Moving Frames for Lie Pseudo–Groups

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Abstract.

We propose a new, constructive theory of moving frames for Lie pseudo-group actions on submanifolds. The moving frame provides an effective means for determining complete systems of differential invariants and invariant differential forms, classifying their syzygies and recurrence relations, and solving equivalence and symmetry problems arising in a broad range of applications.

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1. Introduction.

Local Lie groups of transformations and their infinite-dimensional counterparts are collectively known as Lie pseudo-groups, [**18, 19, 33, 34, 53, 56**]. Lie pseudo-groups arise in a wide range of applications, including gauge theories, [**5**], conformal geometry and field theory, [**17, 20**], fluid mechanics, [**7, 47**], and geometric numerical integration, [**43**]. However, development of suitable mathematical theory and computational algorithms has lagged far behind the well-studied and well-understood situation of finite-dimensional Lie group actions.

Given a Lie pseudo-group $\mathcal{G}$ acting on an m-dimensional manifold M, we will study the induced action of $\mathcal{G}$ on submanifolds $S \subset M$. A particularly important case is when the pseudo-group represents the symmetry group of a system of differential equations, and the submanifolds are the graphs of candidate solutions, cf. [**7, 47**]. As in the classical theory of moving frames, [**8, 24**], we will concentrate on the induced action of $\mathcal{G}$ on jets of submanifolds. Equivalence and symmetry properties of submanifolds are then, in accordance with Cartan's general philosophy, completely prescribed by the differential invariants, [**48**]. For these and a host of other applications, the key foundational issue is to understand, in as much detail as possible, the structure of the algebra of differential invariants. In this paper, we develop a theory of moving frames for Lie pseudo-group actions on submanifold jets that algorithmically reveals this structure.

In the finite-dimensional theory, [**21**], a moving frame is defined as an equivariant map $\rho^{(n)}\colon \mathrm{J}^n \to G$ from an open subset of the submanifold jet bundle to the Lie group. For Lie pseudo-groups, we still lack a suitable abstract object that can play the role of the group, and instead we define a moving frame to be an equivariant section of a suitable bundle (or, more accurately, groupoid) $\mathcal{H}^{(n)} \to \mathrm{J}^n$ constructed from the jets of pseudo-group transformations. For finite-dimensional Lie group actions, the existence of a moving frame requires that the action be free, i.e., have trivial isotropy. Clearly, an infinite-dimensional pseudo-group action never has trivial isotropy, and so we must modify the definition of freeness to require that all elements of the isotropy sub-pseudo-group of a point in J^n have the same n^{th} order jet as the identity diffeomorphism. Our freeness condition constrains the dimensions of the groupoids $\mathcal{H}^{(n)}$, and thereby assumes the role of the Spencer cohomological growth conditions imposed by Kumpera, [**32**], in his analysis of differential invariants. A word of caution: Freeness of a prolonged pseudo-group action *does not* reduce to the usual freeness condition when the pseudo-group is a finite-dimensional Lie group! Indeed, an interesting future direction of research would be to investigate the repercussions of this more general notion of freeness for finite-dimensional Lie group actions.

Assuming freeness, the explicit construction of the moving frame is founded on the Cartan normalization procedure associated with a choice of local cross-section to the group orbits in J^n, cf. [**21**]. The moving frame induces an invariantization process that canonically maps general differential functions and differential forms on J^∞ to their invariant counterparts. In particular, invariantization of the standard jet coordinates results in a complete system of functionally independent normalized differential invariants, while invariantization of the horizontal and contact one-forms yields an invariant coframe. The corresponding dual invariant total differential operators will map invariants to invariants of higher order. The structure of the algebra of differential invariants, including the specification of a finite generating set of differential invariants and the syzygies or differential relations among the generators, will then follow from the recurrence relations that relate the differentiated and normalized differential invariants. Remarkably, this final step requires only linear algebra and differentiation based on the infinitesimal generators of the pseudo-group action, and not the explicit formulae for either the differential invariants, the invariant differential operators, or even the moving frame. In the final section of the paper, we develop an alternative computational technique based on formal power series expansions, that can be effectively used to compactly specify complete systems of moving frame normalizations and recurrence relations.

We shall illustrate all our constructions with two elementary examples, which, nevertheless, already underscore many of the underlying features of the theory. More substantial applications, in geometry, physics, symmetries of differential equations, and so on will appear elsewhere, [**12**, **13**, **14**]. Extensions of these methods to Cartesian product pseudo-group actions, leading to joint invariants and joint differential invariants, as in [**50**], and multi-invariants and invariant numerical approximations, [**51**], are readily incorporated into our general "moving framework".

2. Prolongation of Diffeomorphisms.

Throughout this paper, M will be a smooth m-dimensional manifold, and we study its regular, smooth submanifolds $S \subset M$ of a fixed dimension $0 < p < n$. We will assume the reader is familiar with basic jet bundle constructions as presented, for example, in [**1**, **47**, **48**]. Our first task is to analyze the prolonged action of the diffeomorphism pseudo-group on jets of submanifolds, with the aim of placing the implicit differentiation formulae of multivariable calculus in a conducive geometric setting. To ease the reader into the formalism, let us look at the simplest situation: regular plane curves.

Remark: As in [**53**], we will consistently follow Cartan's notational convention that lower case letters z, x, u, etc., refer to source coordinates, while their capitalized counterparts Z, X, U, refer to the target coordinates of local diffeomorphisms $Z = \varphi(z)$.

Example 2.1. Let $M = \mathbb{R}^2$ have coordinates $z = (x, u)$. For $0 \leq n \leq \infty$, the n-jet of a local diffeomorphism $X = \chi(x, u)$, $U = \psi(x, u)$, at a source point in $\mathbb{R}^2$, is prescribed by its derivatives (Taylor coefficients) up to order n, which we denote by

$$X_x = \frac{\partial \chi}{\partial x}, \quad X_u = \frac{\partial \chi}{\partial u}, \quad U_x = \frac{\partial \psi}{\partial x}, \quad U_u = \frac{\partial \psi}{\partial u}, \quad X_{xx} = \frac{\partial^2 \chi}{\partial x^2}, \quad X_{xu} = \frac{\partial^2 \chi}{\partial x\, \partial u}, \quad \ldots .$$

We use

$$(z, Z^{(n)}) = (x, u, X^{(n)}, U^{(n)}) = (x, u, X, U, X_x, X_u, U_x, U_u, X_{xx}, X_{xu}, X_{uu}, U_{xx}, \ \ldots), \quad (2.1)$$

to denote the induced local coordinates on the diffeomorphism jet bundle $\mathcal{D}^{(n)} = \mathcal{D}^{(n)}(\mathbb{R}^2)$, which is the subbundle of $\mathrm{J}^n(\mathbb{R}^2, \mathbb{R}^2)$ that is specified by the local invertibility constraint $X_x U_u - X_u U_x \neq 0$. In view of the chain rule, composition and inversion of diffeomorphisms induce composition and inversion operations on their jets, endowing $\mathcal{D}^{(n)}$ with the structure of a *groupoid*, cf. [**18**, **40**, **53**]. The term "groupoid" refers to the fact that composition of jets (Taylor polynomials/series) is only well defined when the target of the initial jet matches the source of its successor.

Consider the action of local diffeomorphisms on curves $C \subset \mathbb{R}^2$, that is, $p = 1-$dimensional regular smooth submanifolds. Since our viewpoint is local, we can focus our attention on curves which are the graphs of smooth functions $u = f(x)$. (Curves with vertical tangents are handled by a different choice of local coordinates, e.g., interchanging the roles of independent and dependent variables. Extensions to general parametrized curves are straightforward.) The n-jet of such a curve is prescribed by its derivatives, denoted

$$u_x = f'(x), \qquad u_{xx} = f''(x), \qquad \ldots \qquad u_n = f^{(n)}(x),$$

and so

$$z^{(n)} = (x, u^{(n)}) = (x, u, u_x, u_{xx}, \ \ldots \ , u_n) \tag{2.2}$$

are the induced local coordinates on the curve jet space $\mathrm{J}^n = \mathrm{J}^n(\mathbb{R}^2, 1)$. The action of local diffeomorphisms on curves induces an action on their jets, known as the *prolonged action* of the diffeomorphism pseudo-group. Moreover, as a consequence of the chain rule, the n-jet of the transformed curve only depends on the n-jet of the diffeomorphism, and so there is an induced action of the diffeomorphism groupoid $\mathcal{D}^{(n)}$ on J^n.

The explicit formulae for the prolonged action are, as usual, obtained by implicit differentiation. We will use

$$\widehat{Z}^{(n)} = (X, \widehat{U}^{(n)}) = (X, U, \widehat{U}_X, \widehat{U}_{XX}, \ldots , \widehat{U}_n)$$

to denote the jet coordinates of the transformed curve:

$$\widehat{U} = U = F(X), \qquad \widehat{U}_X = F'(X), \qquad \widehat{U}_{XX} = F''(X), \qquad \ldots \qquad \widehat{U}_n = F^{(n)}(X).$$

(The hats are added to avoid confusion with the diffeomorphism jet coordinates U_x, U_u, $U_{xx}, U_{xu}, \ \ldots$.) Let

$$\begin{aligned}
\mathbb{D}_x &= \frac{\partial}{\partial x} + X_x \frac{\partial}{\partial X} + U_x \frac{\partial}{\partial U} + X_{xx} \frac{\partial}{\partial X_x} + X_{xu} \frac{\partial}{\partial X_u} + U_{xx} \frac{\partial}{\partial U_x} + U_{xu} \frac{\partial}{\partial U_u} + \ \cdots \ , \\
\mathbb{D}_u &= \frac{\partial}{\partial u} + X_u \frac{\partial}{\partial X} + U_u \frac{\partial}{\partial U} + X_{xu} \frac{\partial}{\partial X_x} + X_{uu} \frac{\partial}{\partial X_u} + U_{xu} \frac{\partial}{\partial U_x} + U_{uu} \frac{\partial}{\partial U_u} + \ \cdots \ ,
\end{aligned} \tag{2.3}$$

be the total derivative operators on the diffeomorphism jet bundle $\mathcal{D}^{(n)}$, cf. [**53**; (2.12)]. Further, let

$$\mathrm{D}_x = \mathbb{D}_x + u_x \, \mathbb{D}_u + u_{xx} \frac{\partial}{\partial u_x} + u_{xxx} \frac{\partial}{\partial u_{xx}} + \ \cdots \tag{2.4}$$

be the total derivative operator with respect to all variables — both diffeomorphism jets (2.1) and curve jets (2.2). (For the moment, we defer the discussion of precisely which bundle this operator lives on.) The required implicit differentiation operator is then given by

$$\mathrm{D}_X = \frac{1}{\mathrm{D}_x X} \, \mathrm{D}_x = \frac{1}{X_x + u_x \, X_u} \, \mathrm{D}_x. \tag{2.5}$$

Indeed, the local coordinate formulae for the prolonged action of a diffeomorphism jet on a curve jet are found by recursively applying D_X to the dependent variable U:

$$\begin{aligned}
\widehat{U}_X &= \mathrm{D}_X U = \frac{\mathrm{D}_x U}{\mathrm{D}_x X} = \frac{U_x + u_x\, U_u}{X_x + u_x\, X_u}\,, \\
\widehat{U}_{XX} &= \mathrm{D}_X^2 U = \frac{\mathrm{D}_x^2 U\, \mathrm{D}_x X - \mathrm{D}_x U\, \mathrm{D}_x^2 X}{(\mathrm{D}_x X)^3} \\
&= (X_x + u_x\, X_u)^{-3}\big[(U_{xx} + 2\, u_x\, U_{xu} + u_x^2\, U_{uu} + u_{xx}\, U_u)(X_x + u_x\, X_u) \\
&\qquad\qquad - (U_x + u_x\, U_u)(X_{xx} + 2\, u_x\, X_{xu} + u_x^2\, X_{uu} + u_{xx}\, X_u)\big],
\end{aligned} \tag{2.6}$$

and so on, reproducing the well-known implicit differentiation formulae of elementary calculus.

