d\alpha & \alpha\, d\delta - \gamma\, d\beta \end{pmatrix}. \tag{3.4}$$

Similarly, if $G = \mathrm{E}(2) = \mathrm{O}(2) \ltimes \mathbb{R}^2$ is the Euclidean group in the plane, then we can identify $\mathrm{E}(2) \subset \mathrm{GL}(3)$ as a subgroup of $\mathrm{GL}(3)$ by identifying $(R, \mathbf{a}) \in \mathrm{E}(2)$ with the 3×3 matrix

$$\begin{pmatrix} R & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & a \\ \sin\phi & \cos\phi & b \\ 0 & 0 & 1 \end{pmatrix}.$$

Substituting into (3.3) leads to

$$\boldsymbol{\mu} = \begin{pmatrix} R^{-1} & -R^{-1}\mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} dR & d\mathbf{a} \\ \mathbf{0} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -d\phi & \cos\phi\, da + \sin\phi\, db \\ d\phi & 0 & -\sin\phi\, da + \cos\phi\, db \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, the three independent Euclidean Maurer–Cartan forms are

$$\mu_1 = d\phi, \qquad \mu_2 = \cos\phi\, da + \sin\phi\, db, \qquad \mu_3 = -\sin\phi\, da + \cos\phi\, db. \tag{3.5}$$

In cases when the group is explicitly realized as a local group of transformations on a manifold M, and not necessarily as a matrix Lie group, it is useful to have a direct method for determining the Maurer–Cartan forms. Given $g \in G$ and $z \in M$, we explicitly write the group transformation $\bar z = g \cdot z$ in coordinate form:

$$\bar z^i = H^i(z, g), \qquad i = 1, \ldots, m.$$

We then compute the differentials of the group transformations:

$$d\bar z^i = \sum_{k=1}^m \frac{\partial H^i}{\partial z^k}\, dz^k + \sum_{j=1}^r \frac{\partial H^i}{\partial g^j}\, dg^j, \qquad i = 1, \ldots, m,$$

or, more compactly,

$$d\bar z = H_z\, dz + H_g\, dg. \tag{3.6}$$

Next, set $d\bar z = 0$ in (3.6), and solve the resulting system of linear equations for the differentials dz^k. This leads to the formulae

$$-dz = F\, dg = (H_z^{-1} \cdot H_g)\, dg,$$

or, in full detail,

$$-\, dz^k = \sum_{j=1}^r F_j^k(z, g)\, dg^j, \qquad k = 1, \ldots, m. \tag{3.7}$$

Then, for each k and each fixed $z_0 \in M$, the one-form

$$\mu_0 = \sum_{j=1}^{r} F_j^k(z_0, g)\, dg^j, \tag{3.8}$$

is a left-invariant Maurer–Cartan form on the group G. Alternatively, if one expands the right hand side of (3.7) in a power series (or Fourier series, or ...) in z,

$$\sum_{j=1}^{r} F_j^k(z, g)\, dg^j = \sum_{\#I=0}^{\infty} z^I\, \mu_I, \tag{3.9}$$

then each coefficient μ_I also forms a left-invariant Maurer–Cartan form on G. In particular, when G acts locally effectively, the resulting collection of one-forms spans the space of Maurer–Cartan forms.

Example 3.2. Consider the action of GL(2) given by

$$\bar{x} = \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad \bar{u} = \frac{u}{\gamma x + \delta}, \tag{3.10}$$

as discussed above. Differentiating (3.10), we find, as in (3.6),

$$\begin{aligned} d\bar{x} &= \frac{(\gamma x + \delta)(\alpha\, dx + x\, d\alpha + d\beta) - (\alpha x + \beta)(\gamma\, dx + x\, d\gamma + d\delta)}{(\gamma x + \delta)^2} \\ &= \frac{(\alpha\delta - \beta\gamma)\, dx + (\gamma x + \delta)(x\, d\alpha + d\beta) - (\alpha x + \beta)(x\, d\gamma + d\delta)}{(\gamma x + \delta)^2}, \\ d\bar{u} &= \frac{(\gamma x + \delta)\, du + u(\gamma\, dx + x\, d\gamma + d\delta)}{(\gamma x + \delta)^2}. \end{aligned}$$

Setting $d\bar{x} = 0 = d\bar{u}$ and solving for dx and du, we obtain

$$\begin{aligned} -dx &= \frac{\delta\, d\beta - \beta\, d\delta}{\alpha\delta - \beta\gamma} + \left(\frac{\delta\, d\alpha + \gamma\, d\beta - \alpha\, d\delta - \beta\, d\gamma}{\alpha\delta - \beta\gamma} \right) x + \left(\frac{\gamma\, d\alpha - \alpha\, d\gamma}{\alpha\delta - \beta\gamma} \right) x^2, \\ -du &= \left(\frac{\alpha\, d\delta - \gamma\, d\beta}{\alpha\delta - \beta\gamma} \right) u + \left(\frac{\alpha\, d\gamma - \gamma\, d\alpha}{\alpha\delta - \beta\gamma} \right) xu. \end{aligned} \tag{3.11}$$

Note that the coefficients of 1, x and x^2 in the first formula, i.e.,

$$\widehat{\mu}_1 = \frac{\delta\, d\beta - \beta\, d\delta}{\alpha\delta - \beta\gamma}, \qquad \widehat{\mu}_2 = \frac{\delta\, d\alpha + \gamma\, d\beta - \alpha\, d\delta - \beta\, d\gamma}{\alpha\delta - \beta\gamma}, \qquad \widehat{\mu}_3 = \frac{\gamma\, d\alpha - \alpha\, d\gamma}{\alpha\delta - \beta\gamma}, \tag{3.12}$$

recover three of the Maurer–Cartan forms in (3.4), while the coefficient of either u or xu in the second formula provides the remaining one.

Remark: The coefficients in (3.11) are, in fact, immediately found in terms of the coefficients of the infinitesimal generators for the transformation group. See [**15**] for details.

If G does not act effectively on M, then the forms computed by this method will form a basis for the annihilator

$$(\mathfrak{g}_M)^\perp = \{ \, \omega \in \mathfrak{g}^* \ \big| \ \langle \, \omega \, ; \mathbf{v} \, \rangle = 0 \qquad \text{for all} \qquad \mathbf{v} \in \mathfrak{g}_M \, \}$$

of the Lie algebra of the global isotropy subgroup

$$G_M = \{ \, g \in G \ | \ g \cdot z = z \qquad \text{for all} \qquad z \in M \, \},$$

and thus can be identified with the Maurer–Cartan forms for the effectively acting quotient group $\widetilde{G} = G/G_M$. For example, if we only treat the linear fractional transformations in x in (3.10), then the resulting three Maurer–Cartan forms (3.12) all annihilate the generator $\mathbf{v} = \alpha\partial_\alpha + \beta\partial_\beta + \gamma\partial_\gamma + \delta\partial_\delta$ of the isotropy subgroup $\{\lambda 1\!\!1\} \subset \mathrm{GL}(2)$ consisting of scalar multiples of the identity matrix. Hence, the three one-forms can be identified with a basis for the Maurer–Cartan forms of the effectively acting projective linear group $\mathrm{PSL}(2) = \mathrm{GL}(2)/\{\lambda 1\!\!1\}$.

4. Compatible Lifts and Moving Coframes.

In this section, we begin our development of the moving coframe method, strting with the simplest problems and gradually work our way up to more complicated situations. Throughout this section, we assume that G is an r-dimensional Lie group which acts locally effectively and transitively on an m-dimensional manifold M. (As remarked above, we can always assume local effectiveness by quotienting by the global isotropy subgroup.) We begin by choosing a convenient "base point" $z_0 \in M$.

Definition 4.1. A smooth map $\rho: M \to G$ is called a *compatible lift* with base point z_0 if it satisfies

$$\rho(z) \cdot z_0 = z. \tag{4.1}$$

In order to compute the most general compatible lift, we solve the system of m equations (4.1) for m of the group parameters in terms of the coordinates z on M and the remaining $r - m = \dim G - \dim M$ group parameters, which we denote by h. This leads to a general formula $g = \rho_0(z, h)$ for the solution to the compatibility equations (4.1). In other words, by solving the compatibility conditions (4.1), we have effectively "normalized" m of the original group parameters. Since our considerations are always local, in practice, we only need to solve the compatibility equations (4.1) near z_0. In accordance with Cartan's terminology, [**6**], we will call the general compatible lift $\rho_0(z, h)$ the *moving frame of order zero* for the given transformation group. If $\iota: X \to M$ defines a parametrized submanifold $N = \iota(X)$, then one can view the composition $\rho_0(\iota(x), h)$ as a restriction of the order zero moving frame to the submanifold N, where the unnormalized parameters h determine the degree of indeterminacy of the moving frame on N. In geometrical situations, such restrictions can be identified with the classical moving frames; see also Section 7 below.

Example 4.2. Consider the planar action (2.3) of the general linear group GL(2):

$$A \cdot (x, u) = \left(\frac{\alpha x + \beta}{\gamma x + \delta}, \frac{u}{\gamma x + \delta} \right), \qquad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2), \tag{4.2}$$

The action (4.2) is transitive on $M = \mathbb{R}^2 \setminus \{u = 0\}$. Choose the base point to be $z_0 = (0,1)$. Since $A \cdot z_0 = (\beta/\delta, 1/\delta)$, any compatible lift $A = \rho(x,u)$ must satisfy $\beta/\delta = x$, $1/\delta = u$, and hence the solution to (4.1) is

$$\beta = \frac{x}{u}, \qquad \delta = \frac{1}{u}. \tag{4.3}$$

The most general compatible lift thus has the form

$$\rho_0(x,u,\alpha,\gamma) = \begin{pmatrix} \alpha & x/u \\ \gamma & 1/u \end{pmatrix}, \tag{4.4}$$

where $\alpha = \alpha(x,u)$, $\gamma = \gamma(x,u)$ are arbitrary functions, subject only to the condition $\alpha \neq x\gamma$, so that the determinant of (4.4) does not vanish, and hence ρ_0 does take its values in the group GL(2).

Note that since G acts transitively, we can locally identify $M \simeq G/H$ with a homogeneous space, where $H = G_{z_0}$ is the isotropy group of the base point. Therefore, a compatible lift is merely a (local) section of the fiber bundle $G \to G/H$.

Proposition 4.3. *Two maps $\rho, \widehat{\rho}\colon M \to G$ are compatible lifts with the same base point if and only if they satisfy*

$$\widehat{\rho}(z) = \rho(z) \cdot \eta(z),$$

where $\eta\colon M \to H$ is an arbitrary map to the isotropy subgroup of the base point z_0.

Thus, in the previous example, the isotropy subgroup H of the point $z_0 = (0,1)$ consists of all invertible lower triangular matrices of the form $\begin{pmatrix} \alpha' & 0 \\ \gamma' & 1 \end{pmatrix}$. Indeed, we can rewrite (4.4) in the factored form

$$\rho_0(x,u,\alpha,\gamma) = \begin{pmatrix} \alpha & x/u \\ \gamma & 1/u \end{pmatrix} = \begin{pmatrix} 1 & x/u \\ 0 & 1/u \end{pmatrix} \begin{pmatrix} \alpha' & 0 \\ \gamma' & 1 \end{pmatrix}, \tag{4.5}$$

where $\alpha' = \alpha - x\gamma$, $\gamma' = u\gamma$, reconfirming Proposition 4.3 in this particular example.

Although the remaining unspecified group parameters can be identified with the isotropy subgroup coordinates, in any practical implementation of the moving coframe algorithm, it is not necessary to identify the isotropy subgroup explicitly, nor to adopt its particular coordinates to characterize the order zero moving frame. Thus, in the present example, the coordinates α, γ, are just as effective as the subgroup coordinates α', γ'. (The interested reader can follow through the ensuing calculations using the subgroup coordinates instead, reproducing the final result.)

The order zero moving frame $\rho_0(z,h)$, which is the general solution to the compatible lift equations (4.1), defines a map from the zero$^{\rm th}$ order moving frame bundle $\mathcal{B}_0 = M \times H \simeq G/H \times H$, coordinatized by (z,h), to the group G, which is, in fact, a local diffeomorphism $\rho_0\colon \mathcal{B}_0 \xrightarrow{\sim} G$. There is an induced action of G on the moving frame bundle $\mathcal{B}_0$ that makes ρ_0 into a G-equivariant map: $\rho_0(g \cdot (z,h)) = g \cdot \rho_0(z,h)$. Thus, the action on the unnormalized group parameters h can be explicitly determined by multiplying the moving frame on the left by a group transformation. The action of G on $\mathcal{B}_0$ projects to the original action of G on M, so that $g \cdot (z,h) = (g \cdot z, \eta(g,z,h))$ for $g \in G$.

In the present example, the induced action of GL(2) on the unspecified parameters α, γ, is found by multiplying the moving frame (4.4) on the left by a group element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2)$; explicitly,

$$\begin{pmatrix} \overline{\alpha} & \bar{x}/\bar{u} \\ \overline{\gamma} & 1/\bar{u} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \alpha & x/u \\ \gamma & 1/u \end{pmatrix} = \begin{pmatrix} a\alpha + b\gamma & (ax+b)/u \\ c\alpha + d\gamma & (cx+d)/u \end{pmatrix}. \tag{4.6}$$

Therefore, the action of $G = \mathrm{GL}(2)$ on the moving frame bundle $\mathcal{B}_0$ is given by

$$\bar{x} = \frac{ax+b}{cx+d}, \qquad \bar{u} = \frac{u}{cx+d}, \qquad \overline{\alpha} = a\alpha + b\gamma, \qquad \overline{\gamma} = c\alpha + d\gamma. \tag{4.7}$$

Note that the (x,u) transformations coincide with the original action (2.3), as they should.

Remark: In practical implementations of the moving coframe algorithm, we do *not* have to explicitly compute this group action. We do this here so as to provide the reader with some justification for our claims.

Remark: The action of G on $\mathcal{B}_0 = M \times H$ does *not* project to an action on the isotropy subgroup H, even if we use the associated subgroup coordinates. In the present example, we find (4.7) implies that the subgroup coordinates α', γ' in (4.5) transform according to

$$\overline{\alpha}' = \frac{ad-bc}{cx+d}\,\alpha', \qquad \overline{\gamma}' = \gamma' + \frac{cu}{cx+d}\,\alpha'.$$

The next step is to characterize the group transformations by a collection of differential forms. In the finite-dimensional situation that we are currently considering, these will be obtained by pulling back the left-invariant Maurer–Cartan forms $\boldsymbol{\mu}$ on G to the order zero moving frame bundle $\mathcal{B}_0$ using the compatible lift. The resulting one-forms $\boldsymbol{\zeta}_0 = (\rho_0)^* \boldsymbol{\mu}$ will provide an invariant coframe on $\mathcal{B}_0$, which we name the *moving coframe* of order zero. The moving coframe forms $\boldsymbol{\zeta}_0$ clearly satisfy the same Maurer–Cartan structure equations (3.2).