Let us now discuss how to properly formalize this basic example in a general framework. For $0 \leq n \leq \infty$, let $\mathrm{J}^n = \mathrm{J}^n(M,p)$ denote the n^{th} order extended[1] jet bundle consisting of equivalence classes of p-dimensional submanifolds $S \subset M$ under the equivalence relation of n^{th} order contact, cf. [**48**]. We use the standard local coordinates

$$z^{(n)} = (x, u^{(n)}) = (\ \ldots\ x^i\ \ldots\ u_J^\alpha\ \ldots\ ) \tag{2.7}$$

on J^n induced by a splitting of the local coordinates $z = (x,u) = (x^1, \ldots, x^p, u^1, \ldots, u^q)$ on M into p independent and $q = m - p$ dependent variables, [**47, 48**]. When $k > n$, we let $\widetilde{\pi}_n^k \colon \mathrm{J}^k \to \mathrm{J}^n$ denote the usual projection, so $\widetilde{\pi}_n^k(z^{(k)}) = z^{(n)}$.

The choice of independent variables induces a decomposition of the differential one-forms on J^∞. The basis horizontal forms are the differentials $dx^1, \ldots, dx^p$ of the independent variables, while the basis contact forms are denoted by

$$\theta_J^\alpha = du_J^\alpha - \sum_{i=1}^p u_{J,i}^\alpha\, dx^i, \qquad \alpha = 1, \ldots, q, \qquad \#J \geq 0. \tag{2.8}$$

This decomposition[2] splits the differential $d = d_H + d_V$ on J^∞ into horizontal and vertical (or contact) components, and endows the space of differential forms with the structure of a variational bicomplex[3], [**1, 31, 61**]. In particular, given a differential function $F \colon \mathrm{J}^n \to \mathbb{R}$,

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[1] In other words, we are not assuming that M has any preassigned bundle structure, so as to allow jets of arbitrary embedded p-dimensional submanifolds $S \subset M$. Since all our constructions are local, they are equally valid when $M \to N$ is a fiber bundle with p-dimensional base N, and $\mathrm{J}^n M \subset \mathrm{J}^n(M,p)$ is the dense open subbundle prescribed by jets of sections.

[2] The decomposition only works at infinite order, which is one of the main reasons for passing to the infinite jet bundle.

[3] We re-emphasize that this construction is only valid in a local coordinate chart, relying, further, on the selection of independent and dependent variables. See Itskov's thesis, [**26, 27**], for an intrinsic reformulation based on the $\mathcal{C}$–spectral sequence induced by the contact filtration of differential forms on J^∞.

its horizontal differential is

$$d_H F = \sum_{j=1}^{p} (\mathrm{D}_{x^j} F)\, dx^j, \qquad \text{where} \qquad \mathrm{D}_{x^j} = \frac{\partial}{\partial x^j} + \sum_{\alpha=1}^{q} \sum_{\# J \geq 0} u^\alpha_{J,j} \frac{\partial}{\partial u^\alpha_J}\,, \tag{2.9}$$

are the usual total derivative operators, while its vertical differential

$$d_V F = \sum_{\alpha=1}^{q} \sum_{\# J \geq 0} \frac{\partial F}{\partial u^\alpha_J}\, \theta^\alpha_J \tag{2.10}$$

can be interpreted as its "first variation", [**48**].

For example, in the planar situation of Example 2.1, the basis contact forms are

$$\theta = du - u_x\, dx, \qquad \theta_x = du_x - u_{xx}\, dx, \qquad \theta_{xx} = du_{xx} - u_{xxx}\, dx, \qquad \ldots\,. \tag{2.11}$$

The exterior derivative of a differential function $F(x, u^{(n)})$ accordingly splits into horizontal and contact constituents:

$$\begin{aligned} dF &= \frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial u}\, du + \frac{\partial F}{\partial u_x}\, du_x + \frac{\partial F}{\partial u_{xx}}\, du_{xx} + \cdots \\ &= (D_x F)\, dx + \left(\frac{\partial F}{\partial u}\, \theta + \frac{\partial F}{\partial u_x}\, \theta_x + \frac{\partial F}{\partial u_{xx}}\, \theta_{xx} + \cdots \right) = d_H F + d_V F. \end{aligned}$$

Let $\mathcal{D} = \mathcal{D}(M)$ be the pseudo-group of all local diffeomorphisms[4] $\varphi\colon M \to M$. For each $n \geq 0$, let $\mathcal{D}^{(n)} = \mathcal{D}^{(n)}(M) \subset \mathrm{J}^n(M, M)$ denote the subbundle formed by their n^{th} order jets. Composition and inversion of local diffeomorphisms induces the composition and inversion of their jets, so

$$\mathrm{j}_n \psi|_{\varphi(z)} \cdot \mathrm{j}_n \varphi|_z = \mathrm{j}_n(\psi \circ \varphi)|_z, \qquad \left(\mathrm{j}_n \varphi|_z \right)^{-1} = \mathrm{j}_n\left(\varphi^{-1} \right)|_{\varphi(z)}, \tag{2.12}$$

whenever $z \in \operatorname{dom} \varphi$ and $\varphi(z) \in \operatorname{dom} \psi$. In particular, the product of jets is only defined when the target of the initial jet matches the source of its successor. The resulting operations endow each $\mathcal{D}^{(n)}$ with the structure of a groupoid.

Local coordinates of a diffeomorphism jet in $\mathcal{D}^{(n)}$ are indicated by $(z, Z^{(n)})$, where $z = (x, u) = \boldsymbol{\sigma}^{(n)}(z, Z^{(n)})$ are the source coordinates on M, while the fiber jet coordinates

$$\begin{aligned} &Z^{(n)} = (\ \ldots\ Z^b_A\ \ldots\ ) = (X^{(n)}, U^{(n)}) = (\ \ldots\ X^i_A\ \ldots\ U^\alpha_A\ \ldots\ ), \qquad \text{where} \\ &\qquad b = 1, \ldots, m, \qquad i = 1, \ldots, p, \qquad \alpha = 1, \ldots, q, \\ &\qquad A = (a_1, \ldots, a_k), \qquad \text{with} \qquad 1 \leq a_\nu \leq m \qquad \text{and} \qquad 0 \leq k = \# A \leq n, \end{aligned} \tag{2.13}$$

indicate partial derivatives of the target coordinates $Z = (X, U) = \boldsymbol{\tau}^{(n)}(z, Z^{(n)})$ with respect to all source variables $z = (x, u)$. The *source map* $\boldsymbol{\sigma}^{(n)}$ and *target map* $\boldsymbol{\tau}^{(n)}$ serve

[4] Our notational conventions allow the domain of such a map to be a proper open subset: $\operatorname{dom} \varphi \subset M$. Also, when we write $\varphi(z)$ we implicitly assume $z \in \operatorname{dom} \varphi$.

to define the double fibration

$$\begin{tikzcd} & \mathcal{D}^{(n)} \arrow[dl, "\boldsymbol{\sigma}^{(n)}"'] \arrow[dr, "\boldsymbol{\tau}^{(n)}"] & \\ M & & M \end{tikzcd} \tag{2.14}$$

A local diffeomorphism $\varphi \in \mathcal{D}$ preserves the contact equivalence relation between p-dimensional submanifolds $S \subset M$, and thus induces an action on the jet bundle $\mathrm{J}^n = \mathrm{J}^n(M,p)$, known as the n^{th} *prolonged action.* As in (2.6), the chain rule implies that the n–jet of the transformed submanifold depends only on the n–jet of the diffeomorphism, and hence there is a corresponding action of the diffeomorphism jet groupoid $\mathcal{D}^{(n)}$ on J^n, given by

$$\mathrm{j}_n\varphi|_z \cdot \mathrm{j}_n N|_z = \mathrm{j}_n \varphi(N)|_{\varphi(z)}. \tag{2.15}$$

As we saw in (2.6), the local coordinate formulae for the prolonged action of $\mathcal{D}^{(n)}$ on J^n involve both sets of jet coordinates. Together, they naturally coordinatize the pull-back bundle $\mathcal{E}^{(n)} \to \mathrm{J}^n$ of the diffeomorphism jet bundle $\mathcal{D}^{(n)} \to M$ via the standard projection $\widetilde{\pi}^n_0 \colon \mathrm{J}^n \to M$. For $k > n$ we let $\widehat{\pi}^k_n \colon \mathcal{E}^{(k)} \to \mathcal{E}^{(n)}$ denote the projection induced by $\widetilde{\pi}^k_n \colon \mathrm{J}^k \to \mathrm{J}^n$ and $\pi^k_n \colon \mathcal{D}^{(k)} \to \mathcal{D}^{(n)}$. Points in $\mathcal{E}^{(n)}$ are characterized by two quantities:

- a jet $z^{(n)} \in \mathrm{J}^n$ of a p-dimensional submanifold passing through $z = \widetilde{\pi}^n_0(z^{(n)}) \in M$, and,
- a jet $(z, Z^{(n)}) \in \mathcal{D}^{(n)}$ of a local diffeomorphism based at the same point $z = \boldsymbol{\sigma}^{(n)}(z, Z^{(n)})$.

The combined actions of local diffeomorphisms on submanifold jets, (2.15), and on diffeomorphism jets, (2.12), induces an action of $\mathcal{D}$, and hence also the diffeomorphism jet groupoid $\mathcal{D}^{(n)}$, on the bundle $\mathcal{E}^{(n)}$.