Theorem 4.4. *The order zero moving coframe forms completely characterize the group transformations on the bundle $\mathcal{B}_0$. In other words, a map $\Psi: \mathcal{B}_0 \to \mathcal{B}_0$ satisfies $\Psi^* \boldsymbol{\zeta}_0 = \boldsymbol{\zeta}_0$ if and only if $\Psi(z,h) = g \cdot (z,h)$ coincides with the action of a group element $g \in G$ on $\mathcal{B}_0$.*

In the present example, we substitute the formulae (4.3) characterizing our compatible lift (4.4) into the Maurer–Cartan forms (3.4). The result is the order zero moving coframe

$$\begin{aligned} \zeta_1 &= \frac{d\alpha - x\,d\gamma}{\alpha - \gamma x}, & \zeta_2 &= \frac{dx}{u(\alpha - \gamma x)}, \\ \zeta_3 &= \frac{u(\alpha\,d\gamma - \gamma\,d\alpha)}{\alpha - \gamma x}, & \zeta_4 &= -\frac{\gamma\,dx}{\alpha - \gamma x} - \frac{du}{u}, \end{aligned} \tag{4.8}$$

which forms a basis for the space of one-forms on $\mathcal{B}_0$. The skeptical reader can explicitly check that these four one-forms really do completely characterize the group action (4.7), as described in Theorem 4.4.

Let us now consider a curve $N \subset M$. For simplicity, we shall assume that the curve coincides with the graph of a function $u = u(x)$. However, this restriction is not essential for the method to work, and later we show how parametrized curves can also be readily handled by the general method. We restrict the moving coframe forms to the curve, which amounts to replacing the differential du by its "horizontal" component $u_x\, dx$. If we interpret the derivative u_x as a coordinate on the first jet space $\mathrm{J}^1 = \mathrm{J}^1 M \simeq \mathbb{R}^3$ of curves in M, then the restriction of a differential form to the curve can be reinterpreted as the natural projection of the one-form du on J^1 to its horizontal component, using the canonical decomposition of differential forms on the jet space into horizontal and contact components. Indeed, the vertical component of the form du is the contact form $du - u_x\, dx$, which vanishes on all prolonged sections of the first jet bundle $\mathrm{J}^1 M$. We refer the reader to [**38**; Chapter 4] for a comprehensive review of the contact geometry of jet bundles. Therefore, the restricted (or horizontal) moving coframe forms are explicitly given by

$$\eta_1 = \frac{d\alpha - x\, d\gamma}{\alpha - \gamma x}, \qquad \eta_2 = \frac{dx}{u(\alpha - \gamma x)}, \qquad \eta_3 = \frac{u(\alpha\, d\gamma - \gamma\, d\alpha)}{\alpha - \gamma x}, \qquad \eta_4 = \frac{\gamma(x u_x - u) - \alpha u_x}{u(\alpha - \gamma x)}\, dx, \tag{4.9}$$

which now depend on first order derivatives.

The next step in the procedure is to look for invariant combinations of coordinates and group parameters. Each such invariant combination will either provide us with a basic differential invariant for the problem, or, in the case that it explicitly depends on the remaining group parameters, a "lifted invariant" which can be normalized and thereby eliminate one of the remaining group parameters, as discussed below. Specifically, in the present example, a function $J(\alpha, \gamma, x, u, u_x)$ will be a *lifted invariant* provided it is unaffected by the group action on its arguments, meaning that

$$J(\overline{\alpha}, \overline{\gamma}, \bar{x}, \bar{u}, \bar{u}_{\bar{x}}) = J(\alpha, \gamma, x, u, u_x), \tag{4.10}$$

wherever $\overline{\alpha}, \overline{\gamma}, \bar{x}, \bar{u}$, are related to α, γ, x, u, according to the induced action (4.7) of the group on the moving frame bundle, and $\bar{u}_{\bar{x}}$ is related to u_x according to the standard prolongation, [**38**], of the action of G on M to the first jet bundle J^1. In the present case, if $\bar{x}, \bar{u}$ are given by (4.7), then a straightforward chain rule computation provides the prolonged action of $\mathrm{GL}(2)$ on the derivative coordinate:

$$\bar{u}_{\bar{x}} = \frac{(cx + d)u_x - cu}{ad - bc}. \tag{4.11}$$

In other words, we interpret $\alpha, \gamma, x, u, u_x$ as coordinates on a bundle $\widetilde{\mathcal{B}}_0 \to \mathrm{J}^1$ over the first jet space, which is merely the pull-back $\widetilde{\mathcal{B}}_0 = (\pi^1_0)^* \mathcal{B}_0$ of the zero$^{\text{th}}$ order moving frame bundle via the standard projection $\pi^1_0 \colon \mathrm{J}^1 \to M$. There is an induced action of G on $\widetilde{\mathcal{B}}_0$ which projects to its prolonged action on J^1. A (first order) lifted invariant, then, is just a function $J: \widetilde{\mathcal{B}}_0 \to \mathbb{R}$ which is invariant under the action of G on $\widetilde{\mathcal{B}}_0$. If the lifted invariant $J = J(x, u, u_x)$ does not, in fact, depend on the group parameters α, γ, then it will be a (first order) differential invariant. (However, in the present example, there are no non-constant first order differential invariants, since GL(2) acts transitively on J^1.) Alternatively, if J actually depends on either α or γ then it can be used in the normalization procedure.

Fortunately, the lifted invariants can be determined without explicitly computing the prolonged group action, or solving any differential equations. They appear in the linear dependencies among the restricted (horizontal) moving coframe forms! Indeed, because the one-forms are invariant, each coefficient J_i in a linear relation $\eta_0 = J_1\,\eta_1 + \cdots + J_k\,\eta_k$, in which the forms η_i on the right hand side are linearly independent, is automatically invariant under the action of the group. In our example, we note that, among the restricted one-forms (4.9), there is one linear dependency, namely $\eta_4 = J\eta_2$, where

$$J = \gamma(x u_x - u) - \alpha u_x. \tag{4.12}$$

One can explicitly verify that J is indeed a lifted invariant, meaning that it satisfies (4.10) whenever $\overline{\alpha}, \overline{\gamma}, \bar{x}, \bar{u}, \bar{u}_{\bar{x}}$, are related to $\alpha, \gamma, x, u, u_x$, according to (4.7), (4.11).

The ultimate goal of the moving frame method is to eliminate all the ambiguities, i.e., the undetermined group parameters, in the original moving frame, in a suitably invariant manner. Cartan's crucial observation is that, we can, without loss of generality, *normalize* any lifted invariant by setting it equal to any convenient constant value,

$$J(\alpha, \gamma, x, u, u_x) = c, \tag{4.13}$$

without affecting the equivalence problem. In (4.13), c can be any constant, subject only to the requirement that the solutions to (4.13) remain in the group, e.g., that the determinant of any resulting matrix (4.4) remains nonzero. Typically, c is taken to be $0, 1$, or -1, although other values can be chosen to simplify the resulting formulae. Assuming that J does actually depend on the parameters α, γ, we can solve the normalization equation (4.13) for one of them; e.g., $\alpha = \alpha(\gamma, x, u, u_x)$. Because J is an invariant, such a normalization will not alter the solution to the equivalence problem, and hence we can use it to eliminate α from the original formulae for the moving frame and moving coframe. The result is a first order moving frame, depending on one fewer unnormalized group parameter. This produces a corresponding first order moving coframe, to which one can apply the same procedure, leading to a chain of successive normalizations and reductions, eventually enabling one to completely eliminate all the undetermined parameters and specify a uniquely defined moving frame on some suitable jet bundle $\mathrm{J}^n = \mathrm{J}^n M$.

In accordance with the general procedure, then, we can normalize our particular lifted invariant (4.12) by setting it equal to zero; the solution to the normalization equation $J = 0$ is then given by

$$\alpha = \left(\frac{x u_x - u}{u_x} \right) \gamma. \tag{4.14}$$

Substituting (4.14) into (4.4) produces the *first order moving frame*

$$\rho_1(x, u, u_x, \gamma) = \begin{pmatrix} (x u_x - u)\gamma/u_x & x/u \\ \gamma & 1/u \end{pmatrix}, \tag{4.15}$$

which now depends on first order derivatives of u, and just one unnormalized group parameter. We can regard the coordinates (x, u, u_x, γ) as parametrizing a bundle $\mathcal{B}_1 \to \mathrm{J}^1$ sitting over the first jet space, which is realized as a G-invariant subbundle of $\widetilde{\mathcal{B}}_0$, namely $\mathcal{B}_1 = J^{-1}\{0\} \subset \widetilde{\mathcal{B}}_0$. As before, one can restrict the first order moving frame to a curve $u = u(x)$ by restricting the map ρ_1 to the first prolongation or jet of the curve, i.e., we set $u = u(x)$, $u_x = u'(x)$, in (4.15), with γ indicating the remaining ambiguity. There is an induced action of GL(2) on $\mathcal{B}_1$, which projects to the usual first prolonged action $G^{(1)}$ of the group on J^1, cf. (4.7), (4.11), and makes the first order moving frame $\rho_1 : \mathcal{B}_1 \xrightarrow{\sim} G$ into a local G-equivariant diffeomorphism. In our case, the explicit transformation rules on $\mathcal{B}_1$ are given by

$$\bar{x} = \frac{ax+b}{cx+d}, \quad \bar{u} = \frac{u}{cx+d}, \quad \bar{u}_{\bar{x}} = \frac{(cx+d)u_x - cu}{ad-bc}, \quad \overline{\gamma} = \left(\frac{(cx+d)u_x - cu}{ad-bc}\right)\gamma, \qquad (4.16)$$

which coincide with left multiplication of the first order moving frame (4.15) by the given group element. (Again, these explicit formulae are provided for illustration only, and are not essential for application of the method.) Furthermore, substituting (4.14) into (4.8), we find the first order moving coframe

$$\begin{aligned} \zeta_1 &= \frac{d\gamma}{\gamma} - \frac{du_x}{u_x} + \frac{du - u_x\,dx}{u}, & \zeta_2 &= -\frac{u_x\,dx}{\gamma u^2}, \\ \zeta_3 &= \frac{\gamma u\,du_x}{u_x} - \gamma(du - u_x\,dx), & \zeta_4 &= \frac{du - u_x\,dx}{u}. \end{aligned} \qquad (4.17)$$

As in the order zero case, cf. Theorem 4.4, the first order moving coframe completely characterizes the group transformations on $\mathcal{B}_1$.

As before, we determine new lifted invariants by restricting the first order moving coframe one-forms to a curve $u = u(x)$. This amounts to replacing du and du_x by their horizontal components $u_x\,dx$ and $u_{xx}\,dx$ respectively, leading to the restricted forms

$$\eta_1 = \frac{d\gamma}{\gamma} - \frac{u_{xx}\,dx}{u_x}, \qquad \eta_2 = -\frac{u_x\,dx}{\gamma u^2}, \qquad \eta_3 = \frac{\gamma u u_{xx}\,dx}{u_x}, \qquad \eta_4 = 0, \qquad (4.18)$$

that now depend on second order derivatives. Alternatively, one could deduce these restricted forms by substituting the normalization (4.14) into the previous restricted forms (4.9). Note in particular that the fact that η_4 vanishes is an automatic consequence of our normalization condition $\eta_4 = J\eta_2 = 0$; alternatively, we note that ζ_4 is an invariant contact form, which hence vanishes when restricted to any submanifold. Now there is an additional dependency, namely $\eta_3 = K\eta_2$, where

$$K = -\frac{\gamma^2 u^3 u_{xx}}{u_x^2},$$

is a new lifted invariant. Again, the reader can check that K is invariant under the prolonged action of GL(2) on the bundle $\widetilde{\mathcal{B}}_1 = (\pi_1^2)^* \mathcal{B}_1 \to \mathrm{J}^2$, where $\pi_1^2 : \mathrm{J}^2 \to \mathrm{J}^1$ is the natural projection, and is provided by (4.16) and the second order prolongation (chain rule) formula

$$\bar{u}_{\bar{x}\bar{x}} = \frac{(cx+d)^3 u_{xx}}{(ad-bc)^2}. \tag{4.19}$$

We can normalize $K = -1$ by setting

$$\gamma = \frac{u_x}{\sqrt{u^3 u_{xx}}}\,. \tag{4.20}$$

Note that we *cannot* normalize $K = 0$ since this would require $\gamma = 0$, but then the lift (4.15) would have zero determinant, violating the group conditions. The final lift

$$\rho_2(x,u,u_x,u_{xx}) = \begin{pmatrix} \dfrac{x u_x - u}{\sqrt{u^3 u_{xx}}} & \dfrac{x}{u} \\ \dfrac{u_x}{\sqrt{u^3 u_{xx}}} & \dfrac{1}{u} \end{pmatrix} \tag{4.21}$$

defines the second order moving frame. The moving frame (4.21) provides an explicit G-equivariant identification $\rho_2\colon \mathcal{V}^2 \xrightarrow{\sim} G$ of the open subset $\mathcal{V}^2 = \{u u_{xx} \neq 0\} \subset \mathrm{J}^2$ of the second jet bundle with an open subset of the group G, identifying the prolonged action of $G^{(2)}$ on J^2 with the ordinary left multiplication on G; thus

$$\rho_2(g^{(2)} \cdot z^{(2)}) = g \cdot \rho_2(z^{(2)}), \qquad g \in \mathrm{GL}(2), \quad z^{(2)} = (x,u,u_x,u_{xx}) \in \mathcal{V}^2.$$

Substituting (4.20) into (4.18) produces the final set of invariant one-forms

$$\begin{aligned} \zeta_1 &= -\frac{du_{xx}}{2u_{xx}} - \frac{du}{2u} - \frac{u_x\,dx}{u}, & \zeta_2 &= -\sqrt{\frac{u_{xx}}{u}}\,dx, \\ \zeta_3 &= \frac{du_x}{\sqrt{u u_{xx}}} - \frac{u_x(du - u_x\,dx)}{\sqrt{u^3 u_{xx}}}, & \zeta_4 &= \frac{du - u_x\,dx}{u}, \end{aligned} \tag{4.22}$$

which form the second order moving coframe. Note that the second order moving frame (4.21) provides an equivalence, $\rho_2^* \mu_i = \zeta_i$, mapping the moving coframe forms on the second order jet space to the Maurer–Cartan forms (3.4) on the group. Consequently, the forms ζ_i uniquely characterize the second order prolonged action of $\mathrm{GL}(2)$ on $\mathcal{V}^2 \subset \mathrm{J}^2$.