Local coordinates on $\mathcal{E}^{(n)}$ are indicated by $\mathbf{Z}^{(n)} = (z^{(n)}, Z^{(n)})$, where $z^{(n)} = (x, u^{(n)})$ are identified with the usual coordinates (2.7) on J^n, while $Z^{(n)} = (X^{(n)}, U^{(n)})$ are identified with the fiber coordinates (2.13) of the diffeomorphism jet bundle. For instance, in the plane curve case of Example 2.1, the coordinates on $\mathcal{E}^{(n)}$ are

$$\begin{aligned} \mathbf{Z}^{(n)} &= (z^{(n)}, Z^{(n)}) = (x, u^{(n)}, X^{(n)}, U^{(n)}) \\ &= (x, u, u_x, u_{xx}, \ \ldots \ , X, U, X_x, X_u, U_x, U_u, X_{xx}, X_{xu}, X_{uu}, U_{xx}, \ \ldots \ ), \end{aligned}$$

where $u_x, u_{xx}, \ldots$ are curve jet coordinates, whereas $X, U, X_x, X_u, U_x, U_u, \ X_{xx}, \ldots$ are diffeomorphism jet coordinates.

The groupoid structure on $\mathcal{E}^{(n)}$ is induced by that on $\mathcal{D}^{(n)}$, namely composition and inversion of jets of diffeomorphisms, (2.12), coupled with the prolonged action of diffeomorphisms on submanifold jets (2.15). For the associated double fibration

$$\begin{tikzcd} & \mathcal{E}^{(n)} \arrow[dl, "\widetilde{\boldsymbol{\sigma}}^{(n)}"'] \arrow[dr, "\widetilde{\boldsymbol{\tau}}^{(n)}"] & \\ \mathrm{J}^n & & \mathrm{J}^n \end{tikzcd} \tag{2.16}$$

the source map is merely the projection, $\widetilde{\boldsymbol{\sigma}}^{(n)}(z^{(n)}, Z^{(n)}) = z^{(n)}$, while the target is defined by the prolonged action of $\mathcal{D}^{(n)}$ on J^n, namely

$$(X, \widehat{U}^{(n)}) = \widehat{Z}^{(n)} = \widetilde{\boldsymbol{\tau}}^{(n)}(\mathbf{Z}^{(n)}) = \widetilde{\boldsymbol{\tau}}^{(n)}(z^{(n)}, Z^{(n)}) = \mathbf{Z}^{(n)} \cdot z^{(n)}. \tag{2.17}$$

Here, as noted above, we place hats on the target submanifold jet coordinates to avoid confusion with the diffeomorphism jet coordinates. In local coordinates, the entries of the target map encode the implicit differentiation formulae

$$\widehat{U}^\alpha_J = F^\alpha_J(z^{(n)}, Z^{(n)}) = F^\alpha_J(x, u^{(n)}, X^{(n)}, U^{(n)}) \tag{2.18}$$

for the jets of transformed submanifolds, which we now determine.

The bundle structure $\widetilde{\boldsymbol{\sigma}}^{(\infty)}\colon \mathcal{E}^{(\infty)} \to \mathrm{J}^\infty$ induces a splitting of its cotangent bundle $T^*\mathcal{E}^{(\infty)}$ into jet and group components, spanned, respectively, by[5] the *jet forms*, consisting of the horizontal and contact one-forms

$$dx^i, \qquad \theta^\alpha_J, \qquad i = 1, \ldots, p, \qquad \alpha = 1, \ldots, q, \qquad \#J \geq 0, \tag{2.19}$$

from the submanifold jet bundle J^∞, and the contact one-forms

$$\Upsilon^b_A = d_G\, Z^b_A = dZ^b_A - \sum_{c=1}^{m} Z^b_{A,c}\, dz^c, \qquad b = 1, \ldots, m, \qquad \#A \geq 0, \tag{2.20}$$

from the diffeomorphism jet bundle $\mathcal{D}^{(\infty)} \subset \mathrm{J}^\infty(M, M)$, cf. [**53**]. We will call the latter *group forms*, in order to distinguish them from the contact forms on the submanifold jet bundle. For instance, in the planar case of Example 2.1, the group forms are

$$\begin{aligned} \Upsilon^1 &= dX - X_x\, dx - X_u\, du, & \Upsilon^2 &= dU - U_x\, dx - U_u\, du, \\ \Upsilon^1_x &= dX_x - X_{xx}\, dx - X_{xu}\, du, & \Upsilon^1_u &= dX_u - X_{xu}\, dx - X_{uu}\, du, \end{aligned} \tag{2.21}$$

and so on. We accordingly decompose the differential on $\mathcal{E}^{(\infty)}$ into jet and group components, the former further splitting into horizontal and vertical components:

$$d = d_J + d_G = d_H + d_V + d_G\,. \tag{2.22}$$

The resulting operators satisfy

$$d_J^2 = d_G^2 = d_H^2 = d_V^2 = 0, \tag{2.23}$$
$$d_J\, d_G = -\, d_G\, d_J, \quad d_H\, d_V = -\, d_V\, d_H, \quad d_H\, d_G = -\, d_G\, d_H, \quad d_V\, d_G = -\, d_G\, d_V,$$

and so form a pseudo-group generalization of the "lifted tricomplex" introduced in [**30, 31**].

The horizontal differential of a function $F(\mathbf{Z}^{(n)}) = F(z^{(n)}, Z^{(n)})$ has the local coordinate formula

$$d_H\, F = \sum_{j=1}^{p} (\mathrm{D}_{x^j} F)\, dx^j, \tag{2.24}$$

where

$$\mathrm{D}_{x^j} = \mathbb{D}_{x^j} + \sum_{\alpha=1}^{q} \left(u^\alpha_j\, \mathbb{D}_{u^\alpha} + \sum_{\#J \geq 1} u^\alpha_{J,j}\, \frac{\partial}{\partial u^\alpha_J} \right) \tag{2.25}$$

[5] In all cases, to avoid unnecessary clutter, we identify functions and forms with their pull-backs to $\mathcal{E}^{(\infty)}$ under the appropriate bundle projection.

are the *lifted total derivative operators* on $\mathcal{E}^{(\infty)}$, which are obtained from the usual total derivatives (2.9) by replacing the order zero partial derivatives[6] $\partial/\partial x^j, \partial/\partial u^\alpha$ by the corresponding total derivative operators

$$\mathbb{D}_{z^a} = \frac{\partial}{\partial z^a} + \sum_{b=1}^{m} \sum_{\# A \geq 0} Z^b_{A,a} \frac{\partial}{\partial Z^b_A}, \qquad a = 1, \ldots, m, \tag{2.26}$$

on the diffeomorphism jet bundle $\mathcal{D}^{(\infty)}$. We use the same notation for the total derivative operators on J^n and $\mathcal{E}^{(n)}$ since they coincide when $F(z^{(n)}) = F(x, u^{(n)})$ does not actually depend upon the diffeomorphism jet coordinates. When computing, it is important to remember that the horizontal differential d_H also includes differentiation with respect to the pseudo-group parameters. The local coordinate formulas for the vertical and group differentials are given by

$$d_V F = \sum_{\alpha=1}^{q} \left[(\mathbb{D}_{u^\alpha} F)\, \theta^\alpha + \sum_{\# J \geq 1} \frac{\partial F}{\partial u^\alpha_J}\, \theta^\alpha_J \right], \qquad d_G F = \sum_{b=1}^{m} \sum_{\# A \geq 0} \frac{\partial F}{\partial Z^b_A}\, \Upsilon^b_A. \tag{2.27}$$

In the planar case, the differentials of $F(x, u, u_x, u_{xx}, \ldots, X_x, X_u, U_x, \ldots)$ are

$$\begin{aligned} d_H F &= (\mathrm{D}_x F)\, dx \\ d_V F &= (\mathbb{D}_u F)\, \theta + \frac{\partial F}{\partial u_x}\, \theta_x + \frac{\partial F}{\partial u_{xx}}\, \theta_{xx} + \frac{\partial F}{\partial u_{xxx}}\, \theta_{xxx} + \cdots, \\ d_G F &= \frac{\partial F}{\partial X}\, \Upsilon^1 + \frac{\partial F}{\partial U}\, \Upsilon^2 + \frac{\partial F}{\partial X_x}\, \Upsilon^1_x + \frac{\partial F}{\partial X_u}\, \Upsilon^1_u + \frac{\partial F}{\partial U_x}\, \Upsilon^2_x + \cdots, \end{aligned}$$

where D_x is the total derivative operator (2.4), while $\mathbb{D}_u$ is given in (2.3).

Recall that the capitalized notation $Z = (X, U)$ refers to the target coordinates of the diffeomorphism, and hence its entries can be viewed as functions on $\mathcal{D}^{(\infty)}$, and, through a further pull-back, on $\mathcal{E}^{(\infty)}$. We use the target independent variables X^i on $\mathcal{E}^{(\infty)}$ to construct the *lifted horizontal coframe*

$$d_H X^i = \sum_{j=1}^{p} (\mathrm{D}_{x^j} X^i)\, dx^j, \qquad i = 1, \ldots, p, \tag{2.28}$$

whose coefficients

$$\mathrm{D}_{x^j} X^i = X^i_{x^j} + \sum_{\alpha=1}^{q} u^\alpha_j\, X^i_{u^\alpha}$$

depend linearly on the first order jet coordinates $Z^{(1)} = (X^{(1)}, U^{(1)}) \in \mathcal{D}^{(1)}$ and on the submanifold jet coordinates $u^{(1)}$. In local coordinate computations, to ensure that the one-forms (2.28) are linearly independent, we restrict our attention to the dense open subset

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[6] We only need to replace the order zero partial derivatives because we are dealing with pseudo-groups of point transformations. With a little extra work, our methods can be straightforwardly extended to pseudo-groups of (first order) contact transformations, [**48**].

where the total Jacobian determinant is non-zero,

$$\det\big(\,\mathrm{D}_{x^j} X^i\,\big) \neq 0, \tag{2.29}$$

which excludes jets of submanifolds which no longer intersect the vertical fibers transversally when acted on by the diffeomorphism jet. Again, the excluded submanifolds can be handled by adopting an alternative system of local coordinates. The horizontal differentiation formula

$$d_H\, F = \sum_{i=1}^{p} (\mathrm{D}_{X^i} F)\, d_H X^i, \tag{2.30}$$

which is valid for any differential function $F(z^{(n)}, Z^{(n)})$, serves to define the dual total differentiation operators

$$\mathrm{D}_{X^i} = \sum_{j=1}^{p} W_i^j\, \mathrm{D}_{x^j}, \qquad \text{where} \qquad \big(\,W_i^j\,\big) = \big(\,\mathrm{D}_{x^j} X^i\,\big)^{-1} \tag{2.31}$$

indicates the entries of the inverse total Jacobian matrix. For instance, in the planar case, the horizontal one-form

$$d_H\, X = (\mathrm{D}_x X)\, dx = (X_x + u_x\, X_u)\, dx \quad \text{has dual differentiation} \quad \mathrm{D}_X = \frac{1}{X_x + u_x\, X_u}\, \mathrm{D}_x,$$

as noted above in (2.5).