Finally, the restricted (horizontal) moving coframe forms become

$$\eta_1 = -\frac{u u_{xxx} + 3u_x u_{xx}}{2u u_{xx}}\,dx, \qquad \eta_2 = \sqrt{\frac{u_{xx}}{u}}\,dx, \qquad \eta_3 = -\eta_2, \qquad \eta_4 = 0.$$

There is one final linear dependency, namely $\eta_1 = -I\,\eta_2$, where

$$I = \frac{u u_{xxx} + 3u_x u_{xx}}{2\sqrt{u u_{xx}^3}} \tag{4.23}$$

is the fundamental differential invariant of the transformation group, also known as the group-invariant curvature. The remaining one-form $ds = \eta_2$ is the fundamental invariant one-form, or group-invariant arc length element. All higher order differential invariants can be found by differentiating the curvature invariant with respect to the invariant arc length; for instance, the fundamental fourth order differential invariant is

$$J = \frac{\partial I}{\partial \eta_2} = \frac{dI}{ds} = \sqrt{\frac{u}{u_{xx}}}\,\frac{dI}{dx} \\ = \frac{2u^2 u_{xx} u_{xxxx} - 3u^2 u_{xxx}^2 - 2u u_x u_{xx} u_{xxx} + 6u u_{xx}^3 - 3u_x^2 u_{xx}^2}{4u u_{xx}^3}. \tag{4.24}$$

From the general theory, we conclude that every differential invariant for the group (2.3) is a function of the curvature and its successive derivatives with respect to the arc length. On the regular part $\mathcal{V}^2$ of the jet space J^2, all GL(2) invariant ordinary differential equations can be written in terms of these invariants; for instance the most general invariant third order ordinary differential equation has the form

$$u u_{xxx} + 3 u_x u_{xx} = k\sqrt{u u_{xx}^3}, \tag{4.25}$$

for some constant k.

Applications to the equivalence problem for curves (which includes the equivalence problem for first order Lagrangians as well as that of classical invariant theory) follow directly from the general theorems. Given a function $u = u(x)$, we define its *classifying curve* $\mathcal{C}$ to be the planar curve parametrized by the fundamental differential invariants $I(x), J(x)$. The general result states that two curves are mapped to each other by a group transformation (2.3), so $\overline{C} = g \cdot C$, if and only if their classifying curves are identical, $\overline{\mathcal{C}} = \mathcal{C}$. A curve C is maximally symmetric if and only if its classifying curve reduces to a point; in this case the original curve is, in fact, an orbit of a one-parameter subgroup of GL(2). Thus, we have, in a very simple and direct manner, recovered the results in [**36**] on the equivalence and symmetry of binary forms, which were found by a much less direct approach based on the standard Cartan equivalence problem for particle Lagrangians.

There are a few technical points that should have been addressed during the preceding discussion. First, one needs to impose certain conditions on the function $u(x)$ in order to ensure that the computation is valid. For instance, the normalization (4.14) requires $u_x \neq 0$, i.e., the curve does not have a horizontal tangent. (We have already assumed that it does not have a vertical tangent by requiring that it be the graph of a smooth function.) If $u_x = 0$, then we can still normalize $J = 0$ as long as $u \neq x u_x$, in which case we normalize by solving for γ instead of α. Actually, both cases can be simultaneously handled by the normalization $\alpha = \lambda(x u_x - u)$, $\gamma = \lambda u_x$, where $\lambda \neq 0$ is a new parameter whose normalization will be specified at the next stage of the procedure. The reader can check that this alternative procedure leads to the same lift and differential invariants as before. In the second normalization, we have assumed[†] $u_{xx} > 0$ in order to take the square root. For $u_{xx} < 0$ we would need to normalize $K = +1$, and use $\sqrt{-u_{xx}}$ instead. Thus the problem actually separates into two branches, with the inflection points $u_{xx} = 0$ being interpreted as singular points for the group action. The straight lines, for which $u_{xx} \equiv 0$, form a special class and must be analyzed separately. Finally, the square root itself has a

[†] In the complex-valued problem, there is no sign restriction.

sign ambiguity (or, in the complex case, an ambiguity in its choice of branch). Both signs must, in fact, be allowed in the final expression for the lift and the differential invariants. Such branching and ambiguous sign phenomena will be familiar to practitioners of the Cartan equivalence method; see [**38**] for a detailed discussion of these issues.

Let us finish this section by summarizing the basic method of moving coframes, in a form which will apply to more general problems. The basic steps are:

(*a*) Determine the general invariant lift, or moving frame of order zero, by choosing a base point and solving (4.1) for the given group action.

(*b*) Determine the invariant forms. In the finite-dimensional case, they are the Maurer–Cartan forms, which can be computed either by using the matrix approach, or by direct use of the transformation group formulae.

(*c*) Use the invariant lift to pull-back the invariant forms, leading to the moving coframe of order zero.

(*d*) Determine lifted invariants by finding linear dependencies among the restricted or horizontal components of the moving coframe forms.

(*e*) Normalize any group-dependent invariants to convenient constant values by solving for some of the unspecified parameters.

(*f*) Successively eliminate parameters by substituting the normalization formulae into the moving coframe and recomputing dependencies.

(*g*) After the parameters have all been normalized, the differential invariants will appear through any remaining dependencies among the final moving coframe elements. The invariant differential operators are found as the dual differential operators to a basis for the invariant coframe forms.

Note that we do not need the explicit isotropy groups for the transformation group actions, nor do we need compute explicit formulae for the prolonged group action in order to successfully apply the method.

Remark: If one is solely interested in the final differential invariants and invariant horizontal one-forms (i.e., invariant forms on the submanifold itself), then one need only determine the effect of the normalizations on the horizontal components of the moving coframe forms during the computation. The moving coframe itself will also include invariant contact forms, which vanish upon restriction, but which, nevertheless, play an important role in other aspects of the geometry. See [**38**, **40**, **20**], for applications of invariant contact forms to the study of invariant evolution equations, with applications to image processing. Applications to the computation of the invariant cohomology of the variational bicomplex, cf. [**2**], are also of particular importance in the analysis of symmetries and conservation laws of variational problems.

Remark: The proposed method of moving coframes has the same basic structure as the Cartan equivalence method, [**11**, **16**, **38**], in that one deals with a system of differential forms depending on arbitrary parameters, and seeks to normalize all the parameters by a suitable collection of lifted invariants. One can, indeed, view the two methods as particular cases of a completely general equivalence procedure. However, it is worth pointing out a few of the differences between the two. First, the Cartan method only deals with lifted coframes, whose constituents are linearly independent differential forms, whereas the differential forms occurring in the moving coframe method are linearly dependent. The invariant combinations (lifted invariants) used to normalize the parameters are found via linear dependencies in the moving coframe method, whereas they arise as unabsorbed torsion coefficients in the differentials of the lifted coframe forms in the Cartan equivalence method. In the moving coframe method, the differentials of the moving coframe one-forms satisfy the Maurer–Cartan structure equations, and hence do not provide any nonconstant invariants. Finally, and perhaps most significantly, the group parameters g only occur algebraically in the lifted coframe elements in the Cartan equivalence method, whereas in the moving frame problems their differentials dg occur as well, since they appear in the Maurer–Cartan forms. One can, of course, imagine solving hybrid equivalence problems, in which aspects of both problems occur during the normalization procedure, although we are not currently aware of any interesting examples where these occur naturally.

5. Intransitive Lie Group Actions.

Our next task is to extend the moving coframe method to the case of finite-dimensional Lie groups whose action is no longer transitive. In the intransitive case, we still assume that G is an r-dimensional Lie group acting effectively, and regularly, which implies that its orbits, which we take to have dimension s, form a foliation of M. We choose a local *cross-section* $\mathcal{K} \subset M$ to this foliation, i.e., a submanifold of dimension $m-s$ intersecting the orbits transversally, and introduce a *compatible lift* $\rho: M \to G$ by requiring that, for each z near $\mathcal{K}$, the lift ρ satisfies

$$z = \rho(z) \cdot z_0, \qquad \text{for some} \qquad z_0 \in \mathcal{K}. \tag{5.1}$$

The general solution to the compatible lift equations (5.1) will be of the form $\rho(z,h)$ depending on $r-s$ parameters h. Note that unless the isotropy subgroups at each point in the cross-section happen to be identical, we cannot identify the unspecified parameters as local coordinates on any subgroup $H \subset G$, leading us beyond any principal bundle-theoretic interpretation of the method. Nevertheless, the Implicit Function Theorem will allow us to locally write the general compatible lift in this form. In addition, the group admits (locally) $m-s$ functionally independent invariants, $I_1(z), \ldots, I_{m-s}(z)$, whose level sets characterize the orbits. The zero$^{\text{th}}$ order moving frame will then be the map

$$\rho_0(z,h) = (\rho(z,h), I(z)), \tag{5.2}$$

whose first components $g = \rho(z,h)$ are those of the general compatible lift (for the given cross-section) and, in addition, has the invariants $w = I(z) = (I_1(z), \ldots, I_{m-s}(z))$ as further components. Note that ρ_0 is only locally defined, since z must lie near the cross-section $\mathcal{K}$, and, moreover, the remaining parameters h are determined in accordance with the Implicit Function Theorem.

Note: We can view the range $G \times \mathbb{R}^{m-s}$ of ρ_0 as having the structure of a Cartesian product Lie group, the additive group structure on the second factor formalizing the fact that we can add invariants.

The moving coframe forms in this case are constructed from the Maurer–Cartan forms $\boldsymbol{\mu}$ on the group G, together with the coordinate one-forms $d\mathbf{w} = \{dw_1, \ldots, dw_{m-s}\}$ on

$\mathbb{R}^{m-s}$. The group transformations are then characterized by the conditions

$$\Phi^* \overline{\mathbf{w}} = \mathbf{w}, \qquad \Phi^* d\,\overline{\mathbf{w}} = d\mathbf{w}, \qquad \Phi^*\,\overline{\boldsymbol{\mu}} = \boldsymbol{\mu}. \tag{5.3}$$

Using the moving frame lift $g = \rho(z,h)$, $w = I(z)$, to pull back these one-forms, we are led to the zero$^{\text{th}}$ order moving coframe, consisting of the pulled-back Maurer–Cartan forms $(\rho_0)^*\boldsymbol{\mu}$, along with the differentials $(\rho_0)^* d\mathbf{w}_\kappa = dI_\kappa$ of the group invariants. At this stage, the set up of the intransitive problem is complete, and one proceeds, as in the transitive case, to look for dependencies among the restricted coframe forms, and then normalize the resulting lifted invariants.

Example 5.1. The intransitive action

$$A: \quad (x,u) \longmapsto \left(x, \frac{\alpha u + \beta}{\gamma u + \delta} \right), \qquad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2), \tag{5.4}$$

of the special linear group SL(2) on $M = \mathbb{R}^2$ arises in complex function theory, [**21**]. (We restrict to SL(2) in order to maintain local effectiveness.) The group orbits are vertical lines and so the basic invariant is merely $I(x,u) = x$. We choose the cross-section $\mathcal{K} = \{u = 0\}$. Solving the equation $A(x,u) \cdot z_0 = (x,u)$, where $z_0 = (y,0)$, leads to the general compatible lift

$$A_0(x,u,\alpha,\delta) = \begin{pmatrix} \alpha & \delta u \\ \dfrac{\alpha\delta - 1}{\delta u} & \delta \end{pmatrix}, \tag{5.5}$$

which forms the group component of the zero$^{\text{th}}$ order moving frame. The other component is just the invariant

$$w = I(x,u) = x. \tag{5.6}$$

Pulling back the Maurer–Cartan forms $\boldsymbol{\mu} = A^{-1}\,dA$ and dw via the lift (5.5), (5.6), leads to the order zero moving coframe

$$\begin{aligned} \zeta_1 &= (\alpha\delta - 1)\frac{du}{u} - \frac{d\delta}{\delta}, & \zeta_2 &= \delta^2\,du, \\ \zeta_3 &= \frac{u\,d(\alpha\delta) + \alpha\delta(1-\alpha\delta)\,du}{\delta^2 u^2}, & \zeta_4 &= dx. \end{aligned} \tag{5.7}$$

As before, we restrict to a curve $u = u(x)$ by replacing du by its horizontal component $u_x\,dx$. Letting η_i denote the horizontal component of ζ_i, we find that there is one resulting linear dependency, namely

$$\eta_2 = \delta^2 u_x\,dx = J\,dx = J\eta_4.$$

The leads to the first normalization $\delta = 1/\sqrt{u_x}$ resulting from setting $J = 1$. Substituting this normalized value into (5.5), (5.6), provides the first order moving frame. Furthermore, substituting into (5.7) produces the second order moving coframe, with horizontal components

$$\eta_1 = \left(\frac{2\alpha u_x^{3/2} + u u_{xx} - 2u_x^2}{2uu_x} \right) dx, \qquad \eta_2 = \eta_4 = dx, \qquad \eta_3 = \frac{\sqrt{u_x}}{u}(\,d\alpha - \alpha\,\eta_1). \tag{5.8}$$

Now we normalize the coefficient of η_1 to 0 by setting $\alpha = u_x^{-3/2}(u_x^2 - \frac{1}{2} u u_{xx})$. The final moving frame (of order 2) is

$$A_2 = \frac{1}{u_x^{3/2}} \begin{pmatrix} u_x^2 - \frac{1}{2} u u_{xx} & u u_x \\ -\frac{1}{2} u_{xx} & u_x \end{pmatrix}, \qquad w = x. \tag{5.9}$$

The corresponding restricted moving coframe has reduced to

$$\eta_2 = \eta_4 = dx, \qquad \eta_3 = -\tfrac{1}{4} S\, \eta_2, \qquad \eta_1 = 0, \tag{5.10}$$

where

$$S = \frac{2 u_x u_{xxx} - 3 u_{xx}^2}{u_x^2} \tag{5.11}$$

is the classical Schwarzian derivative of the function $u(x)$, whose invariance under linear fractional transformations is of fundamental importance in complex function theory. Since the one-form dx is invariant, all the higher order differential invariants are found by differentiating S with respect to x.

Actually, the preceding computation can be slightly simplified by extending our general method to non-effective actions. We consider (5.4) as defining a non-effective (and intransitive) action of the general linear group GL(2) on $\mathbb{R}^2$. We may apply the second algorithm for computing the required Maurer–Cartan forms, leading to the three one-forms (3.12) that annihilate the global isotropy subalgebra. We substitute the compatible lift formulae $\beta = \delta u$ for the order zero moving coframe, which is now $A_0 = \begin{pmatrix} \alpha & \delta u \\ \gamma & \delta \end{pmatrix}$, into (3.12), leading to the restricted moving coframe forms

$$\begin{aligned} \widehat{\eta}_1 &= \frac{\delta u_x\, dx}{\alpha - u\gamma}, & \widehat{\eta}_2 &= \frac{d\alpha - u\, d\gamma + \gamma u_x\, dx}{\alpha - u\gamma} - \frac{d\delta}{\delta}, \\ \widehat{\eta}_3 &= \frac{\gamma\, d\alpha - \alpha\, d\gamma}{\delta(\alpha - u\gamma)}, & \widehat{\eta}_4 &= dx. \end{aligned} \tag{5.12}$$

The first dependency between $\widehat{\eta}_1$ and $\widehat{\eta}_4$ leads to the reduction $\delta = (\alpha - u\gamma)/u_x$. Substituting into $\widehat{\eta}_3$ leads to a second dependency, and the resulting normalization yields $\alpha = \gamma(u - 2u_x^2/u_{xx})$. At this stage, even though we have not normalized the final parameter γ, it no longer appears in the coframe, which coincides with our earlier one, (5.10). It does, of course, occur in the final moving frame lift, which is obtained by multiplying the matrix A_2 in (5.9) by γ. However, γ plays no other role in the problem, and merely reflects a final indeterminacy stemming from the ineffectiveness of the group action. The main point in this solution method is that one does not have to explicitly implement an effective action, as was done in the original lift (5.5), in order to solve the problem. Indeed, in more complicated examples, it may be relatively straightforward to write down the compatible lift for an ineffective group action, whereas doing the same for the effectively acting quotient group G/G_M may be considerably more complicated.