With all this in hand, the chain rule formulae (2.18) for the higher-order prolonged action of $\mathcal{D}^{(n)}$ on J^n, i.e., the target map $\widetilde{\boldsymbol{\tau}}^{(n)} : \mathcal{E}^{(n)} \to \mathrm{J}^n$, are obtained by successively differentiating the target dependent variables U^α with respect to the target independent variables X^i, whereby

$$\widehat{U}_J^\alpha = \mathrm{D}_X^J U^\alpha = \mathrm{D}_{X^{j_1}} \cdots \mathrm{D}_{X^{j_k}} U^\alpha. \tag{2.32}$$

These are the multi-dimensional versions of the implicit differentiation formulae (2.6).

For later purposes, we introduce the right-invariant contact one-forms on $\mathcal{D}^{(\infty)}$, which, according to [**53**], are to be interpreted as the *Maurer–Cartan forms* for the diffeomorphism pseudo-group. To this end, we use the product bundle structure of $\mathcal{D}^{(\infty)} \subset \mathrm{J}^\infty(M, M)$ to split its differential $d = d_M + d_G$ into horizontal and group (or vertical or contact) components — as in the standard variational bicomplex construction noted above. This splitting is invariant under right composition of diffeomorphisms. Since the target coordinates Z^a are obviously right-invariant, so are their horizontal differentials

$$\sigma^a = d_M\, Z^a = \sum_{i=1}^{m} Z_i^a\, dz^i, \qquad a = 1, \ldots, m. \tag{2.33}$$

Let $\mathbb{D}_{Z^1}, \ldots, \mathbb{D}_{Z^m}$ denote the corresponding dual right-invariant total derivative operators, so that

$$d_M\, F = \sum_{a=1}^{m} (\mathbb{D}_{Z^a} F)\, \sigma^a \qquad \text{whenever} \qquad F : \mathcal{D}^{(\infty)} \to \mathbb{R}. \tag{2.34}$$

Then the basis Maurer–Cartan forms are obtained by successively Lie differentiating the (right-invariant) order 0 contact forms $\Upsilon^b = d_G Z^b$:

$$\mu_A^b = \mathbb{D}_Z^A \Upsilon^b, \qquad \begin{array}{lll} b = 1, \ldots, m, & A = (a_1, \ldots, a_k), & 1 \le a_\nu \le m, \\ \text{where} \quad \mathbb{D}_Z^A = \mathbb{D}_{Z^{a_1}} \cdots \mathbb{D}_{Z^{a_k}}, & & k = \#A \ge 0. \end{array} \tag{2.35}$$

The complete collection of one-forms σ^a, μ_A^b in (2.33, 35) forms a right-invariant coframe on $\mathcal{D}^{(\infty)}$. See [**53**] for the explicit form of the resulting diffeomorphism structure equations.

Example 2.2. In the planar case of Example 2.1, the right-invariant horizontal forms (2.33) on $\mathcal{D}^{(\infty)}(\mathbb{R}^2)$ are

$$\sigma^1 = d_M X = X_x \, dx + X_u \, du, \qquad \sigma^2 = d_M U = U_x \, dx + U_u \, du, \tag{2.36}$$

with dual total derivative operators

$$\mathbb{D}_X = \frac{U_u \, \mathbb{D}_x - U_x \, \mathbb{D}_u}{X_x U_u - X_u U_x}, \qquad \mathbb{D}_U = \frac{- X_u \, \mathbb{D}_x + X_x \, \mathbb{D}_u}{X_x U_u - X_u U_x}. \tag{2.37}$$

The zero$^{\text{th}}$ order Maurer–Cartan forms coincide with the zero$^{\text{th}}$ order contact forms

$$\begin{aligned} \mu &= \Upsilon^1 = d_G X = dX - X_x \, dx - X_u \, du, \\ \nu &= \Upsilon^2 = d_G U = dU - U_x \, dx - U_u \, du, \end{aligned} \tag{2.38}$$

while the higher-order Maurer–Cartan forms are obtained by repeatedly applying the right-invariant differential operators (2.37) to the one-forms (2.38). In particular, the first order Maurer–Cartan forms are expressed in terms of the basis contact forms (2.21) as follows:

$$\begin{aligned} \mu_X &= \mathbb{D}_X \mu = \frac{U_u \, \Upsilon_x^1 - U_x \, \Upsilon_u^1}{X_x U_u - X_u U_x}, & \mu_U &= \mathbb{D}_U \mu = \frac{X_x \, \Upsilon_u^1 - X_u \, \Upsilon_x^1}{X_x U_u - X_u U_x}, \\ \nu_X &= \mathbb{D}_X \nu = \frac{U_u \, \Upsilon_x^2 - U_x \, \Upsilon_u^2}{X_x U_u - X_u U_x}, & \nu_U &= \mathbb{D}_U \nu = \frac{X_x \, \Upsilon_u^2 - X_u \, \Upsilon_x^2}{X_x U_u - X_u U_x}. \end{aligned} \tag{2.39}$$

3. Moving Frames for Pseudo–Groups.

Roughly speaking, a sub-pseudo-group $\mathcal{G} \subset \mathcal{D}$ is called a *Lie pseudo-group* if its local diffeomorphisms $\varphi \in \mathcal{G}$ are a complete system of local solutions to a formally integrable system of partial differential equations. We will impose technical assumptions of regularity and tameness on the pseudo-group; see [**53**] for details. In particular, for each $n^* \le n < \infty$, for a fixed $n^* \ge 0$, the subgroupoid $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ consisting of the pseudo-group jets $\mathrm{j}_n \varphi$ for $\varphi \in \mathcal{G}$ is assumed to form a smooth, embedded subbundle with fiber dimension $r_n = \dim \mathcal{G}^{(n)}|_z$ for any $z \in M$. In the limit, the infinite pseudo-group jet bundle $\mathcal{G}^{(\infty)} \subset \mathrm{J}^\infty(M, M)$ can be identified with the complete determining system of partial differential equations for the pseudo-group. We use

$$\mathbf{g}^{(n)} = (z, g^{(n)}) = (x, u, g^{(n)})$$

to indicate local coordinates of a jet $\mathbf{g}^{(n)} \in \mathcal{G}^{(n)}$, with the "pseudo-group parameters" $g^{(n)} = (g_1, \ldots, g_{r_n})$ parametrizing the jet fiber $\mathcal{G}^{(n)}|_z$.

We let $\mathcal{H}^{(n)} \subset \mathcal{E}^{(n)}$ denote the subgroupoid obtained by pulling back $\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ via the projection $\widetilde{\pi}^n_0 \colon \mathrm{J}^n \to M$. The groupoid structure on $\mathcal{H}^{(n)}$ is induced by that of $\mathcal{E}^{(n)}$; the explicit formulae are obtained either by specializing the general prolonged diffeomorphism transformations (2.17) to the pseudo-group subbundle, or by direct construction via implicit differentiation of the pseudo-group transformations on M.

In this context, the notion of a differential invariant for the pseudo-group $\mathcal{G}$ can be formulated as follows.

Definition 3.1. A *differential invariant* is a differential function[7] $I \colon \mathrm{J}^n \to \mathbb{R}$ which is unaffected by the prolonged action of $\mathcal{G}^{(n)}$ on J^n, and so

$$I(X, \widehat{U}^{(n)}) = I(\mathbf{g}^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)}), \tag{3.1}$$

for all $(x, u^{(n)}) \in \mathrm{J}^n$, and all pseudo-group jets $\mathbf{g}^{(n)} \in \mathcal{G}^{(n)}$ such that both the source and target submanifold jets, namely $(x, u^{(n)})$ and $(X, \widehat{U}^{(n)}) = \mathbf{g}^{(n)} \cdot (x, u^{(n)})$, belong to the domain of I.

In other words, differential invariants are constant on the prolonged pseudo-group orbits in J^n. Morally, the entire collection of differential invariants forms an algebra; however, since they are in general only locally defined, they in fact define a sheaf of algebras over J^∞, [**32**, **63**]. But, to foster intuition at the expense of precision, we will usually refer to the "algebra of differential invariants". One of our main goals is to understand its structure in complete detail.

If G is a finite-dimensional transformation group acting locally effectively on subsets of M, as in [**49**], then, for $n \gg 0$, the bundle $\mathcal{H}^{(n)}$ can be locally identified with the principal bundle $\mathrm{J}^n \times G$ introduced in [**21**]. So, working by analogy with the finite-dimensional version, we define a moving frame to be an equivariant section of this bundle. Therefore, our pseudo-group moving frame construction will include the finite-dimensional version in [**21**, **30**, **31**] as a special subcase.

Definition 3.2. A *moving frame* $\rho^{(n)}$ of *order* n is a $\mathcal{G}^{(n)}$ equivariant local section of the bundle $\mathcal{H}^{(n)} \to \mathrm{J}^n$.

More explicitly, we require $\rho^{(n)} \colon \mathrm{J}^n \to \mathcal{H}^{(n)}$ to satisfy

$$\widetilde{\boldsymbol{\sigma}}^{(n)}(\rho^{(n)}(z^{(n)})) = z^{(n)}, \qquad \rho^{(n)}(\mathbf{g}^{(n)} \cdot z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot (\mathbf{g}^{(n)})^{-1},$$

for all $\mathbf{g}^{(n)} \in \mathcal{G}^{(n)}|_z$, with $z = \widetilde{\pi}^n_0(z^{(n)})$, and groupoid inverse $(\mathbf{g}^{(n)})^{-1} \in \mathcal{G}^{(n)}|_{\boldsymbol{\tau}^{(n)}(\mathbf{g}^{(n)})}$, such that both $z^{(n)}$ and $\mathbf{g}^{(n)} \cdot z^{(n)}$ lie in the domain of definition of $\rho^{(n)}$.

Remark: Definition 3.2 defines a right-equivariant moving frame, [**21**]. Classical moving frames for finite-dimensional Lie group actions, [**8**, **24**], are always left-equivariant. It is not difficult to formulate the notion of a left moving frame in the pseudo-group context. As usual, the inversion map converts between right and left moving frames, and so we can concentrate on the slightly simpler right-equivariant version from here on.