Example 5.2. Consider the elementary similarity group $G = \mathbb{R}^+ \ltimes \mathbb{R}^2$ acting transitively on $M = \mathbb{R}^2$ via

$$A: \quad (x, u) \longmapsto (\alpha x + a, \alpha u + b). \tag{5.13}$$

For the base point $z_0 = (0,0)$, the associated moving frame of order 0 is the lift with $a = x$, $b = u$. The Maurer–Cartan forms $\{d\alpha/\alpha, da/\alpha, db/\alpha\}$ are pulled back to provide the zero$^{\text{th}}$ order moving coframe, whose horizontal (or, more precisely, non-contact) components are

$$\eta_1 = \frac{d\alpha}{\alpha}, \qquad \eta_2 = \frac{dx}{\alpha}, \qquad \eta_3 = \frac{u_x\, dx}{\alpha}. \tag{5.14}$$

There is a single linear dependency $\eta_3 = I\,\eta_2$, but the resulting invariant $I = u_x$ does *not* depend on the remaining group parameter, and hence *cannot* be used to normalize it. To proceed further in such cases, we work in analogy with the preceding intransitive case. Here the intransitivity is on the first order jet bundle, and is an indication of the fact that this particular group exhibits the pathology of "pseudo-stabilization" of its prolonged group orbits, cf. [**38**]. We therefore introduce an additional invariant one-form du_x, whose horizontal component is

$$\eta_4 = u_{xx}\, dx = K\,\eta_2.$$

The resulting dependency leads to the lifted invariant $K = \alpha u_{xx}$ which yields the desired normalization $\alpha = 1/u_{xx}$ and the second order moving frame. The associated invariant coframe is

$$\eta_2 = \eta_4 = u_{xx}\, dx, \qquad \eta_1 = -J\,\eta_2, \qquad \eta_3 = I\,\eta_2, \tag{5.15}$$

yielding two fundamental differential invariants

$$I = u_x, \qquad J = u_{xx}^{-2} u_{xxx}. \tag{5.16}$$

The higher order invariants are found by differentiating J with respect to $\eta_4 = u_{xx}\, dx \simeq du_x$, so that a basic fourth order invariant is

$$K = \frac{dJ}{du_x} = \frac{1}{u_{xx}}\,\frac{dJ}{dx} = \frac{u_{xxxx}}{u_{xx}^2} - 2J^2.$$

Note that $dI/du_x = 1$, so that differentiating I produces nothing new. Thus, in this case, we find two fundamental differential invariants, and require three, namely (I, J, K), to parametrize the classifying curve that solves the associated equivalence problem. We conclude that the phenomenon of pseudo-stabilization of group orbits is reflected in the moving coframe procedure by the premature appearance of differential invariants, whose differentials are required to finish the procedure. See [**38**, **39**] for further discussion.

Remark: Interestingly, if the scaling acts differently on x and u, so the group is

$$A: \quad (x, u) \longmapsto (\alpha x + a, \alpha^k u + b), \tag{5.17}$$

for $k \neq 1$, then pseudo-stabilization does not occur. Such cases can be readily handled via our basic method without any such intransitive normalizations.

6. Reparametrization Pseudo-Groups.

The classical applications of moving frames to curves and surfaces in Euclidean, affine, and projective geometry, cf. [**6**, **8**, **19**], can all be readily implemented using the moving coframe algorithm. In each case, we consider the reparametrization equivalence problem for submanifolds, so that the underlying transformation group is the Cartesian product of an infinite Lie pseudo-group, namely the local diffeomorphism group $\mathcal{D}\mathit{iff}(X)$ of the parameter space, and a finite-dimensional Lie group acting on the manifold M. In this case, in addition to the Maurer–Cartan forms for the group, one also includes the one-forms defining the diffeomorphism pseudo-group. One can then proceed to reduce and normalize as before. For simplicity, we just deal with planar curves, although extensions to surfaces and curves in higher dimensional ambient spaces can also be handled without significant further complications.

Example 6.1. *Euclidean geometry of curves.* The most well-known classical example is the reparametrization equivalence problem for curves in the Euclidean plane, introduced in Example 2.1 above. In this case, we are dealing with a finite-dimensional group, the Euclidean group E(2) on the plane, together with the pseudo-group $\mathcal{D}\mathit{iff}(1)$ consisting of all smooth (local) diffeomorphisms of the line representing the change of parameter. Thus, the entire pseudo-group $\mathcal{G} = \mathcal{D}\mathit{iff}(1) \times \mathrm{E}(2)$ acts on the total space $M = \mathbb{R} \times \mathbb{R}^2$ with coordinates $(t, \mathbf{x}) = (t, x, y)$. For the Euclidean component, we use a compatible lift

$$A_0(x, y, \phi) = \begin{pmatrix} R & \mathbf{x} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{pmatrix}, \tag{6.1}$$

and compute the pull-back of the associated Euclidean Maurer–Cartan forms:

$$\boldsymbol{\zeta} = A_0^{-1} dA_0 = \begin{pmatrix} R^{-1}dR & R^{-1}d\mathbf{x} \\ \mathbf{0} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -d\phi & \cos\phi\, dx + \sin\phi\, dy \\ d\phi & 0 & -\sin\phi\, dx + \cos\phi\, dy \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.2}$$

On the other hand, the pseudo-group $\mathcal{D}\mathit{iff}(1)$ is characterized by the invariance of the canonical one-form $\sigma\, dt$ on the frame bundle $\mathcal{F}(\mathbb{R})$, cf. [**26**], and hence we include this additional one-form in our moving coframe formulation. Restricting these four one-forms to a parametrized curve $(x(t), y(t))$ leads to

$$\eta_1 = d\phi, \qquad \eta_2 = (x_t \cos\phi + y_t \sin\phi)\, dt, \qquad \eta_3 = (-x_t \sin\phi + y_t \cos\phi)\, dt, \qquad \eta_4 = \sigma\, dt. \tag{6.3}$$

Now $\eta_2 = J_1 \eta_4$ and $\eta_3 = J_2 \eta_4$ are the linear dependencies, with associated lifted invariants

$$J_1 = \frac{x_t \cos\phi + y_t \sin\phi}{\sigma}, \qquad J_2 = \frac{-x_t \sin\phi + y_t \cos\phi}{\sigma}.$$

We normalize $J_1 = 1$, $J_2 = 0$ by setting

$$\phi = \tan^{-1}(y_t / x_t), \qquad \sigma = \sqrt{x_t^2 + y_t^2}\,. \tag{6.4}$$

This immediately produces the first order moving frame

$$R = \frac{1}{\sqrt{x_t^2 + y_t^2}} \begin{pmatrix} x_t & -y_t \\ y_t & x_t \end{pmatrix}, \qquad \mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \sigma = \sqrt{x_t^2 + y_t^2}\,. \tag{6.5}$$

The canonical one-form $\sigma\,dt$ has been reduced to the fundamental arc length form $ds = \sqrt{x_t^2 + y_t^2}\,dt$ for the Euclidean group. Substituting into (6.3), we are left with a final set of horizontal one-forms

$$\eta_1 = \kappa\,\eta_4, \qquad \eta_2 = \eta_4 = ds = \sqrt{x_t^2 + y_t^2}\,dt, \qquad \eta_3 = 0. \tag{6.6}$$

Here

$$\kappa = \frac{x_t y_{tt} - x_{tt} y_t}{(x_t^2 + y_t^2)^{3/2}} = \frac{\mathbf{x}_t \wedge \mathbf{x}_{tt}}{|\,\mathbf{x}_t\,|^3} = \mathbf{x}_s \wedge \mathbf{x}_{ss} \tag{6.7}$$

is the fundamental differential invariant for the Euclidean group — the curvature of the plane curve. All higher order differential invariants are obtained by successively differentiating the curvature with respect to arc length.

The classical Frenet equations for curves in the Euclidean plane are reformulations of our final moving frame formulae. (See Section 7 below for more details on the connection with the classical theory.) The rotational component in (6.5) is traditionally written as $R = (\mathbf{e}_1, \mathbf{e}_2)$, where $\mathbf{e}_1$ is the unit tangent and $\mathbf{e}_2$ the unit normal. The translational Maurer–Cartan forms $\eta_2 = ds$, $\eta_3 = 0$ are computed by the original formula as the entries of $R^{-1} d\mathbf{x} = \begin{pmatrix} ds \\ 0 \end{pmatrix}$, which reduces to the first Frenet equation $\dfrac{d\mathbf{x}}{ds} = \mathbf{e}_1$. Similarly, the Maurer–Cartan matrix

$$R^{-1}\,dR = \begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix} ds \qquad \text{implies that} \qquad \frac{dR}{ds} = R \cdot \begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix}.$$

The columns of the latter matrix differential equation complete the system of Frenet equations:

$$\frac{d\mathbf{x}}{ds} = \mathbf{e}_1, \qquad \frac{d\mathbf{e}_1}{ds} = \kappa\,\mathbf{e}_2, \qquad \frac{d\mathbf{e}_2}{ds} = -\kappa\,\mathbf{e}_1. \tag{6.8}$$

Finally, the Maurer–Cartan structure equations (3.2) for the Euclidean group reduce to the classical Frenet–Serret equations for curves. See [**13**; p. 23], [**19**; p. 20], for details.

Remark: One can also compute, as in our original example, the full moving coframe forms on the jet bundle, leading to a corresponding set of fundamental Euclidean-invariant contact forms.

Remark: Actually, since we are dealing with the full pseudo-group $\mathcal{D}\mathit{iff}(1)$ consisting of *all* diffeomorphisms of $\mathbb{R}$, the final one-form $\eta_4 = \sigma\,dt$ in our moving coframe (6.3) is, in fact, irrelevant — one could perform the same normalization (6.4) of the angle ϕ based on the dependency between η_2 and η_3, the lifted invariant now being J_2/J_1 which is normalized to zero by setting $J_2 = 0$, leading to the same final moving coframe and curvature invariant. Thus the calculations for parametrized curves and surfaces can, in fact, be done without invoking the diffeomorphism pseudo-group. Nevertheless, in all examples we have treated, the inclusion of the canonical one-form η_4 on the parameter space leads to an immediate identification of the final invariant arc length element. More generally, the restricted reparametrization equivalence problem does require the introduction of suitable one-forms that characterize the pseudo-group of allowed reparametrizations.

Remark: The problem of Euclidean equivalence of curves with *fixed* parametrizations, as discussed in Example 2.1, can also be formulated and solved in the moving coframe context. Now we are in the intransitive framework, where the parameter t provides a scalar invariant. Consequently, we retain the first three one-forms η_1, η_2, η_3 in (6.3), but replace η_4 by dt to form the moving coframe. We normalize $J_1 = 0$ as before, but now $J_2 = v = \sqrt{x_t^2 + y_t^2}$ forms a first order differential invariant — the speed of the particle. The final moving frame has $\eta_1 = K\,dt$, $\eta_2 = 0$, $\eta_3 = v\,dt$, $\eta_4 = dt$, where $K = x_t y_{tt} - x_{tt} y_t = v^3 \kappa$ is a second order differential invariant. The higher order differential invariants are found by differentiating with respect to t. Note that the arc length $ds = v\,dt$ is also an invariant one-form, being an invariant multiple of dt, and hence one can, without loss of generality, apply the arc length derivative d/ds to produce the higher order differential invariants instead. Thus, in this case, a complete list of differential invariants is provided by v, κ, and their derivatives with respect to arc length.

Example 6.2. The equiaffine geometry of curves in the plane is governed by the special affine group $\mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$, acting on $M = \mathbb{R}^2$ according to

$$g: \quad \mathbf{x} \longmapsto A\mathbf{x} + \mathbf{a}, \qquad \mathbf{x} \in M, \quad A \in \mathrm{SL}(2), \quad \mathbf{a} \in \mathbb{R}^2. \tag{6.9}$$

We shall adopt a vector notation for the matrix $A = (\boldsymbol{\alpha}\,\boldsymbol{\beta}) \in \mathrm{SL}(2)$, so that the column vectors are subject to the unimodularity constraint

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = 1. \tag{6.10}$$

It will be computationally convenient *not* to explicitly implement the unimodularity constraint (6.10) by solving for one of the parameters, but retain it as an additional constraint that is to be respected during the course of the calculation. This method, i.e., treating a subgroup of a larger Lie group via a collection of algebraic constraints, rather than parametrizing it directly, has general applicability, and can be readily implemented as is done in this particular case.

The Maurer–Cartan forms are computed directly as in Section 3, leading to

$$\mu_1 = \boldsymbol{\alpha} \wedge d\boldsymbol{\alpha}, \quad \mu_2 = \boldsymbol{\beta} \wedge d\boldsymbol{\alpha}, \quad \mu_3 = \boldsymbol{\beta} \wedge d\boldsymbol{\beta}, \quad \nu_1 = \boldsymbol{\alpha} \wedge d\mathbf{a}, \quad \nu_2 = \boldsymbol{\beta} \wedge d\mathbf{a}. \tag{6.11}$$

Note that the unimodularity constraint (6.10) implies that

$$\boldsymbol{\alpha} \wedge\, d\boldsymbol{\beta} = \boldsymbol{\beta} \wedge\, d\boldsymbol{\alpha}, \tag{6.12}$$

which means that the matrix of Maurer–Cartan forms $\boldsymbol{\mu} = A^{-1}\,dA$ must be trace free.

Choose the base point to be $\mathbf{x}_0 = 0$. Solving the compatible lift equations $\mathbf{x} = g\cdot\mathbf{x}_0 = \mathbf{a}$ yields the zero$^{\text{th}}$ order moving frame, which sets $\mathbf{a} = \mathbf{x}$. Substituting into the Maurer–Cartan forms (6.11), we find that, for a parametrized curve $\mathbf{x}(t)$, the forms ν_1, ν_2 restrict to the following two horizontal forms:

$$\eta_1 = (\boldsymbol{\alpha} \wedge \mathbf{x}_t)\,dt, \qquad \eta_2 = (\boldsymbol{\beta} \wedge \mathbf{x}_t)\,dt. \tag{6.13}$$

Their ratio produces the lifted invariant $(\boldsymbol{\alpha} \wedge \mathbf{x}_t)/(\boldsymbol{\beta} \wedge \mathbf{x}_t)$, which is normalized to 0 by setting

$$\boldsymbol{\alpha} = \lambda\,\mathbf{x}_t, \tag{6.14}$$

for some scalar parameter λ. Substituting (6.14) into the first Maurer–Cartan form $\mu_1 = \boldsymbol{\alpha} \wedge d\boldsymbol{\alpha}$, leads to the restricted form $\xi_1 = \lambda^2(\mathbf{x}_t \wedge \mathbf{x}_{tt})\, dt$. Assuming $\mathbf{x}_t \wedge \mathbf{x}_{tt} \neq 0$, the latter form can be normalized to equal $-\eta_2$ by setting

$$-\boldsymbol{\beta} \wedge \mathbf{x}_t = \lambda^2(\mathbf{x}_t \wedge \mathbf{x}_{tt}), \qquad \text{or} \qquad \boldsymbol{\beta} = \lambda^2\, \mathbf{x}_{tt} + \mu\, \mathbf{x}_t, \tag{6.15}$$

for some scalar μ. However, applying the unimodularity constraint (6.10) to the normalizations (6.14), (6.15), we deduce that $\lambda^3(\mathbf{x}_t \wedge \mathbf{x}_{tt}) = 1$, and thus

$$\lambda = \frac{1}{\sqrt[3]{\mathbf{x}_t \wedge \mathbf{x}_{tt}}}. \tag{6.16}$$

Note that (6.15), (6.16) reduce the form η_2 to be minus the equi-affine arc length form

$$ds = \sqrt[3]{\mathbf{x}_t \wedge \mathbf{x}_{tt}}\; dt. \tag{6.17}$$

Furthermore, substituting (6.15), (6.16) into the second Maurer–Cartan form, we find it reduces to a multiple of $\xi_1 = ds$, so

$$\xi_2 = \boldsymbol{\beta} \wedge d\boldsymbol{\alpha} = J\, ds,$$

where the lifted invariant

$$J = \mu(\mathbf{x}_t \wedge \mathbf{x}_{tt})^{1/3} + \frac{\mathbf{x}_t \wedge \mathbf{x}_{ttt}}{3(\mathbf{x}_t \wedge \mathbf{x}_{tt})^{4/3}}$$

is normalized to zero in the obvious manner. Therefore, the final moving frame is given by

$$\boldsymbol{\alpha} = \frac{d\mathbf{x}}{ds} = \frac{\mathbf{x}_t}{\sqrt[3]{\mathbf{x}_t \wedge \mathbf{x}_{tt}}}, \qquad \boldsymbol{\beta} = \frac{d^2\mathbf{x}}{ds^2} = \frac{\mathbf{x}_{tt}}{(\mathbf{x}_t \wedge \mathbf{x}_{tt})^{2/3}} - \frac{(\mathbf{x}_t \wedge \mathbf{x}_{ttt})\, \mathbf{x}_t}{3(\mathbf{x}_t \wedge \mathbf{x}_{tt})^{5/3}}, \qquad \mathbf{a} = \mathbf{x}. \tag{6.18}$$

The final Maurer–Cartan form becomes

$$\xi_3 = \boldsymbol{\beta} \wedge d\boldsymbol{\beta} = \kappa\, ds,$$

where

$$\kappa = \mathbf{x}_{ss} \wedge \mathbf{x}_{sss} = \frac{(\mathbf{x}_t \wedge \mathbf{x}_{tttt}) + 4(\mathbf{x}_{tt} \wedge \mathbf{x}_{ttt})}{3(\mathbf{x}_t \wedge \mathbf{x}_{tt})^{5/3}} - \frac{5(\mathbf{x}_t \wedge \mathbf{x}_{ttt})^2}{9(\mathbf{x}_t \wedge \mathbf{x}_{tt})^{8/3}} \tag{6.19}$$

defines the equi-affine curvature. As usual, all higher order differential invariants are obtained by differentiating κ with respect to the equi-affine arc length ds. This reproduces the basic invariants of the equi-affine geometry of curves, [**19**]; see also [**5**] for applications in computer vision.

As with the Euclidean case, we recover the classical Frenet equations as simple reformulations of the final moving frame formulae. We identify the linear part

$$A = (\mathbf{e}_1, \mathbf{e}_2) = (\mathbf{x}_s, \mathbf{x}_{ss})$$

of the final moving frame with the equi-affine frame at a point $\mathbf{x}(t)$ on the curve, so that $\mathbf{e}_1 = \mathbf{x}_s$ is the unit affine tangent vector, whereas $\mathbf{e}_2 = \mathbf{x}_{ss}$ is the unit equi-affine normal. Combining this with the Maurer–Cartan matrix $A^{-1}\,dA = \begin{pmatrix} 0 & 1 \\ \kappa & 0 \end{pmatrix} ds$ leads to the complete Frenet equations of planar equi-affine geometry, [**13**; p. 27]:

$$\frac{d\mathbf{x}}{ds} = \mathbf{e}_1, \qquad \frac{d\mathbf{e}_1}{ds} = \mathbf{e}_2, \qquad \frac{d\mathbf{e}_2}{ds} = \kappa\,\mathbf{e}_1. \tag{6.20}$$

See [**19**; § 7–3] for further details.

Example 6.3. The most complicated example treated in the literature, [**7**], is the projective geometry of curves in the plane. Here the group is SL(3), acting on $M = \mathbb{RP}^2$ according to

$$g\colon (x,u) \longmapsto \left(\frac{\alpha x + \beta u + \gamma}{\rho x + \sigma u + \tau}, \frac{\lambda x + \mu u + \nu}{\rho x + \sigma u + \tau} \right), \qquad \det A = \det \begin{vmatrix} \alpha & \beta & \gamma \\ \lambda & \mu & \nu \\ \rho & \sigma & \tau \end{vmatrix} = 1. \tag{6.21}$$

For simplicity, we deal with curves which can be expressed as the graphs of functions, $u = u(x)$, although the general case of parametrized curves can be handled via the same sequence of normalizations. Choose the base point to be $z_0 = (0,0)$. Solving $g \cdot (0,0) = (x,u)$ leads to the order zero moving frame in the form†

$$A = \begin{pmatrix} \alpha & \beta & x\tau \\ \lambda & \mu & u\tau \\ \rho & \sigma & \tau \end{pmatrix}, \qquad \text{where} \qquad \alpha = \frac{1 + \tau[\beta(\lambda - \rho u) + x(\mu\rho - \lambda\sigma)]}{\tau(\mu - \sigma x)}. \tag{6.22}$$

The one-forms in the first order moving coframe are the entries of the pull-back of the Maurer–Cartan matrix $A^{-1}\,dA$, which we label (in row order) as $\eta_1, \ldots, \eta_8$, the final entry being $\eta_9 = -\eta_1 - \eta_5$, reflecting the unimodularity of A. For simplicity, we just indicate the salient features of the computation without dwelling on the details. (These computations were done with the aid of some Mathematica routines written for this purpose.) The first normalization comes from the ratio η_3/η_6, whose vanishing requires

$$\mu = \sigma(u - x u_x) - \beta u_x.$$

Plugging this normalization back into the moving coframe forms and recomputing, we find that we can normalize $\eta_6 = \eta_2$ by requiring

$$\beta = \sigma x - u_{xx}^{-1/3}.$$

In the next stage, we set η_5 to zero by normalizing

$$\sigma = \frac{\tau(\rho u - \lambda) u_{xx}^{1/3}}{u_x} - \frac{u_x u_{xxx} - 3u_{xx}^2}{3 u_x u_{xx}^{4/3}}.$$

† In this example, we have chosen to implement the unimodularity constraint explicitly.

At the next step, we can no longer just look at one-forms depending only on dx — these do not produce any further invariants. However, we discover that $\eta_8 = J\eta_2 + \eta_4$, and hence the rather complicated lifted invariant J can be normalized to zero, leading to

$$\lambda = \rho u - \frac{u_x u_{xxx} - 3u_{xx}^2 + u_x \sqrt{18\rho\tau u_{xx}^{8/3} - P_4}}{3\tau u_{xx}^{5/3}},$$

where

$$P_4 = 3u_{xx}u_{xxxx} - 5u_{xxx}^2.$$

Next, the normalization $\eta_2 = -\eta_7$ requires

$$\tau = \sqrt[3]{\frac{L_5}{54\, u_{xx}^4}}, \qquad \text{where} \qquad L_5 = 9u_{xx}^2 u_{xxxxx} - 45u_{xx}u_{xxx}u_{xxxx} + 40u_{xxx}^3.$$

The final normalization

$$\rho = \frac{M_6^2 + P_4 L_5^2}{3\sqrt[3]{4\, u_{xx}^4 L_5^7}}, \qquad \text{where} \qquad M_6 = (u_{xx}D_x - 4u_{xxx})L_5$$

comes from setting η_1 to zero. The final moving frame is explicitly given by

$$\begin{aligned}
\alpha &= \frac{\lambda + \rho(xu_x - u)}{u_x} - \frac{3}{u_x}\sqrt[3]{\frac{2\, u_{xx}^5}{L_5}}, && \beta = x\mu - u_{xx}^{-1/3}, && \gamma = x\tau \\
\lambda &= \frac{uM_6^2 + 6u_x u_{xx} L_5 M_6 + K_4 L_5^2}{3\sqrt[3]{4\, u_{xx}^4 L_5^7}}, && \mu = u\sigma - \frac{u_x}{u_{xx}^{1/3}}, && \nu = u\tau \\
\rho &= \frac{M_6^2 + P_4 L_5^2}{3\sqrt[3]{4\, u_{xx}^4 L_5^7}}, && \sigma = \frac{M_6}{3u_{xx}^{4/3} L_5}, && \tau = \frac{L_5^{1/3}}{3\sqrt[3]{2\, u_{xx}^4}}.
\end{aligned} \tag{6.23}$$

The corresponding final coframe has

$$\eta_2 = \eta_6 = -\eta_7 = ds = \frac{\sqrt[3]{L_5}}{3\sqrt[3]{2}\, u_{xx}}\, dx = \sqrt[3]{\frac{9u_{xx}^2 u_{xxxxx} - 45u_{xx}u_{xxx}u_{xxxx} + 40u_{xxx}^3}{54\, u_{xx}^3}}\, dx, \tag{6.24}$$

which determines the well-known projective arc length element, while $\eta_4 = \eta_8 = -\kappa\, ds$ yields the projective curvature invariant

$$\kappa = \frac{6\, u_{xx} L_5 D_x M_6 - 32\, u_{xxx} L_5 M_6 - 7M_6^2 - P_4 L_5^2}{\sqrt[3]{2}\, L_5^{8/3}}. \tag{6.25}$$

Again, all higher order differential invariants are found by differentiating the projective curvature κ with respect to the projective arc length ds. This relatively straightforward computation reproduces the moving frame and the fundamental invariants for the projective geometry of curves. Cartan, [**7**], presents a variety of alternative methods to arrive at the same basic result. See also [**38**] for a Lie-theoretic approach, and Wilczynski, [**49**], for an approach based on differential operators.

In the classical moving frame method, one identifies the columns of the 3×3 moving frame matrix $A = (\mathbf{P}_2, \mathbf{P}_1, \mathbf{P})$ as homogeneous coordinates for three points in the projective plane $\mathbb{RP}^2$, the last column $\mathbf{P} = \tau \cdot (x, u, 1)^T$ representing the point on the curve. The Maurer–Cartan matrix

$$A^{-1}\, dA = \begin{pmatrix} 0 & 1 & 0 \\ -\kappa & 0 & 1 \\ -1 & -\kappa & 0 \end{pmatrix} ds$$

reduces to the full set of projective Frenet equations,

$$\frac{d\mathbf{P}}{ds} = \mathbf{P}_1, \qquad \frac{d\mathbf{P}_1}{ds} = -\kappa\, \mathbf{P} + \mathbf{P}_2, \qquad \frac{d\mathbf{P}_2}{ds} = -\mathbf{P} - \kappa\, \mathbf{P}_1. \tag{6.26}$$

See also [**13**; pp. 33ff.] for applications to projective curvature evolutions and computer vision.

7. Connections with the Classical Moving Frames Method.

Our initial identification of a moving frame as an equivariant lift from the underlying space to the Lie group will be familiar to readers of the modern formulations of Griffiths, [**18**], and Jensen, [**23**]. However, since this point of view is not completely standard, it is worth reviewing how it relates to the more usual geometric approaches, e.g., [**19, 50**]. Traditionally, a moving frame is realized as a collection of vectors (or, in the projective case, points) in the underlying space. The reason that this works in the classical cases, including Euclidean, affine, and projective geometry of submanifolds, is that it is possible to identify the components of the group itself with objects in the underlying transformation space. For example, in the Euclidean case, one identifies a Euclidean group element $(R, \mathbf{a}) \in \mathrm{E}(m) \simeq \mathrm{O}(m) \ltimes \mathbb{R}^m$ with a vector $\mathbf{a} \in \mathbb{R}^m$, together with an orthonormal frame determined by the columns of the orthogonal matrix R. The zero$^{\text{th}}$ order moving frame, then, uses the lift $\mathbf{a} = \mathbf{x}$, where $\mathbf{x}$ is a point on the submanifold $N \subset \mathbb{R}^m$, and the orthogonal matrix is identified with an orthonormal frame in the ambient space based at the point. The remaining ambiguity in the frame is up to orthogonal transformations, which must then be resolved in an invariant manner. Similarly, in the equi-affine case, one identifies a group element $(A, \mathbf{a}) \in \mathrm{SA}(m) \simeq \mathrm{SL}(m) \ltimes \mathbb{R}^m$ with a vector $\mathbf{a} \in \mathbb{R}^m$ together with a unimodular frame determined by the columns of the matrix A. Again, the zero$^{\text{th}}$ order moving frame takes $\mathbf{a} = \mathbf{x}$ to be a point on the submanifold, and the unimodular frame becomes a set of vectors based at the point $\mathbf{x}$. In both cases, the moving coframe method introduces the Maurer–Cartan forms $\boldsymbol{\mu} = (\boldsymbol{\sigma}, \boldsymbol{\nu})$, where $\boldsymbol{\sigma} = A^{-1}\, dA$, $\boldsymbol{\nu} = A^{-1}\, d\mathbf{a}$, leading to the initial structure equations

$$d\mathbf{x} = A \cdot \boldsymbol{\nu}, \qquad dA = A \cdot \boldsymbol{\sigma}. \tag{7.1}$$

The moving coframe forms satisfy the usual Maurer–Cartan structure equations (3.2), which, in the classical cases, become the fundamental Cartan structure equations for Euclidean or affine geometry.

Usually, one bypasses the order zero moving frame entirely, and proceeds directly to the first order moving frame, in which the frame at the point $\mathbf{x} \in N$ is split into two parts, so that (using column vector notation)

$$A = (E, F) = (\mathbf{e}_1, \ldots, \mathbf{e}_n, \mathbf{f}_1, \ldots, \mathbf{f}_{m-n}), \tag{7.2}$$

where the first $n = \dim N$ frame vectors form a basis for the tangent space TN to the submanifold, while the remainder are left arbitrary, subject to the entire frame satisfying the proper orthonormality or unimodularity constraint. Thus, in the Euclidean case, the vectors $\{\mathbf{f}_1, \ldots, \mathbf{f}_{m-n}\}$ form an orthonormal basis for the normal space to N, whereas in the equi-affine case they are left arbitrary subject only to the condition that the determinant of the matrix (7.2) be unity. If we parametrize the submanifold by $\mathbf{x}(t_1, \ldots, t_n)$, then the most general first order moving frame (7.2) will have the form

$$E = (\mathbf{e}_1, \ldots, \mathbf{e}_n) = V \cdot B, \tag{7.3}$$

where

$$V = (\mathbf{v}_1, \ldots, \mathbf{v}_n), \qquad \mathbf{v}_i = \frac{\partial \mathbf{x}}{\partial t_i}\,, \tag{7.4}$$

is the $m \times n$ Jacobian matrix, whose columns span the tangent space to N, while B is an invertible $n \times n$ matrix. (In the Euclidean case, the matrix B is restricted so that the columns of E are orthonormal, leaving an $\mathrm{O}(n)$ ambiguity.)