[7] We continue to use the convention that functions need only be defined on an open subset of their domain space.

In the finite-dimensional construction, [**21**], the existence of a moving frame requires that the group action be free and regular on an open subset of the jet space. Similar conditions are required in the pseudo-group framework. The crucially important freeness condition is defined as follows. Let

$$\mathcal{G}^{(n)}_z = \left\{ \mathbf{g}^{(n)} \in \mathcal{G}^{(n)} \;\middle|\; \boldsymbol{\tau}^{(n)}(\mathbf{g}^{(n)}) = \boldsymbol{\sigma}^{(n)}(\mathbf{g}^{(n)}) = z \right\} \subset \mathcal{G}^{(n)}|_z$$

denote the n^{th} order *isotropy jet subgroup* of the point $z \in M$, which, as a consequence of our definition of a Lie pseudo-group, is a finite-dimensional Lie group when $n < \infty$. The *isotropy subgroup* of $z^{(n)} = (x, u^{(n)}) \in \mathrm{J}^n$ is then defined as the closed Lie subgroup

$$\mathcal{G}^{(n)}_{z^{(n)}} = \left\{ \mathbf{g}^{(n)} \in \mathcal{G}^{(n)}_z \;\middle|\; \mathbf{g}^{(n)} \cdot z^{(n)} = z^{(n)} \right\} \subset \mathcal{G}^{(n)}_z, \qquad \text{where} \qquad z = \widetilde{\pi}^n_0(z^{(n)}). \tag{3.2}$$

Thus, the isotropy subgroup of a submanifold n-jet only contains diffeomorphism jets of order n, and ignores their higher order derivatives. The n–jet of the identity diffeomorphism $\mathbb{1}\colon M \to M$ at z, denoted $\mathbb{1}^{(n)}_z$, clearly lies in $\mathcal{G}^{(n)}_{z^{(n)}}$. Freeness requires that this be the only isotropy jet.

Definition 3.3. The pseudo-group $\mathcal{G}$ acts *freely* at $z^{(n)} \in \mathrm{J}^n$ if $\mathcal{G}^{(n)}_{z^{(n)}} = \{ \mathbb{1}^{(n)}_z \}$, and *locally freely* if $\mathcal{G}^{(n)}_{z^{(n)}}$ is a discrete subgroup of $\mathcal{G}^{(n)}_z$. The pseudo-group $\mathcal{G}$ is said to act (*locally*) *freely at order* n if it acts (locally) freely on an open subset $\mathcal{V}^n \subset \mathrm{J}^n$, called the set of *regular* n*-jets*.

In other words, freeness of the action means that every pseudo-group transformation that fixes the jet $z^{(n)} \in \mathrm{J}^n|_z$ must have the same derivatives (jet) as the identity map up to order n, irrespective of the values of its derivatives of order $> n$. Note that the freeness condition for a pseudo-group is, in fact, equivalent to the freeness of the action of the isotropy jet subgroup $\mathcal{G}^{(n)}_z$ on the jet fiber $\mathrm{J}^n|_z$. At order $n = 0$, any pseudo-group action trivially satisfies the freeness condition, because $\mathcal{G}^{(0)}_z = \{ \mathbb{1}^{(0)}_z \}$. Thus, freeness is only of interest when $n \geq 1$.

Warning: According to the standard definition, [**21**], any (locally) free action of a finite-dimensional Lie group satisfies the (local) freeness condition of Definition 3.3, but the converse is *not* valid. For instance, the four-dimensional Lie group

$$(x, u) \longmapsto (x + a, u + b\,x^2 + c\,x + d)$$

defines a free pseudo-group action on $\mathrm{J}^n(\mathbb{R}^2, 1)$ for *all* $n \geq 0$. But, as a Lie group, the action is only free when $n \geq 2$. In this paper, even for finite-dimensional Lie group actions, we will use "free" in the more general sense of Definition 3.3. An interesting project would be to revisit the study of differential invariants of finite-dimensional Lie group actions using this more refined notion of freeness.

Let

$$\mathcal{O}^{(n)}_{z^{(n)}} = \left\{ \mathbf{g}^{(n)} \cdot z^{(n)} \;\middle|\; \mathbf{g}^{(n)} \in \mathcal{G}^{(n)}|_z,\ z = \widetilde{\pi}^n_0(z^{(n)}) \right\} \subset \mathrm{J}^n \tag{3.3}$$

denote the prolonged pseudo-group orbit passing through the submanifold jet $z^{(n)} \in \mathrm{J}^n$. The tameness condition of [**53**; Definition 5.2] implies, by a theorem of Sussmann, [**60**], that the pseudo-group orbits are immersed submanifolds. Regularity requires that the orbits form a regular foliation, i.e., its leaves intersect small open sets in pathwise connected subsets. Further details can be found in [**52**].

Proposition 3.4. *The pseudo-group $\mathcal{G}$ acts locally freely on the subset*

$$\left\{ z^{(n)} \in \mathrm{J}^n \;\middle|\; \dim \mathcal{O}^{(n)}_{z^{(n)}} = r_n \right\}, \tag{3.4}$$

consisting of those jets whose orbit dimension equals the fiber dimension of the n^{th} order jet groupoid $\mathcal{G}^{(n)} \to M$.

Thus, freeness of the pseudo-group at order n requires, at the very least, that the fiber dimension satisfy the inequality

$$r_n = \dim \mathcal{G}^{(n)}|_z \leq \dim \mathrm{J}^n = p + (m-p)\binom{p+n}{p}. \tag{3.5}$$

Therefore, freeness is an alternative — and simpler — means of quantifying the Spencer cohomological growth conditions imposed by Kumpera, [**32**]. Pseudo-groups having too large a fiber dimension r_n will, typically, act transitively on (a dense open subset of) J^n, and thus possess no non-constant differential invariants. A familiar example is the pseudo-group of canonical transformations of a symplectic manifold. In such cases, all (generic) submanifolds are locally equivalent, and the local theory is trivial. But there are, of course, deep global issues not addressed by the local moving frame theory, [**23**].

In a forthcoming paper, [**54**], we will establish the following fundamental result, thereby rigorously justifying the general constructions used in this paper.

Theorem 3.5. *Let $\mathcal{G}$ be a regular pseudo-group acting on an m-dimensional manifold M. If $\mathcal{G}$ acts (locally) freely at $z^{(n)} \in \mathrm{J}^n$ for some $n > 0$, then it acts (locally) freely at any $z^{(k)} \in \mathrm{J}^k$ with $\widetilde{\pi}^k_n(z^{(k)}) = z^{(n)}$, for $k \geq n$.*

As in the finite-dimensional Lie group version, [**21**], moving frames are constructed through a normalization procedure based on a choice of *cross-section* to the pseudo-group orbits, i.e., a transverse submanifold of the complementary dimension.

Theorem 3.6. *Suppose $\mathcal{G}^{(n)}$ acts freely and regularly on $\mathcal{V}^n \subset \mathrm{J}^n$. Let $K^n \subset \mathcal{V}^n$ be a (local) cross-section to the pseudo-group orbits. Given $z^{(n)} \in \mathcal{V}^n$, define $\rho^{(n)}(z^{(n)}) \in \mathcal{H}^{(n)}$ to be the unique groupoid jet such that $\widetilde{\boldsymbol{\tau}}^{(n)}(\rho^{(n)}(z^{(n)})) \in K^n$ (when such exists). Then $\rho^{(n)} \colon \mathrm{J}^n \to \mathcal{H}^{(n)}$ is a moving frame for $\mathcal{G}$ defined on an open subset of $\mathcal{V}^n$. The local cross-section coordinates of the induced map $I^{(n)} = \widetilde{\boldsymbol{\tau}}^{(n)} \circ \rho^{(n)} \colon \mathrm{J}^n \to K^n$ provide a complete system of functionally independent n^{th} order differential invariants on the domain of definition of the moving frame.*

In most practical situations, we select a coordinate cross-section, defined by fixing the values of r_n of the individual jet coordinates $z^{(n)} = (x, u^{(n)})$. We first write out the implicit differentiation formulae for the prolonged pseudo-group action

$$(X, \widehat{U}^{(n)}) = F^{(n)}(x, u^{(n)}, g^{(n)}) \tag{3.6}$$

in terms of the submanifold jet coordinates $(x, u^{(n)})$ and a convenient system of group parameters $g^{(n)} = (g_1, \ldots, g_{r_n})$ which, when combined together, serve to coordinatize the bundle $\mathcal{H}^{(n)} \to \mathrm{J}^n$. The r_n components of (3.6) corresponding to our choice of cross-section variables serve to define the *normalization equations*

$$F_1(x, u^{(n)}, g^{(n)}) = c_1, \qquad \ldots \qquad F_{r_n}(x, u^{(n)}, g^{(n)}) = c_{r_n}. \tag{3.7}$$

Solving them for the group parameters

$$g^{(n)} = \gamma^{(n)}(x, u^{(n)}) \tag{3.8}$$

(a solution is assured by their arising from a bona fide cross-section) yields the explicit formula for the moving frame section:

$$\rho^{(n)}(x, u^{(n)}) = (x, u^{(n)}, \gamma^{(n)}(x, u^{(n)})).$$

Substituting the moving frame formulae for the pseudo-group parameters (3.8) into the unnormalized components of (3.6) yields the *normalized differential invariants*:

$$I^{(n)}(x, u^{(n)}) = F^{(n)}(x, u^{(n)}, \gamma^{(n)}(x, u^{(n)})) = \big(\ \ldots\ H^i(x, u^{(n)})\ \ldots\ I^\alpha_K(x, u^{(n)})\ \ldots\ \big). \tag{3.9}$$

The r_n components of $I^{(n)}$ appearing in the normalization equations (3.7) will be constant, and are known as the *phantom differential invariants*. The remaining $s_n = \dim \mathrm{J}^n - r_n$ components are the cross-section coordinates, and hence form a complete system of functionally independent differential invariants of order $\leq n$.

Definition 3.7. A moving frame $\rho^{(k)}\colon \mathrm{J}^k \to \mathcal{H}^{(k)}$ of order $k > n$ is *compatible* with a moving frame $\rho^{(n)}\colon \mathrm{J}^n \to \mathcal{H}^{(n)}$ of order n provided $\widehat{\pi}^k_n \circ \rho^{(k)} = \rho^{(n)} \circ \widetilde{\pi}^k_n$ where defined.

A *complete moving frame* is provided by a mutually compatible collection of moving frames of all orders $k \geq n$. To avoid technical problems with shrinking domains of definition, we further assume that the lowest order moving frame $\rho^{(n)}$ is defined on a domain $\mathcal{V}^n \subset \mathrm{J}^n$, while each higher order compatible moving frame $\rho^{(k)}$ is defined on $\mathcal{V}^k = (\widetilde{\pi}^k_n)^{-1}(\mathcal{V}^n)$. In applications, we typically deal with complete moving frames, and we use $\rho^{(\infty)}\colon \mathrm{J}^\infty \to \mathcal{H}^{(\infty)}$ to denote the limiting equivariant local section. Before continuing, let us understand how the moving frame algorithm works in two basic examples.

Example 3.8. Consider the intransitive pseudo-group action

$$X = f(x), \qquad\qquad Y = y, \qquad\qquad U = \frac{u}{f'(x)}\,, \tag{3.10}$$

on $M = \mathbb{R}^3 \setminus \{\, u = 0 \,\}$. This pseudo-group was introduced by Lie, [**36**; p. 373], in his study of second order partial differential equations integrable by the method of Darboux, and also considered by Vessiot, [**62**], in his paper on group splitting and automorphic systems. More recently, Kumpera, [**32**], again employed this pseudo-group as the one example used to illustrate his Spencerian formalization of the Lie theory of differential invariants. Our methods reproduce Kumpera's final results with minimal effort, and, subsequently, elucidate the structure of its differential invariant algebra, which was not exposed in previous treatments of this elementary example.

We are interested in the induced action of (3.10) on surfaces $S \subset M$, which, for simplicity, we assume to be the graph of a function $u = h(x, y)$. (Extending the method to more general parametric surfaces, cf. [**21**], is not difficult.) We adopt the Taylor coordinates $f, f_x, f_{xx}, \ldots$ of the diffeomorphism $f(x)$ to parametrize the pseudo-group. The lifted horizontal coframe is

$$d_H X = f_x \, dx, \qquad\qquad d_H Y = dy, \tag{3.11}$$

and hence the dual implicit differentiations are

$$\mathrm{D}_X = \frac{1}{f_x} \mathrm{D}_x, \qquad\qquad \mathrm{D}_Y = \mathrm{D}_y. \tag{3.12}$$

The prolonged pseudo-group transformations on the surface jet bundle $\mathrm{J}^n = \mathrm{J}^n(M, 2)$ are obtained by repeated application of the implicit differentiation operators (3.12) to $U = u / f_x$, and so

$$\begin{gathered} X = f, \qquad Y = y, \qquad \widehat{U} = U = \frac{u}{f_x}, \qquad \widehat{U}_X = \frac{u_x}{f_x^2} - \frac{u\, f_{xx}}{f_x^3}, \qquad \widehat{U}_Y = \frac{u_y}{f_x}, \\ \widehat{U}_{XX} = \frac{u_{xx}}{f_x^3} - \frac{3\, u_x\, f_{xx}}{f_x^4} - \frac{u\, f_{xxx}}{f_x^4} + \frac{3\, u\, f_{xx}^2}{f_x^5}\,, \qquad \widehat{U}_{XY} = \frac{u_{xy}}{f_x^2} - \frac{u_y\, f_{xx}}{f_x^3}\,, \qquad \widehat{U}_{YY} = \frac{u_{yy}}{f_x}\,, \end{gathered} \tag{3.13}$$

and so on. Since $u \neq 0$, the isotropy subgroup $\mathcal{G}^{(n)}_{z^{(n)}}$ of any $(x, u^{(n)}) \in \mathrm{J}^n$ consists only of the identity jet, $f = x, f_x = 1, f_{xx} = 0, \ldots$, and hence the pseudo-group acts freely at every order $n \geq 0$.

We choose the coordinate cross-section

$$x = 0, \qquad u = 1, \qquad u_x = u_{xx} = u_{xxx} = \cdots = 0.$$

The associated moving frame map is found by solving the corresponding normalization equations

$$X = 0, \qquad U = 1, \qquad \widehat{U}_X = 0, \qquad \widehat{U}_{XX} = 0, \qquad \ldots\,, \tag{3.14}$$

for the group parameters:

$$f = 0, \qquad f_x = u, \qquad f_{xx} = u_x, \qquad f_{xxx} = u_{xx}, \qquad \ldots\,. \tag{3.15}$$

Substituting (3.15) into the prolonged transformation formulae (3.13) yield the normalized second order differential invariants; those corresponding to the normalization variables (3.14) are the constant phantom differential invariants, while the remainder, namely

$$Y \longmapsto y, \quad \widehat{U}_Y \longmapsto J = \frac{u_y}{u}, \quad \widehat{U}_{XY} \longmapsto J_1 = \frac{u\, u_{xy} - u_x\, u_y}{u^3}, \quad \widehat{U}_{YY} \longmapsto J_2 = \frac{u_{yy}}{u}, \tag{3.16}$$

form a complete system of functionally independent second order differential invariants. Moreover, substitution of the moving frame formulae (3.15) into the lifted horizontal forms (3.11), i.e., pulling back by the moving frame, leads to the basic invariant horizontal coframe

$$d_H X \longmapsto u\, dx, \qquad\qquad d_H Y \longmapsto dy, \tag{3.17}$$

and corresponding dual invariant differential operators

$$\mathcal{D}_1 = \frac{1}{u}\,\mathrm{D}_x, \qquad\qquad \mathcal{D}_2 = \mathrm{D}_y. \tag{3.18}$$

As we shall subsequently prove — see Examples 7.2, 8.3, and 8.5 — all the higher-order normalized differential invariants can be obtained by successively applying the invariant operators (3.18) to the basic differential invariant J. For example,

$$\mathcal{D}_1 J = \frac{u\,u_{xy} - u_x\,u_y}{u^3} = J_1, \qquad\qquad \mathcal{D}_2 J = \frac{u\,u_{yy} - u_y^2}{u^2} = J_2 - J^2. \tag{3.19}$$

Later, we will learn how to algorithmically derive such recurrence formulae relating the differentiated invariants to the normalized differential invariants.

Example 3.9. Consider the action of the pseudo-group

$$X = f(x), \qquad\qquad Y = f'(x)\,y + g(x), \qquad\qquad U = u + \frac{f''(x)\,y + g'(x)}{f'(x)}, \tag{3.20}$$

on surfaces $u = h(x,y)$. To obtain the prolonged pseudo-group transformations, we begin with the lifted horizontal coframe,

$$d_H\,X = f_x\,dx, \qquad\qquad d_H\,Y = e_x\,dx + f_x\,dy, \tag{3.21}$$

where, for convenience, we set

$$e(x,y) = f'(x)\,y + g(x), \qquad\quad \text{and so} \qquad\quad e_y = f_x, \quad f_y = 0.$$

The prolonged pseudo-group transformations are found by applying the dual implicit differentiations

$$\mathrm{D}_X = \frac{1}{f_x}\,\mathrm{D}_x - \frac{e_x}{f_x^2}\,\mathrm{D}_y, \qquad\qquad \mathrm{D}_Y = \frac{1}{f_x}\,\mathrm{D}_y,$$

successively to

$$\widehat{U} = U = u + \frac{e_x}{f_x} = u + \frac{f_{xx}\,y + g_x}{f_x}\,,$$

so that

$$\begin{aligned}
\widehat{U}_X &= \frac{u_x}{f_x} + \frac{e_{xx} - e_x\,u_y}{f_x^2} - 2\,\frac{f_{xx}\,e_x}{f_x^3}\,, \qquad\qquad \widehat{U}_Y = \frac{u_y}{f_x} + \frac{f_{xx}}{f_x^2}\,,\\
\widehat{U}_{XX} &= \frac{u_{xx}}{f_x^2} + \frac{e_{xxx} - e_{xx}\,u_y - 2\,e_x\,u_{xy} - f_{xx}\,u_x}{f_x^3} + \\
&\qquad\quad + \frac{e_x^2\,u_{yy} + 3\,e_x f_{xx}\,u_y - 4\,e_{xx}\,f_{xx} - 3\,e_x\,f_{xxx}}{f_x^4} + 8\,\frac{e_x\,f_{xx}^2}{f_x^5}\,,\\
\widehat{U}_{XY} &= \frac{u_{xy}}{f_x^2} + \frac{f_{xxx} - f_{xx}\,u_y - e_x\,u_{yy}}{f_x^3} - 2\,\frac{f_{xx}^2}{f_x^4}\,, \qquad\quad \widehat{U}_{YY} = \frac{u_{yy}}{f_x^2}\,,
\end{aligned} \tag{3.22}$$

and so on. The pseudo-group cannot act freely on J^1 since $r_1 = \dim \mathcal{G}^{(1)}|_z = 6 > \dim \mathrm{J}^1 = 5$. On the other hand, $r_2 = \dim \mathcal{G}^{(2)}|_z = 8 = \dim \mathrm{J}^2$, and the action on J^2 is, in fact, locally free and transitive on the sets $\mathcal{V}_+^2 = \mathrm{J}^2 \cap \{u_{yy} > 0\}$ and $\mathcal{V}_-^2 = \mathrm{J}^2 \cap \{u_{yy} < 0\}$. Moreover, in accordance with Theorem 3.5, $\mathcal{G}^{(n)}$ acts locally freely on the corresponding open subsets of J^n for any $n \geq 2$.

To construct the moving frame, we restrict our attention to[8] $\mathcal{V}_+^2$ and adopt the following normalizations:

$$\begin{aligned}
X &= 0, & f &= 0,\\
Y &= 0, & e &= 0,\\
U &= 0, & e_x &= -\,u\, f_x,\\
\widehat{U}_Y &= 0, & f_{xx} &= -\,u_y\, f_x,\\
\widehat{U}_X &= 0, & e_{xx} &= (u\, u_y - u_x)\, f_x,\\
\widehat{U}_{YY} &= 1, & f_x &= \sqrt{u_{yy}}\,,\\
\widehat{U}_{XY} &= 0, & f_{xxx} &= -\sqrt{u_{yy}}\,\left(u_{xy} + u u_{yy} - u_y^2\right),\\
\widehat{U}_{XX} &= 0, & e_{xxx} &= -\sqrt{u_{yy}}\,\left(u_{xx} - u u_{xy} - 2\,u^2 u_{yy} - 2u_x u_y + u u_y^2\right).
\end{aligned} \tag{3.23}$$

At this stage, we have normalized enough parameters to find the first two fundamental differential invariants of the pseudo-group, namely,

$$\widehat{U}_{XYY} \longmapsto J_1 = \frac{u_{xyy} + u u_{yyy} + 2\,u_y u_{yy}}{u_{yy}^{3/2}}\,, \qquad \widehat{U}_{YYY} \longmapsto J_2 = \frac{u_{yyy}}{u_{yy}^{3/2}}\,. \tag{3.24}$$

The two remaining third order jet coordinates can be normalized to $\widehat{U}_{XXX} = \widehat{U}_{XXY} = 0$, to produce formulae for the pseudo-group parameters f_{xxxx} and e_{xxxx}. In general, for $n \geq 2$, there are

$$\dim \mathrm{J}^n - r_n = \left[\,\frac{(n+1)(n+2)}{2} + 2\,\right] - (2\,n+4) = \frac{(n+1)(n-2)}{2}$$

functionally independent differential invariants of order $\leq n$.