Let us show how this preliminary normalization to a first order moving frame is an immediate consequence of our general normalization procedure. Using the zero$^{\text{th}}$ order moving frame lift, the pull-backs of the subset of Maurer–Cartan forms given by the entries of $\boldsymbol{\nu} = A^{-1}\, d\mathbf{a}$ can be written in matrix form as

$$\boldsymbol{\nu} = A^{-1}\, d\mathbf{x} = A^{-1} V\, d\mathbf{t}.$$

Precisely n of the m one-forms $\boldsymbol{\nu}$ are linearly independent, and hence we can normalize so that the last $m - n$ of these forms vanish. This requires that the matrix A satisfy the block matrix equation

$$A^{-1} V = \begin{pmatrix} D \\ 0 \end{pmatrix}, \tag{7.5}$$

where D is a nonsingular $n \times n$ matrix, while 0 denotes the zero matrix of size $(m-n) \times n$. Writing $A = (E, F)$ in block form (7.2), we see that (7.5) requires

$$V = E \cdot D, \qquad \text{or} \qquad E = V \cdot C, \qquad \text{where} \qquad C = D^{-1},$$

thereby recovering (7.3). Thus, we can see that in such cases, the first order frame recovered by an order zero normalization coincides with the traditional first order frame involving tangent and normal directions.

Similar considerations apply to the projective case. According to Cartan, [**7**], the zero$^{\text{th}}$ order frame can be identified with a set of $n+1$ linearly independent points in the projective space which are identified with the columns of the matrix $A \in \mathrm{SL}(n+1)$. The zero$^{\text{th}}$ order lift, as in (6.22), amounts to identifying one of the columns with the point on the curve $\mathbf{x}$. More precisely, the column is a vector with $n+1$ components, which are interpreted as the homogeneous coordinates of $\mathbf{x}$.

In more sophisticated versions, one realizes the moving frame on the submanifold $N \subset M$ as a section of the frame bundle $\mathcal{F}(M)$ of M, pulled back to N, i.e., a section $\psi \colon N \to \mathcal{F}(M)$. One can also try to handle cases that do not so readily fit into this simple framework by reinterpreting them as sections of a suitable higher order frame bundle $\mathcal{F}^k(M)$ over N, cf. [**26**]. Although this is possible for all (regular, transitive) transformation groups, the original geometrical realization has now been obscured, and such a reformulation does not, we think, offer much insight or help in the explicit implementation of the method.

Consequently, the method of moving coframes includes all the classical constructions based on the indicated identification of group elements with geometric objects on the transformation space. However, once one goes beyond the traditional cases, such identifications become much less apparent, and, in our opinion, attempting to mimic the Euclidean, affine, and projective constructions directly on the transformation space has hindered the development of any significant extensions of the method. Furthermore, once one steps outside the realm of "classical" moving frame geometries, one can no longer use the identification of the first order frame with tangent and normal directions. Our non-traditional examples all illustrate this — the first order frames do not include the tangent spaces to the submanifolds in any obvious manner, because their naïve identification with subspaces of Euclidean space is not necessarily invariant with respect to the given transformation group. It is our view that, in order to attain their full range of applicability, the constructions must be viewed in the purely group- or, more generally, bundle-theoretic framework that we have presented here and develop in detail in part II.

8. Joint Differential Invariants.

New applications in image processing and object recognition, [**35**], have demonstrated the need for classification and computation of the joint differential invariants or, as they are known in computer vision, *semi-differential invariants*, for a given transformation group. Specifically, one is given a Lie group (or pseudo-group) G acting on M and considers its diagonal action $g \cdot (z^1, \ldots, z^k) = (g \cdot z^1, \ldots, g \cdot z^k)$ on the k-fold Cartesian product $M^{\times k} = M \times \cdots \times M$. The invariants $I(z^1, \ldots, z^k)$ of such a Cartesian product action are known as the *k-point joint invariants* of the transformation group. Note that for $j < k$, any j-point invariant can be regarded as a k-point invariant, in several different ways. For example, the two-point invariant $I(z_1, z_2)$ produces three invariants on $M^{\times 3}$, namely $\widehat{I}(z_1, z_2, z_3) = I(z_1, z_2)$ or $I(z_1, z_3)$ or $I(z_2, z_3)$. If I is not symmetric in its arguments, these in turn lead to 3 further invariants by interchanging the points. To avoid this trivial extension, we will reserve the term k-point invariant for a joint invariant which cannot be written as as one depending on fewer than k arguments.

Similarly, the invariants of the prolonged diagonal action of $G^{(n)}$ on a k-fold Cartesian product of jet spaces $(\mathrm{J}^n)^{\times k}$ are the *joint differential invariants* of k different submanifolds $N_1, \ldots, N_k \subset M$, which we view as a single submanifold $N_1 \times \ldots \times N_k$ of the Cartesian product space $M^{\times k}$. In applications, the submanifolds $N_j = N$ are identical, but the joint differential invariants are measured at k different points along the given submanifold.

The method of moving coframes readily adapts to this slightly more general situation, and immediately provides complete classifications of joint differential invariants for all of the standard geometric transformation groups.

Example 8.1. *Euclidean joint differential invariants.* Consider the Euclidean group $\mathrm{E}(2)$ acting on the plane $M = \mathbb{R}^2$. We consider two-point differential invariants, corresponding to the Cartesian product action

$$(\mathbf{x},\mathbf{y}) \longmapsto (R\cdot\mathbf{x}+\mathbf{a}, R\cdot\mathbf{y}+\mathbf{a}), \qquad \mathbf{x},\mathbf{y}\in M, \quad (R,a)\in \mathrm{E}(2), \tag{8.1}$$

on $M^{\times 2}\simeq \mathbb{R}^4$. Note that the action is intransitive on $M^{\times 2}$, with the interpoint distance

$$r = |\,\mathbf{z}\,|, \qquad \text{where} \qquad \mathbf{z} = \mathbf{x}-\mathbf{y}, \tag{8.2}$$

being the fundamental joint Euclidean invariant. (See [**48**] for a proof that all Euclidean joint invariants can be written in terms of the elementary two-point invariants.) We can choose the cross-section to the orbits given by $\mathbf{x}_0 = 0$, $\mathbf{y}_0 = (r, 0)$, which leads to the compatible lift with

$$\mathbf{a} = \mathbf{x}, \qquad (r\cos\phi, r\sin\phi) = \mathbf{z} = \mathbf{x}-\mathbf{y}. \tag{8.3}$$

Therefore, all the group parameters are normalized by the initial compatible lift, and it only remains to substitute (8.3) into the Euclidean Maurer–Cartan forms (3.5). The net result is the following system of invariant forms:

$$\zeta_1 = \mathbf{z}\cdot d\mathbf{x}, \qquad \zeta_2 = \mathbf{z}\cdot d\mathbf{y}, \qquad \zeta_3 = r^2\, d\phi = \mathbf{z}\wedge d\mathbf{z}. \tag{8.4}$$

Note that the forms (8.4) include the differential of the joint invariant (8.2) since $r\,dr = \zeta_1+\zeta_2$. Therefore, given two parametrized curves

$$\mathbf{x} = \mathbf{x}(t), \qquad \mathbf{y} = \mathbf{y}(s), \tag{8.5}$$

the first two one-forms (8.4) restrict to define two invariant one-forms

$$\eta_1 = (\mathbf{z}\cdot\mathbf{x}_t)\, dt, \qquad \eta_2 = (\mathbf{z}\cdot\mathbf{y}_s)\, ds, \tag{8.6}$$

while $\eta_3 = I_1\,\eta_1 + I_2\,\eta_2$, where

$$I_1 = \frac{\mathbf{z}\wedge\mathbf{x}_t}{\mathbf{z}\cdot\mathbf{x}_t}, \qquad I_2 = \frac{\mathbf{z}\wedge\mathbf{y}_s}{\mathbf{z}\cdot\mathbf{y}_s}, \tag{8.7}$$

are the two fundamental first order differential invariants, which, along with the original joint invariant (8.2), form a complete system of first order joint differential invariants. The vector identity

$$(\mathbf{a}\cdot\mathbf{b})^2 + (\mathbf{a}\wedge\mathbf{b})^2 = |\,\mathbf{a}\,|^2|\,\mathbf{b}\,|^2 \tag{8.8}$$

demonstrates that

$$J_1 = \frac{|\,\mathbf{x}_t\,|}{\mathbf{z}\cdot\mathbf{x}_t} = \frac{\sqrt{1+(I_1)^2}}{r}$$

is also a joint differential invariant, and hence (in the orientation-preserving case) one can replace the one-forms (8.6) by the two Euclidean arc-length forms

$$\omega_1 = J_1\,\eta_1 = |\,\mathbf{x}_t\,|\, dt, \qquad \omega_2 = J_2\,\eta_2 = |\,\mathbf{y}_s\,|\, ds. \tag{8.9}$$

Theorem 8.2. *Every two-point Euclidean joint differential invariant is a function of the interpoint distance* $r = |\,\mathbf{x}-\mathbf{y}\,|$ *and its derivatives with respect to the two arc length forms* (8.9).

For example, to recover the Euclidean curvature $\kappa_1 = |\,\mathbf{x}_t\,|^{-3}(\mathbf{x}_t \wedge \mathbf{x}_{tt})$ of the first curve, we differentiate

$$\frac{\partial I_1}{\partial \eta_1} = \frac{\mathbf{z} \wedge \mathbf{x}_{tt}}{(\mathbf{z} \cdot \mathbf{x}_t)^2} - \frac{(\mathbf{z} \wedge \mathbf{x}_t)[(\mathbf{z} \cdot \mathbf{x}_{tt}) + |\,\mathbf{x}_t\,|^2]}{(\mathbf{z} \cdot \mathbf{x}_t)^3} = \frac{(\mathbf{z} \wedge \mathbf{x}_{tt})(\mathbf{z} \cdot \mathbf{x}_t) - (\mathbf{z} \wedge \mathbf{x}_t)(\mathbf{z} \cdot \mathbf{x}_{tt})}{(\mathbf{z} \cdot \mathbf{x}_t)^3} - I_1 J_1^2$$
$$= \frac{(\mathbf{x}_t \wedge \mathbf{x}_{tt})|\,\mathbf{z}\,|^2}{(\mathbf{z} \cdot \mathbf{x}_t)^3} - I_1 J_1^2 = \kappa_1 r^2 - I_1 J_1^2,$$

where we have used the first of the following equivalent determinantal identities

$$\begin{aligned} (\mathbf{a} \wedge \mathbf{b})(\mathbf{c} \cdot \mathbf{d}) + (\mathbf{b} \wedge \mathbf{c})(\mathbf{a} \cdot \mathbf{d}) - (\mathbf{a} \wedge \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) &= 0, \\ (\mathbf{a} \wedge \mathbf{b})(\mathbf{c} \wedge \mathbf{d}) + (\mathbf{b} \wedge \mathbf{c})(\mathbf{a} \wedge \mathbf{d}) - (\mathbf{a} \wedge \mathbf{c})(\mathbf{b} \wedge \mathbf{d}) &= 0. \end{aligned} \tag{8.10}$$

Example 8.3. *Equi-affine joint differential invariants.* A more substantial example is provided by the two-point differential invariants for the special affine group $\mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$, acting on $M = \mathbb{R}^2$. The Cartesian product action

$$(\mathbf{x}, \mathbf{y}) \longmapsto (A\mathbf{x} + \mathbf{a}, A\mathbf{y} + \mathbf{a}), \qquad \mathbf{x}, \mathbf{y} \in M, \quad A \in \mathrm{SL}(2), \quad \mathbf{a} \in \mathbb{R}^2, \tag{8.11}$$

is transitive on $M^{\times 2}$. As in Example 6.2, we use the vector notation $A = (\boldsymbol{\alpha}\,\boldsymbol{\beta}) \in \mathrm{SL}(2)$, where $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = 1$.

In view of (8.11), we can choose the base point $\mathbf{x}_0 = 0$, $\mathbf{y}_0 = (1, 0)$, noting that the diagonal $\Delta = \{\mathbf{x} = \mathbf{y}\} \subset M^{\times 2}$ is a singular two-dimensional orbit. This leads to the compatible lift with

$$\mathbf{a} = \mathbf{x}, \qquad \boldsymbol{\alpha} = \mathbf{z} = \mathbf{x} - \mathbf{y}. \tag{8.12}$$

Substituting into the Maurer–Cartan forms (6.11), we find that, for a pair of parametrized curves as in (8.5), the following horizontal forms:

$$(\mathbf{z} \wedge \mathbf{x}_t)\, dt, \qquad (\boldsymbol{\beta} \wedge \mathbf{x}_t)\, dt, \qquad (\mathbf{z} \wedge \mathbf{y}_s)\, ds, \qquad (\boldsymbol{\beta} \wedge \mathbf{y}_s)\, ds,$$

the first two being the pull-backs of ν_1, ν_2, and the latter being that of $\nu_1 - \mu_1$, $\nu_2 - \mu_2$. Generically (i.e., provided $\mathbf{x} - \mathbf{y}$ is not parallel to $\mathbf{x}_t$) we can normalize the second form to zero, leading, in view of (8.12) and the unimodularity constraint, to

$$\boldsymbol{\beta} = \frac{\mathbf{x}_t}{\mathbf{z} \wedge \mathbf{x}_t}, \tag{8.13}$$

which, combined with (8.12) provides the complete moving frame. The remaining one-forms are

$$\eta_1 = \mathbf{z} \wedge d\mathbf{x} = (\mathbf{z} \wedge \mathbf{x}_t)\, dt, \qquad \eta_2 = \mathbf{z} \wedge d\mathbf{y} = (\mathbf{z} \wedge \mathbf{y}_s)\, ds, \tag{8.14}$$

which provide the two fundamental invariant one-forms, and

$$\eta_3 = \boldsymbol{\beta} \wedge d\mathbf{y} = \left[\frac{\mathbf{x}_t \wedge \mathbf{y}_s}{\mathbf{z} \wedge \mathbf{x}_t}\right] ds, \qquad \eta_4 = \boldsymbol{\beta} \wedge d\boldsymbol{\beta} = \left[\frac{\mathbf{x}_t \wedge \mathbf{x}_{tt}}{(\mathbf{z} \wedge \mathbf{x}_t)^2}\right] dt.$$

The resulting linear dependencies provide the two basic differential invariants, consisting of a single first order invariant

$$I = \frac{\mathbf{x}_t \wedge \mathbf{y}_s}{(\mathbf{z} \wedge \mathbf{x}_t)(\mathbf{z} \wedge \mathbf{y}_s)}\,, \tag{8.15}$$

and the first of the two second order invariants

$$J_1 = \frac{\mathbf{x}_t \wedge \mathbf{x}_{tt}}{(\mathbf{z} \wedge \mathbf{x}_t)^3}\,, \qquad J_2 = \frac{\mathbf{y}_s \wedge \mathbf{y}_{ss}}{(\mathbf{z} \wedge \mathbf{y}_s)^3}\,. \tag{8.16}$$

Clearly J_2 can be obtained from J_1 by the interchange symmetry $\mathbf{x} \leftrightarrow \mathbf{y}$. Alternatively, we use (8.10) to compute

$$\frac{\partial I}{\partial \eta_1} - I^2 = -\frac{(\mathbf{x}_{tt} \wedge \mathbf{y}_s)(\mathbf{z} \wedge \mathbf{x}_t) - (\mathbf{x}_t \wedge \mathbf{y}_s)(\mathbf{z} \wedge \mathbf{x}_{tt})}{(\mathbf{z} \wedge \mathbf{x}_t)^3 (\mathbf{z} \wedge \mathbf{y}_s)} = \frac{(\mathbf{x}_{tt} \wedge \mathbf{x}_t)(\mathbf{z} \wedge \mathbf{y}_s)}{(\mathbf{z} \wedge \mathbf{x}_t)^3 (\mathbf{z} \wedge \mathbf{y}_s)} = J_1,$$

so that (8.16) are equivalent to the invariant first order derivatives of the single basic joint invariant I.

Theorem 8.4. *Every two-point equi-affine joint differential invariant is a function of the fundamental first order invariant* (8.15) *and its derivatives with respect to the two "joint arc length" forms* (8.14).

The reader is invited to try to express the ordinary affine curvature in terms of the derivatives of I. The same method readily extends to multi-point invariants of more general groups, including the projective group, as well as joint invariants for surfaces and higher dimensional submanifolds. Additional examples and applications will appear elsewhere.

9. Pseudo-Group Actions.

The next case is that of infinite Lie pseudo-groups, cf. [**28**, **30**, **10**, **42**, **43**]. See also [**29**, **45**], for classical results on differential invariants of Lie pseudo-groups, and Kumpera, [**27**], for a modern treatment. These are readily fit into the same general framework as follows. Assume, initially, that the pseudo-group $\mathcal{G}$ acts transitively on the space M. By definition, a Lie pseudo-group consists of an infinite-dimensional family of invertible (local) transformations that form the general solution to an involutive system of partial differential equations. We can always characterize the transformations $\psi: M \to M$ in $\mathcal{G}$ as the projections of bundle maps $\Psi: \mathcal{B} \to \mathcal{B}$, defined on a principal fiber bundle $\mathcal{B} \to M$, that preserve a system of one-forms $\boldsymbol{\zeta} = \{\zeta_1, \ldots, \zeta_k\}$ defined on $\mathcal{B}$:

$$\Psi^* \boldsymbol{\zeta} = \boldsymbol{\zeta}. \tag{9.1}$$

The forms $\boldsymbol{\zeta}$ will play the role of the moving coframe forms for the pseudo-group, and the fiber coordinates of the bundle $\mathcal{B}$ will play the role of the undetermined group parameters. Of course, in this case $\boldsymbol{\zeta}$ does not form a full coframe on $\mathcal{B}$. (It cannot, because the symmetry group of a coframe is necessarily a finite-dimensional Lie group, [**38**].) A compatible lift, or moving frame of order zero, is just an arbitrary section $\rho_0: M \to \mathcal{B}$. Such a section defines a corresponding moving frame $\rho = \rho_0 \circ \iota: X \to \mathcal{B}$ on any parametrized submanifold $\iota: X \to M$. With these provisos, the normalization and reduction procedure proceeds as in the finite-dimensional situation.

Example 9.1. Consider the pseudo-group $\mathcal{G}$ consisting of (local) diffeomorphisms on $M = \mathbb{R}^2$ of the form

$$\bar{x} = f(x), \qquad \bar{u} = \frac{u}{f'(x)}. \tag{9.2}$$

The Lie algebra of $\mathcal{G}$ is generated by vector fields of the form

$$\mathbf{v}_h = h(x)\partial_x - uh'(x)\partial_u.$$

This pseudo-group was first introduced by Lie, [**28**; p. 353, **32**], in his classification of infinite-dimensional pseudo-groups acting on the plane. We are interested in the action of $\mathcal{G}$ on curves which, for simplicity, we assume are graphs of functions $u = u(x)$.

The first step is to construct a bundle $\mathcal{B}$ and one-forms on the bundle whose invariance characterizes the pseudo-group transformations. In this case, away the axis $u = 0$, the group transformations (9.2) form the general solution to the defining system of partial differential equations

$$z_u = 0, \qquad z_x = \frac{u}{w}, \qquad w_u = \frac{w}{u}, \tag{9.3}$$

for $\bar{x} = z(x,u)$, $\bar{u} = w(x,u)$, cf. [**46**; p. 325]. The system (9.3) defines a submanifold $\Phi\colon \mathcal{R} \hookrightarrow J^1(\mathbb{R}^2,\mathbb{R}^2)$ of the first jet space, parametrized by the coordinates (x,u,z,w,w_x). The pull-backs of the basic contact forms on $J^1(\mathbb{R}^2,\mathbb{R}^2)$ to the equation submanifold $\mathcal{R}$ are given by

$$\begin{aligned} \theta_z &= \Phi^*(dz - z_x\,dx - z_u\,du) = dz - \frac{u}{w}\,dx, \\ \theta_w &= \Phi^*(dw - w_x\,dx - w_u\,du) = dw - w_x\,dx - \frac{w}{u}\,du. \end{aligned} \tag{9.4}$$

The Pfaffian system

$$\theta_z = 0, \qquad \theta_w = 0,$$

with independence condition $dx \wedge du \neq 0$ is involutive on $\mathcal{R}$, cf. [**9**], [**3, 38**]. Indeed, the first Cartan character is $s_1 = 1$, as it should be. Following a general procedure† presented by Kamran, [**24**], we set $dz = dw = 0$, which amounts to pulling back to a level set of $\mathcal{R}$ where $u = u_0$ and $w = w_0$ are constant. Choosing $w_0 = 1$ we find that the contact forms (9.4) reduce to the invariant one-forms

$$\zeta_1 = -u\,dx, \qquad \zeta_2 = -w_x\,dx - \frac{du}{u}.$$

Therefore, the desired bundle $\mathcal{B} \simeq M \times \mathbb{R}$ will be coordinatized by x,u, and the remaining jet coordinate, which we rewrite as $\alpha = w_x$ for clarity. In other words, the zero$^{\text{th}}$ order moving coframe forms for the pseudo-group (9.2) will be

$$\zeta_1 = u\,dx, \qquad \zeta_2 = \alpha\,dx + \frac{du}{u}. \tag{9.5}$$

Restricting to a curve $u = u(x)$, and letting η_i denote the horizontal component of ζ_i, we have the relation

$$\eta_2 = \frac{u\alpha + u_x}{u}\,dx = \frac{u\alpha + u_x}{u^2}\,\eta_1,$$

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† Interestingly, this method is similar to our construction of the Maurer–Cartan forms in the finite-dimensional case.

and so we normalize $\alpha = -u_x/u$. Thus the final invariant moving coframe is

$$\zeta_1 = u\,dx, \qquad \zeta_2 = \frac{du - u_x\,dx}{u}, \tag{9.6}$$

the first providing a pseudo-group invariant arc length form, and the latter an invariant contact form. Note that there are no dependencies among these one-forms, and hence there are *no* differential invariants in this example. Indeed, it is not hard to see that the prolonged actions of $\mathcal{G}$ are transitive on every jet space $\mathrm{J}^n M$, justifying the preceding statement.

Example 9.2. We now extend the pseudo-group discussed in the previous example to an intransitive action obtained by augmenting the transformation rules (9.2) by an additional invariant coordinate y, so that the pseudo-group now has the form

$$\bar{x} = f(x), \qquad \bar{y} = y, \qquad \bar{u} = \frac{u}{f'(x)}. \tag{9.7}$$

This pseudo-group was introduced by Lie, [**31**; p. 373], in his study of second order partial differential equations integrable by the method of Darboux. In his paper on group splitting and automorphic systems, Vessiot, [**46**], used (9.7) as one of two principal examples illustrating his method. More recently, Kumpera, [**27**] again employed this pseudo-group to illustrate his formalization of the Lie theory of differential invariants. Now we are interested in the equivalence problem and differential invariants for surfaces $u = u(x, y)$ under the pseudo-group (9.7). The Maurer–Cartan forms are given by supplementing (9.5) by an additional coframe element $\zeta_0 = dy$. The linear dependency

$$\eta_2 = -\frac{u\alpha + u_x}{u^2}\,\eta_1 - \frac{u_y}{u}\,dy$$

again produces the normalization $\alpha = -u_x/u$, along with the basic first order differential invariant

$$I = \frac{u_y}{u}.$$

The final invariant moving coframe is

$$\zeta_0 = dy, \qquad \zeta_1 = u\,dx, \qquad \zeta_2 = \frac{du - u_x\,dx}{u}. \tag{9.8}$$

The invariant total differential operators associated with the first two horizontal forms are

$$\frac{\partial}{\partial \zeta_0} = D_y, \qquad \frac{\partial}{\partial \zeta_1} = \frac{1}{u}\,D_x. \tag{9.9}$$

Applying them to the fundamental invariant I produce the second order differential invariants

$$J_1 = \frac{u u_{yy} - u_y^2}{u^2}, \qquad J_2 = \frac{u u_{xy} - u_x u_y}{u^3},$$

agreeing with the classical formulae. All higher order differential invariants are obtained by successively applying the invariant total derivative operators (9.9) to the invariant I.

Similarly, the classifying surface associated with a generic surface $u(x, y)$ is parametrized by the four invariants (y, I, J_1, J_2); two surfaces are congruent under a pseudo-group transformation if and only if their classifying surfaces are identical. Surfaces with higher order† symmetry occur when I is a function of y only, so that $u(x, y) = f(x) g(y)$ is multiplicatively separable. Finally, the most general second order partial differential equation admitting (9.7) as a symmetry group can be written in the form

$$H\left(y, \frac{u_y}{u}, \frac{u u_{yy} - {u_y}^2}{u^2}, \frac{u u_{xy} - u_x u_y}{u^3} \right) = 0. \tag{9.10}$$

These are the class of equations considered by Lie, [**31**; p. 374].

In his classification of planar second order partial differential equations which admit symmetry pseudo-groups, Medolaghi, [**34**], treats the same example, but rewritten in a slightly different coordinate system. The group transformations take the form

$$\bar{x} = f(x), \qquad \bar{y} = y + \log f'(x), \qquad \bar{u} = u. \tag{9.11}$$

Applying the same method (or merely changing variables) leads to the invariant moving coframe

$$\zeta_1 = e^{-y}\, dx, \qquad \zeta_2 = \frac{u_x}{u_y}\, dx + dy, \qquad \zeta_3 = du.$$

The basic differential invariants are

$$u, \qquad I = u_y, \qquad J_1 = u_{yy}, \qquad J_2 = e^y (u_y u_{xy} - u_x u_{yy}),$$

the latter two being obtained by applying the invariant differential operators

$$D_y, \qquad e^y \big(D_x - (u_x / u_y) D_y \big),$$

to I. This recovers Medolaghi's form, [**34**; p. 249],

$$H\left(u, u_y, u_{yy}, e^y (u_y u_{xy} - u_x u_{yy}) \right) = 0, \tag{9.12}$$

of Lie's equation (9.10). The pseudo-group (9.11) is the second of nine different pseudo-groups acting on a three-dimensional space that are isomorphic to the diffeomorphism pseudo-group $\mathcal{D}\mathit{iff}(1)$, as classified by Medolaghi, [**34**; p. 242]. The other eight pseudo-groups can be handled by the same method, reproducing the differential invariants and invariant differential equations catalogued there.

Example 9.3. Consider the infinite Lie pseudo-group

$$\bar{x} = f(x)\ , \quad \bar{y} = y f'(x) + g(x), \quad \bar{u} = u + \frac{f''(x) y + g'(x)}{f'(x)}, \tag{9.13}$$

acting on the space $M \simeq \mathbb{R}^3$ with coordinates (x, y, u). Here $f(x)$ and $g(x)$ are arbitrary smooth functions of a single variable x. The case $g \equiv 0$ corresponds to the third of

† See [**15**] for more details on higher order submanifolds, including an interpretation as "non-reducible partially invariant solutions" to partial differential equations, cf. [**41**].

Medolaghi's pseudo-groups, [**34**]; the present generalization was introduced by J. Pohjanpelto (personal communication). The pseudo-group transformations can be characterized in terms of an involutive system of invariant one-forms on a rank five bundle $\mathcal{B} \to M$, with coordinates $(x, y, u, \alpha, \beta, \gamma, \delta, \varepsilon)$. These can be found by a similar method to that used in Example 9.1:

$$\begin{aligned} \zeta_1 &= -\alpha\, dx, & \zeta_4 &= \frac{d\alpha}{\alpha} - \frac{\gamma}{\alpha}\, dx, \\ \zeta_2 &= -\alpha\, dy + u\alpha\, dx, & \zeta_5 &= \frac{d\beta}{\alpha} + \frac{u}{\alpha}\, d\gamma - \frac{\delta - u\varepsilon}{\alpha}\, dx - \frac{\varepsilon}{\alpha}\, dy, \\ \zeta_3 &= -du - \beta\, dx - \gamma\, dy, & \zeta_6 &= \frac{d\gamma}{\alpha} - \frac{\varepsilon}{\alpha}\, dx. \end{aligned} \tag{9.14}$$

It is easy to check that a local diffeomorphism $\Psi\colon \mathcal{B} \to \mathcal{B}$ satisfies $\Psi^*\zeta_i = \zeta_i$, $i = 1, \ldots, 6$, if and only if it is a bundle map whose projection $\psi\colon M \to M$ has the form (9.13).

We now consider the equivalence problem for surfaces $u = u(x, y)$ under the pseudo-group (9.13). In order to invariantly normalize the bundle parameters, we replace du by its horizontal component $u_x\, dx + u_y\, dy$, which leads to the linear relation

$$\eta_3 = J_1\, \eta_1 + J_2\, \eta_2,$$

among the horizontal components η_i of ζ_i. The lifted invariants are

$$J_1 = \frac{u_x + \beta + u(u_y + \gamma)}{\alpha}, \qquad J_2 = \frac{u_y + \gamma}{\alpha}.$$

Both J_1 and J_2 can be normalized to zero by choosing $\beta = -u_x$ and $\gamma = -u_y$, which defines the first order moving frame. Substituting these values in the last two moving coframe forms yields

$$\begin{aligned} \eta_5 &= -\frac{u_{xx}\, dx + u_{xy}\, dy}{\alpha} - \frac{u(u_{xy}\, dx + u_{yy}\, dy)}{\alpha} - \frac{\delta - u\varepsilon}{\alpha}\, dx - \frac{\varepsilon}{\alpha}\, dy \\ &= \frac{u_{xx} + 2uu_{xy} + u^2 u_{yy} + \delta}{\alpha^2}\, \eta_1 - \frac{u_{xy} + uu_{yy} + \varepsilon}{\alpha^2}\, \eta_2, \\ \eta_6 &= -\frac{(u_{xy} + \varepsilon)\, dx + u_{yy}\, dy}{\alpha} = -\frac{u_{xy} + uu_{yy} + \varepsilon}{\alpha^2} \eta_1 + \frac{u_{yy}}{\alpha^2} \eta_2. \end{aligned}$$

We can normalize the coefficients of η_1, η_2 in both formulae by choosing

$$\alpha = \sqrt{u_{yy}}, \qquad \varepsilon = -u_{xy} - uu_{yy}, \qquad \delta = -u_{xx} - 2uu_{xy} - u^2 u_{yy},$$

which produces the second order moving frame, given by

$$\alpha = \sqrt{u_{yy}}, \quad \beta = -u_x, \quad \gamma = -u_y, \quad \delta = -u_{xx} - 2uu_{xy} - u^2 u_{yy}, \quad \varepsilon = -u_{xy} - uu_{yy}.$$

Finally, substituting into the last moving coframe form leads to $\eta_4 = -I_1\eta_1 - I_2\eta_2$, where

$$I_1 = \frac{uu_{yyy} + u_{xyy} + 2u_y u_{yy}}{2u_{yy}^{3/2}}, \qquad I_2 = \frac{u_{yyy}}{2u_{yy}^{3/2}}, \tag{9.15}$$

are the principal differential invariants of the pseudo-group. The fundamental invariant horizontal one-forms are

$$\eta_1 = -\sqrt{u_{yy}}\, dx, \qquad \eta_2 = -\sqrt{u_{yy}}\,(dy - u\,dx),$$

so that the invariant total differential operators are

$$\mathcal{D}_1 = \frac{1}{\sqrt{u_{yy}}}\,(D_x + uD_y), \qquad \mathcal{D}_1 = \frac{1}{\sqrt{u_{yy}}}\,D_y.$$

As above, these can be applied to the basic differential invariants (9.15) to generate all higher order differential invariants.