Finally, substituting the pseudo-group normalizations into (3.21) fixes the invariant horizontal coframe

$$d_H X \longmapsto \omega^1 = \sqrt{u_{yy}}\; dx, \qquad d_H Y \longmapsto \omega^2 = \sqrt{u_{yy}}\,(dy - u\,dx). \tag{3.25}$$

The dual invariant total differential operators are

$$\mathcal{D}_1 = \frac{1}{\sqrt{u_{yy}}}\,(\mathrm{D}_x + u\,\mathrm{D}_y), \qquad \mathcal{D}_2 = \frac{1}{\sqrt{u_{yy}}}\,\mathrm{D}_y. \tag{3.26}$$

As we shall subsequently prove — see Example 8.6 — the higher-order differential invariants can be generated by successively applying these differential operators to the pair of basic differential invariants (3.24). According to the general theorem in [**54**], all

[8] To cover $\mathcal{V}_-^2$, just insert an absolute value inside the square root and keep track of signs.

syzygies or functional relations among the differentiated invariants in this example are consequences of the lowest order such syzygy, which is

$$\mathcal{D}_1 J_2 - \mathcal{D}_2 J_1 = 2. \tag{3.27}$$

4. Infinitesimal Generators.

Our subsequent analysis will rely heavily on the infinitesimal generators of the pseudo-group action. Let $\mathcal{X}(M)$ denote the space of locally defined smooth vector fields on M, i.e., local sections of the tangent bundle TM. In terms of the local coordinates $z = (x, u)$ on M, a vector field takes the form

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a} = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x,u) \frac{\partial}{\partial u^\alpha} \ , \tag{4.1}$$

where the coefficients $(\zeta^1, \ldots, \zeta^m) = (\xi^1, \ldots, \xi^p, \varphi^1, \ldots, \varphi^q)$ are smooth functions.

Given $0 \leq n \leq \infty$, let $\mathrm{J}^n TM$ denote the n^{th} order jet bundle of the tangent bundle, whose elements are n-jets $\mathrm{j}_n \mathbf{v}$ of locally defined vector fields $\mathbf{v} \in \mathcal{X}(M)$. In local coordinates, the n-jet $(z, \zeta^{(n)}) \in \mathrm{J}^n TM$ of the vector field (4.1) at a point $z = (x, u)$ is determined by the partial derivatives of its coefficients with respect to all variables $z = (x, u)$ up to order n, which we denote by

$$\begin{gathered} \zeta^{(n)} = (\ \ldots\ \zeta^b_A\ \ldots\ ) = (\xi^{(n)}, \varphi^{(n)}) = (\ \ldots\ \xi^i_A\ \ldots\ \varphi^\alpha_A\ \ldots\ ), \\ \text{where} \quad \begin{aligned} & b = 1, \ldots, m, \qquad i = 1, \ldots, p, \qquad \alpha = 1, \ldots, q, \\ & A = (a_1, \ldots, a_k), \quad \text{with} \quad 1 \leq a_\nu \leq m \quad \text{and} \quad 0 \leq k = \#A \leq n. \end{aligned} \end{gathered} \tag{4.2}$$

Given $\mathbf{v} \in \mathcal{X}(M)$, let $\mathbf{v}^{(n)} \in \mathcal{X}(\mathrm{J}^n)$ denote the corresponding prolonged vector field on the submanifold jet bundle J^n. At each jet $z^{(n)} \in \mathrm{J}^n|_z$, the prolongation operation prescribes a linear map

$$\mathbf{p}^{(n)} = \mathbf{p}^{(n)}_{z^{(n)}} : \ \mathrm{J}^n TM|_z \ \longrightarrow \ T\mathrm{J}^n|_{z^{(n)}}. \tag{4.3}$$

In terms of the local coordinates $z^{(n)} = (x, u^{(n)})$ on J^n, the n^{th} prolongation of the vector field (4.1) has the form

$$\mathbf{v}^{(n)} \ = \ \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\#J \leq n} \widehat{\varphi}^{\,\alpha}_J \frac{\partial}{\partial u^\alpha_J} \ . \tag{4.4}$$

As with the $\widehat{U}^\alpha_J$, we place hats on the prolonged vector field coefficients $\widehat{\varphi}^{\,\alpha}_J$ so as to distinguish them from the partial derivatives (jet coordinates) φ^α_A in (4.2). The coefficients are computed via the usual prolongation formula, cf. [**47, 48**]:

$$\widehat{\varphi}^{\,\alpha}_J = \mathrm{D}^J_x Q^\alpha + \sum_{i=1}^{p} u^\alpha_{J,i}\, \xi^i, \qquad \text{where} \qquad Q^\alpha = \varphi^\alpha - \sum_{i=1}^{p} u^\alpha_i\, \xi^i, \qquad \alpha = 1, \ldots, q, \tag{4.5}$$

are the components of the *characteristic* of $\mathbf{v}$. Consequently, each prolonged vector field coefficient

$$\widehat{\varphi}\,^{\alpha}_{J} = \Phi^{\alpha}_{J}(u^{(n)}, \zeta^{(n)}) \tag{4.6}$$

is a certain universal linear combination of the vector field jet coordinates (4.2), whose coefficients are polynomials in the submanifold jet coordinates u^{β}_{K} for $1 \leq \#K \leq n$.

Example 4.1. On $\mathbb{R}^2$, with coordinates (x, u) as in Example 2.1, the prolongation of a vector field

$$\mathbf{v} = \xi(x,u)\,\frac{\partial}{\partial x} + \varphi(x,u)\,\frac{\partial}{\partial u} \tag{4.7}$$

to $\mathrm{J}^n = \mathrm{J}^n(\mathbb{R}^2, 1)$ takes the familiar form

$$\mathbf{v}^{(\infty)} = \xi\,\frac{\partial}{\partial x} + \varphi\,\frac{\partial}{\partial u} + \widehat{\varphi}\,^{x}\,\frac{\partial}{\partial u_x} + \widehat{\varphi}\,^{xx}\,\frac{\partial}{\partial u_{xx}} + \;\cdots\; + \widehat{\varphi}\,^{n}\,\frac{\partial}{\partial u_n}\,,$$

where

$$\begin{aligned}
\widehat{\varphi}\,^{x} &= \mathrm{D}_x\varphi - u_x\,\mathrm{D}_x\xi = \varphi_x + u_x(\varphi_u - \xi_x) - u_x^2\xi_u,\\
\widehat{\varphi}\,^{xx} &= \mathrm{D}_x^2\varphi - u_x\,\mathrm{D}_x^2\xi - 2\,u_{xx}\,\mathrm{D}_x\xi\\
&= \varphi_{xx} + u_x(2\,\varphi_{xu} - \xi_{xx}) + u_x^2(\varphi_{uu} - 2\,\xi_{xu}) - u_x^3\,\xi_{uu} + u_{xx}(\varphi_u - 2\,\xi_x) - 3\,u_x\,u_{xx}\,\xi_u,
\end{aligned} \tag{4.8}$$

and so on.

Given a pseudo-group $\mathcal{G}$, let $\mathfrak{g} \subset \mathcal{X}(M)$ denote the local Lie algebra consisting of its infinitesimal generators, i.e., the set of locally defined vector fields whose flows belong to the pseudo-group. Let $\mathrm{J}^n\mathfrak{g} \subset \mathrm{J}^nTM$ denote the subbundle[9] prescribed by their jets. In local coordinates, $\mathrm{J}^n\mathfrak{g}$ is defined by a linear system of partial differential equations

$$L^{(n)}(z, \zeta^{(n)}) = 0 \tag{4.9}$$

for the vector field coefficients, called the *linearized* or *infinitesimal determining equations* for the pseudo-group. They are obtained by linearizing the nonlinear determining equations for the pseudo-group transformations at the identity. If $\mathcal{G}$ is the symmetry group of a system of differential equations, then the linearized determining equations (4.9) are the (involutive completion of) the usual determining equations for its infinitesimal generators obtained via Lie's algorithm, [**13**, **14**, **37**, **38**, **47**].

Let

$$\mathfrak{g}^{(n)}|_{z^{(n)}} = \mathbf{p}^{(n)}(\mathrm{J}^n\mathfrak{g}|_z) \subset T\mathrm{J}^n|_{z^{(n)}}$$

denote the subspace spanned by the prolonged infinitesimal generators of the pseudo-group. Assuming tameness of the prolonged pseudo-group action, $\mathfrak{g}^{(n)}|_{z^{(n)}} = T\mathcal{O}^{(n)}_{z^{(n)}}|_{z^{(n)}}$ is equal to the tangent space to the pseudo-group orbit through $z^{(n)}$, [**52**]. The infinitesimal characterization of local freeness of the prolonged pseudo-group action is immediate:

[9] The fact that this forms a subbundle is a consequence of our definitions and local solvability.

Proposition 4.2. *The pseudo-group acts locally freely near* $z^{(n)}$ *if and only if the prolongation map* $\mathbf{p}^{(n)}\colon \mathrm{J}^n\mathfrak{g}|_z \longrightarrow \mathfrak{g}^{(n)}|_{z^{(n)}}$ *is a monomorphism.*

Example 4.3. Consider the pseudo-group

$$X = f(x), \qquad\qquad U = \frac{u}{f'(x)}\,, \tag{4.10}$$

where $f(x) \in \mathcal{D}(\mathbb{R})$ is an arbitrary local diffeomorphism, acting on $M = \mathbb{R}^2 \setminus \{\,u = 0\,\}$. Its infinitesimal generators are the vector fields $\mathbf{v} = \xi\,\partial_x + \varphi\,\partial_u$ that are subject to the linear determining equations

$$\xi_x = -\,\frac{\varphi}{u}\,, \qquad\qquad \xi_u = 0, \qquad\qquad \varphi_u = \frac{\varphi}{u}\,,$$

along with all their differential consequences; see [**53**] for details. When solved, the determining equations yield $\xi = a(x)$, $\varphi = -\,a'(x)\,u$, where $a(x)$ is an arbitrary scalar function, resulting in the explicit formula

$$\mathbf{v} = a(x)\,\frac{\partial}{\partial x} - a'(x)\,u\,\frac{\partial}{\partial u} \tag{4.11}$$

for the infinitesimal generators of this pseudo-group.