Example 9.4. In this example, we show how the well-known equivalence problem of characterizing second order ordinary differential equations under the pseudo-group of fiber-preserving transformations, cf. [**22**, **38**], can be recast into the moving frame formulation, and thereby solved by our moving coframe techniques. This example indicates a general procedure for reformulating all Cartan-type equivalence problems, [**11**, **16**, **38**], as moving frame equivalence problems under a suitable infinite-dimensional Lie pseudo-group.

We consider the trivial bundle $M \simeq \mathbb{R} \times \mathbb{R}$, with coordinates x, u. Let $\mathcal{G}$ denote the pseudo-group of fiber-preserving transformations, i.e., bundle maps

$$\bar{x} = \varphi(x), \qquad \bar{u} = \psi(x,u). \tag{9.16}$$

We let $\mathcal{G}^{(2)}$ denote the associated second prolongation acting on J^2, cf. [**38**]. A (regular) second-order differential equation

$$\Delta(x,u,u_x,u_{xx}) = 0 \tag{9.17}$$

can be identified with a hypersurface $\mathcal{S}_\Delta \subset \mathrm{J}^2$. Two such second order ordinary differential equations are equivalent if and only if their associated surfaces are mapped to each other,

$$g^{(2)}(\mathcal{S}_{\overline{\Delta}}) = \mathcal{S}_\Delta, \tag{9.18}$$

by a prolonged fiber-preserving transformation $g^{(2)} \in \mathcal{G}^{(2)}$.

In order to use the method of moving frames we need the structure equations of the pseudo-group $\mathcal{G}^{(2)}$. These can be found by the Cartan prolongation algorithm, [**11**, **16**, **38**], leading to

$$\begin{aligned}
d\zeta_1 &= \omega_1 \wedge \zeta_1,\\
d\zeta_2 &= \omega_2 \wedge \zeta_2 - \zeta_3 \wedge \zeta_1,\\
d\zeta_3 &= (\omega_2 - \omega_1) \wedge \zeta_3 + \omega_3 \wedge \zeta_2 - \zeta_4 \wedge \zeta_1,\\
d\zeta_4 &= (\omega_2 - 2\omega_1) \wedge \zeta_4 + \omega_4 \wedge \zeta_1 + \omega_5 \wedge \zeta_2 + \omega_6 \wedge \zeta_3,\\
d\omega_1 &= (\omega_6 - 2\omega_3) \wedge \zeta_1,\\
d\omega_2 &= -\pi_2 \wedge \zeta_2 - \omega_3 \wedge \zeta_1,\\
d\omega_3 &= -\pi_1 \wedge \zeta_2 - \pi_2 \wedge \zeta_3 + \omega_3 \wedge \omega_1 - \omega_5 \wedge \zeta_1,\\
d\omega_4 &= -\pi_3 \wedge \zeta_1 - \pi_4 \wedge \zeta_3 - \pi_5 \wedge \zeta_2 - 3\omega_1 \wedge \omega_4 - \omega_4 \wedge \omega_2 + 3(\omega_3 - \omega_6) \wedge \zeta_4,\\
d\omega_5 &= -2\pi_1 \wedge \zeta_3 - \pi_2 \wedge \zeta_4 - \pi_5 \wedge \zeta_1 - \pi_6 \wedge \zeta_2 + 2\omega_5 \wedge \omega_1 - \omega_3 \wedge \omega_6,\\
d\omega_6 &= -2\pi_1 \wedge \zeta_2 - 2\pi_2 \wedge \zeta_3 - \pi_4 \wedge \zeta_1 - \omega_1 \wedge \omega_6 + \omega_5 \wedge \zeta_1.
\end{aligned}$$

The Cartan characters are $s_1 = 5$ and $s_2 = 1$, the kernel dimension is 7, hence this differential system is involutive. The parametric values of the one-forms $\boldsymbol{\zeta}$, $\boldsymbol{\omega}$, are determined by introducing the group transformation matrix

$$S = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & \alpha_3\alpha_2\alpha_1^{-1} & \alpha_2\alpha_1^{-1} & 0 \\ \alpha_4\alpha_2\alpha_1^{-2} & \alpha_5\alpha_2\alpha_1^{-2} & \alpha_6\alpha_2\alpha_1^{-2} & \alpha_2\alpha_1{}^{-2} \end{pmatrix}, \tag{9.19}$$

where α_i, β_i are the fiber coordinates on the prolonged bundle. Equation (9.19) parametrizes the structure group corresponding to the action of the fiber-preserving pseudo-group on J^2; see, for instance, [**38**; p. 398] for the corresponding group on J^1. The first set of lifted forms are

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & \alpha_3\alpha_2\alpha_1^{-1} & \alpha_2\alpha_1^{-1} & 0 \\ \alpha_4\alpha_2\alpha_1^{-2} & \alpha_5\alpha_2\alpha_1^{-2} & \alpha_6\alpha_2\alpha_1^{-2} & \alpha_2\alpha_1{}^{-2} \end{pmatrix} \begin{pmatrix} dx \\ du - u_x dx \\ du_x - u_{xx} dx \\ du_{xx} \end{pmatrix}.$$

Furthermore,

$$\begin{pmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \omega_2 & 0 & 0 \\ 0 & \omega_3 & \omega_2 - \omega_1 & 0 \\ \omega_6 & \omega_5 & \omega_4 & \omega_2 - 2\omega_1 \end{pmatrix} = S^{-1} dS + \Omega,$$

where Ω represents the absorbed torsion terms. The explicit formulas are

$$\begin{aligned}
\omega_1 &= \frac{d\alpha_1}{\alpha_1} + \frac{\alpha_6 - 2\alpha_3}{\alpha_2}\,\zeta_1, \\
\omega_2 &= \frac{d\alpha_2}{\alpha_2} + \frac{\alpha_3}{\alpha_1}\,\zeta_1 - \beta_2\zeta_2, \\
\omega_3 &= \frac{d\alpha_3}{\alpha_2} + \frac{\alpha_3\alpha_6 - \alpha_5 - \alpha_3^2}{\alpha_1^2}\zeta_1 - \beta_1\zeta_2 - \beta_2\zeta_3, \\
\omega_4 &= \frac{\alpha_2}{\alpha_1^3} d\alpha_4 - \beta_3\zeta_1 + \frac{\alpha_6\alpha_5 + \alpha_3\alpha_5 - \alpha_3\alpha_6^2 - \alpha_2^2\beta_5}{\alpha_2^2}\zeta_2 + \frac{\alpha_6^2 - \alpha_5 - \beta_4\alpha_1^2}{\alpha_1^2}\zeta_3 + 3\frac{\alpha_3 - \alpha_6}{\alpha_1}\zeta_4, \\
\omega_5 &= \frac{d\alpha_5}{\alpha_1^2} - \frac{\alpha_3}{\alpha_1^2} d\alpha_4 + \beta_6\zeta_2 - 2\beta_1\zeta_3 - \beta_2\zeta_4 - \beta_5\omega_1, \\
\omega_6 &= \frac{d\alpha_6}{\alpha_1} - \beta_4\zeta_1 - 2\beta_1\zeta_2 - 2\beta_2\zeta_3.
\end{aligned}$$

We now assume, for simplicity, that the second order ordinary differential equation (9.17) is given by the graph of a section $\sigma\colon \mathrm{J}^1 \to \mathrm{J}^2$; this is equivalent to assuming that the equation is normal, and solved

$$u_{xx} = Q(x, u, u_x), \tag{9.20}$$

for its highest order derivative. (However, the moving frame method could be applied without this assumption; doing the corresponding problem for non-normal equations using the Cartan equivalence approach would be harder.) Pulling back the Maurer-Cartan forms under the map σ amounts to substituting for u_{xx} according to (9.20) where-ever it occurs. We denote the pull-back of ζ_i by η_i and of ω_i by ϖ_i. To apply the moving frame method, we look for dependencies among the resulting one-forms. The first of these is

$$\eta_4 = J_1\eta_1 + J_2\eta_2 + J_3\eta_3,$$

where

$$\begin{aligned} J_1 &= \frac{\alpha_2}{\alpha_1^3}\left(\alpha_4 + \frac{dQ}{dx}\right), \\ J_2 &= \frac{1}{\alpha_1^2}\left(\alpha_5 - \alpha_6\alpha_3 + \frac{\partial Q}{\partial u} - \alpha_3\,\frac{\partial Q}{\partial u_x}\right), \\ J_3 &= \frac{1}{\alpha_1}\left(\alpha_6 + \frac{\partial Q}{\partial u_x}\right). \end{aligned} \tag{9.21}$$

Here

$$\frac{dQ}{dx} = \frac{\partial Q}{\partial x} + u_x\,\frac{\partial Q}{\partial u} + Q\,\frac{\partial Q}{\partial u_x}$$

denotes the total derivative of Q, restricted to the equation manifold (9.20). The lifted invariants (9.21) can all be translated to zero by choosing

$$\alpha_4 = -\frac{dQ}{dx}, \qquad \alpha_5 = -\frac{\partial Q}{\partial u}, \qquad \alpha_6 = -\frac{\partial Q}{\partial u_x}.$$

We then pull-back the forms ω_5, ω_6, leading to

$$\left.\begin{aligned} \varpi_5 &\equiv -\left(2\beta_1 + \frac{1}{\alpha_1\alpha_2}Q_{uu_x} - \frac{\alpha_3}{\alpha_2^2}Q_{u_xu_x}\right)\eta_3, \\ \varpi_6 &\equiv -\left(2\beta_2 + \frac{Q_{u_xu_x}}{\alpha_2}\right)\eta_3, \end{aligned}\right. \qquad \operatorname{mod}\{\eta_1, \eta_2\}$$

Translating the coefficients of η_3 to zero in ϖ_5 and ϖ_6 gives

$$\beta_1 = -\frac{1}{2\alpha_1\alpha_2}Q_{uu_x} + \frac{\alpha_3}{2{\alpha_2}^2}Q_{u_xu_x}, \qquad \beta_2 = -\frac{1}{2\alpha_2}Q_{u_xu_x},$$

which then leads to the pulled-back forms

$$\begin{aligned} \varpi_1 &= \frac{d\alpha_1}{\alpha_1} - \left(\frac{Q_{u_x} + 2\alpha_3}{\alpha_1}\right)\eta_1, \\ \varpi_2 &= \frac{d\alpha_2}{\alpha_2} - \frac{\alpha_3}{\alpha_1}\eta_1 + \left(\frac{1}{2\alpha_2}Q_{u_xu_x}\right)\eta_2, \\ \varpi_3 &= \frac{d\alpha_3}{\alpha_1} + \left(\frac{Q_u - \alpha_3^2 - \alpha_3 Q_{u_x}}{\alpha_1^2}\right)\eta_1 + \frac{1}{2\alpha_1\alpha_2}\left(Q_{uu_x} - \alpha_3 Q_{u_xu_x}\right)\eta_2 + \left(\frac{1}{2\alpha_2}Q_{u_xu_x}\right)\eta_3. \end{aligned}$$

At this stage, we have reproduced the system of one-forms obtained via the Cartan equivalence method in [**22**; p. 394]. Further discussion of this example can be found in this reference.

10. Conclusions.

In this paper we have described a systematic procedure for determining moving frames and invariant differential forms for very general Lie group and Lie pseudo-group actions. The moving frame and moving coframe can be used to directly determine a complete system of fundamental differential invariants and invariant differential operators for the given transformation group. These, in turn, have immediate applications, including the solution to equivalence problems, classification of symmetry groups, rigidity theorems, construction of invariant equations and variational principles, and so on. As we have demonstrated, the method not only readily reproduces all of the standard examples of moving frames known in the literature, but is also in a form that can immediately be applied to a host of new and interesting group actions, including intransitive and ineffective actions, infinite-dimensional Lie pseudo-groups, joint actions, and so on. The theoretical foundations of our method will be presented in the second paper in this series, [**15**]. Additional applications — to differential invariants, to the theory of Lie pseudo-groups, to automorphic systems, and to computer vision — will be the subject of subsequent papers in this series. Some extensions that we intend to investigate include:

(1) The moving coframe method, as described in this paper, parallels the explicit "parametric" approach to the solution of Cartan equivalence problems. Gardner, [**16**], showed how, in such situations, one could perform an "intrinsic" computation, based on the infinitesimal group action on the torsion coefficients, and thereby determine the general structure of the solution. An interesting question is whether one can implement an intrinsic version of the moving coframe algorithm.

(2) In [**25**], an inductive approach to complicated equivalence problems, based on the solution to a simpler problem based on a subgroup of the full structure group, was proposed; see also [**38**]. In his thesis, Lisle, [**33**], successfully uses a similar idea in his "frame method" for symmetry classification of partial differential equations. The inductive approach not only simplifies the computations, but also provides direct correspondences between the invariants of the two problems. Is there a similar inductive version of the moving coframe method? For example, does the computation of the moving frame for curves in the plane under, say, the equi-affine group help simplify the corresponding projective computation, thereby expressing the projective arc length and curvature directly in terms of its equiaffine counterparts?

(3) In [**5**], a new scheme for generating invariant numerical approximations to differential invariants based on the use of joint invariants was proposed, and illustrated in the planar Euclidean and equi-affine cases. The computation of joint differential invariants using the moving coframe method strongly indicates that it could be applied to the general problem of invariant numerical formulae for more complicated transformation groups. In particular, determining how joint invariants converge to differential invariants as the points coalesce would be of great importance.

(4) An immediate and important application of the moving method would be to the classification of the differential invariants associated many of the transformation groups arising in physics. As remarked above, to date such classifications have

not been completed, even for some of the most fundamental groups of physical importance.

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