The prolonged infinitesimal generators are obtained by substituting (4.11) into the prolongation formula (4.8):

$$\mathbf{v}^{(n)} = a\,\frac{\partial}{\partial x} - a_x u\,\frac{\partial}{\partial u} - (a_{xx}u + 2\,a_x u_x)\,\frac{\partial}{\partial u_x} - (a_{xxx}u + 3\,a_{xx}u_x + 3\,a_x u_{xx})\,\frac{\partial}{\partial u_{xx}} - \ \cdots\ . \tag{4.12}$$

Since $u \neq 0$, the only vector fields satisfying $\mathbf{v}^{(n)} = 0$ are those with trivial $n^{\rm th}$ order jet: $a = a_x = a_{xx} = \cdots = a_{n+1} = 0$. Proposition 4.2 implies that the pseudo-group acts locally freely on J^n for all $n \geq 0$.

5. Lifted Differential Forms.

The next order of business is to establish complete systems of invariant differential forms on the lifted diffeomorphism jet groupoid $\mathcal{E}^{(\infty)}$. Recall the induced splittings (2.22) of the differential:

$$d = d_J + d_G = d_H + d_V + d_G\,.$$

While the initial split into jet and group components is invariant under the action of the diffeomorphism jet groupoid $\mathcal{D}^{(\infty)}$ on $\mathcal{E}^{(\infty)}$, the finer split into horizontal and vertical components is only invariant under the sub-groupoid generated by the projectable (or fiber-preserving) diffeomorphisms $X = \chi(x)$, $U = \psi(x,u)$. As in [**30, 31**], we decompose the space of differential forms on $\mathcal{E}^{(\infty)}$ into

$$\mathbf{\Omega}^* = \bigoplus\nolimits_{k,l}\,\mathbf{\Omega}^{k,l} = \bigoplus\nolimits_{i,j,l}\,\mathbf{\Omega}^{i,j,l},$$

where l indicates the number of Maurer–Cartan forms μ_A^b, (2.35) (or, equivalently, group forms (2.20)), $k = i + j$ indicates the number of jet forms (2.19), i indicates the number of horizontal forms dx^i, and j indicates the number of contact forms θ_J^α. We let

$$\boldsymbol{\Omega}_J^* = \bigoplus_k \boldsymbol{\Omega}^{k,0} = \bigoplus_{i,j} \boldsymbol{\Omega}^{i,j,0}$$

denote the subspace of *jet forms*, i.e., those containing no Maurer–Cartan forms — although their coefficients are allowed to depend upon the pseudo-group parameters. Let $\pi_J \colon \boldsymbol{\Omega}^* \to \boldsymbol{\Omega}_J^*$ be the natural projection that takes a differential form $\widehat{\Omega}$ on $\mathcal{E}^{(\infty)}$ to its *jet component* $\pi_J(\widehat{\Omega})$. Formally, $\pi_J(\widehat{\Omega})$ is obtained by annihilating all Maurer–Cartan forms in $\widehat{\Omega}$, i.e., by setting all $\mu_A^b \mapsto 0$. If $\widehat{\Omega}$ is invariant under the right action of local diffeomorphisms $\varphi \in \mathcal{D}$ on $\mathcal{E}^{(\infty)}$, so is $\pi_J(\widehat{\Omega})$.

Given any differential form ω on J^∞, its pull-back $\widehat{\Omega} = (\widetilde{\boldsymbol{\tau}}^{(\infty)})^*\omega$ by the target map $\widetilde{\boldsymbol{\tau}}^{(\infty)} \colon \mathcal{E}^{(\infty)} \to \mathrm{J}^\infty$ is automatically invariant. The jet components of the pulled-back forms are also invariant, and play a crucial role in our constructions.

Definition 5.1. The *lift* of a differential form ω on J^∞ is the jet form

$$\Omega = \boldsymbol{\lambda}(\omega) = \pi_J\big[\,(\widetilde{\boldsymbol{\tau}}^{(\infty)})^*\omega\,\big]. \tag{5.1}$$

The lift map is an exterior algebra morphism:

$$\boldsymbol{\lambda}(\omega + \sigma) = \boldsymbol{\lambda}(\omega) + \boldsymbol{\lambda}(\sigma), \qquad \boldsymbol{\lambda}(\omega \wedge \sigma) = \boldsymbol{\lambda}(\omega) \wedge \boldsymbol{\lambda}(\sigma). \tag{5.2}$$

In local coordinates, $\boldsymbol{\lambda}$ maps the jet coordinates x^i, u_K^α, to their lifted counterparts[10] $X^i, \widehat{U}_K^\alpha$, the latter being prescribed by the implicit differentiation formulae (2.32). Similarly, their differentials dx^i, du_K^α lift to the jet differentials of their lifts. In other words, when computing $d_J X^i,\ d_J \widehat{U}_K^\alpha$, we only differentiate with respect to the submanifold jet coordinates x^i, u_J^β, and *not* with respect to the diffeomorphism jet coordinates X_A^i, U_A^α. The resulting one-forms

$$\begin{aligned}
\Omega^i &= \boldsymbol{\lambda}(dx^i) = d_J X^i = \sum_{j=1}^p \mathrm{D}_{x^j} X^i\, dx^j + \sum_{\alpha=1}^q X^i_{u^\alpha}\theta^\alpha, \qquad i = 1, \ldots, p,\\
\Theta^\alpha &= \boldsymbol{\lambda}(\theta^\alpha) = d_J U^\alpha - \sum_{i=1}^p \mathrm{D}_{X^i} U^\alpha\, d_J X^i = \sum_{\beta=1}^q \left(U^\alpha_{u^\beta} - \sum_{i=1}^p X^i_{u^\beta}\, \widehat{U}^\alpha_{X^i} \right) \theta^\beta, \qquad (5.3)\\
\Theta_K^\alpha &= \boldsymbol{\lambda}(\theta_K^\alpha) = \mathrm{D}_X^K \Theta^\alpha = d_J \widehat{U}_K^\alpha - \sum_{i=1}^p \widehat{U}^\alpha_{K,i}\, d_J X^i, \qquad \alpha = 1, \ldots, q, \qquad \#K \geq 0,
\end{aligned}$$

form a basis for the space of lifted jet forms.

---

[10] The jet projection π_J has no effect on functions.

Example 5.2. In the case of plane curves, the lift map takes the curve jet coordinates $x, u, u_x, \ \ldots$ to their lifted counterparts, as given by (2.6):

$$\boldsymbol{\lambda}(x) = X, \qquad \boldsymbol{\lambda}(u) = U, \qquad \boldsymbol{\lambda}(u_x) = \widehat{U}_X, \qquad \boldsymbol{\lambda}(u_{xx}) = \widehat{U}_{XX}, \quad \ldots . \tag{5.4}$$

The lifts of the basis horizontal and contact one-forms, (2.11), are given by[11]

$$\begin{aligned}
\Omega = \boldsymbol{\lambda}(dx) &= \, d_J X = X_x \, dx + X_u \, du = (X_x + u_x \, X_u) \, dx + X_u \, \theta = \mathrm{D}_x X \, dx + X_u \, \theta, \\
\Theta = \boldsymbol{\lambda}(\theta) &= \, d_J U - \widehat{U}_X \, d_J X = \frac{X_x \, U_u - X_u \, U_x}{\mathrm{D}_x X} \, \theta, \\
\Theta_X = \boldsymbol{\lambda}(\theta_x) &= \, d_J \widehat{U}_X - \widehat{U}_{XX} \, d_J X = \mathrm{D}_X \Theta = \frac{1}{\mathrm{D}_x X} \, \mathrm{D}_x \left(\frac{X_x \, U_u - X_u \, U_x}{\mathrm{D}_x X} \, \theta \right).
\end{aligned} \tag{5.5}$$

The higher order lifted contact forms

$$\Theta_n = \boldsymbol{\lambda}(\theta_n) = \mathrm{D}_X^n \Theta$$

are obtained by repeated Lie differentiation with respect to the implicit differentiation operator (2.5).

The formulas for the differentials of a lifted form are of critical importance. The jet differential is straightforward:

Proposition 5.3. *Let $\Omega = \boldsymbol{\lambda}(\omega)$ be a lifted form. Then $d_J \Omega = \boldsymbol{\lambda}(d\omega)$.*

To describe the formula for the group differential, we will extend the lift map to vector field jets. The required construction is most easily explained in local coordinates. With additional effort, it can be placed in a fully intrinsic framework by introducing suitable tensor product bundles. However, the constructions are a bit elaborate, and so, in the interests of brevity, will not be presented here.

We define the *lift* of a vector field jet coordinate (4.2) to be the corresponding Maurer–Cartan form (2.35); specifically,

$$\boldsymbol{\lambda}(\zeta^b_A) = \mu^b_A, \qquad \text{for} \qquad b = 1, \ldots, m, \qquad \#A \geq 0. \tag{5.6}$$

At first sight this definition might strike the reader as a bit odd; however, keep in mind that, at each point, ζ^b_A defines a linear function on the space of vector fields $\mathcal{X}(M)$, and so should be regarded as a kind of differential form. Thus, defining its lift to be another differential form should not be so surprising. Slightly more generally, suppose

$$P(z^{(n)}, \zeta^{(n)}) = \sum_{b=1}^{m} \sum_{0 \leq \#A \leq n} P^b_A(x, u^{(n)}) \, \zeta^b_A$$

is any finite linear combination of vector field jet coordinates whose coefficients are differential functions, i.e., a section of the bundle $(\mathrm{J}^n \widetilde{TM})^* \to \mathrm{J}^n$ which is obtained by pulling

[11] Within this example, Ω is used to denote the lift of dx, and not a generic differential form.

back the dual bundle $(\mathrm{J}^n TM)^* \to M$ via the projection $\widetilde{\pi}^n_0 \colon \mathrm{J}^n \to M$. We define its lift to be the group one-form

$$\boldsymbol{\lambda}\big[\, P(z^{(n)}, \zeta^{(n)}) \,\big] = P(\widehat{Z}^{(n)}, \mu^{(n)}) = \sum_{b=1}^{m} \sum_{0 \le \#A \le n} P^b_A(X, \widehat{U}^{(n)})\, \mu^b_A \tag{5.7}$$

on $\mathcal{E}^{(n)}$. Of particular importance are the lifts of the vector field prolongation coefficients (4.6), which are denoted by

$$\Xi^i = \boldsymbol{\lambda}(\xi^i) = \mu^i, \qquad \widehat{\Psi}^\alpha_J = \boldsymbol{\lambda}(\widehat{\varphi}^{\,\alpha}_J) = \boldsymbol{\lambda}\big[\, \Phi^\alpha_J(u^{(n)}, \zeta^{(n)}) \,\big] = \Phi^\alpha_J(\widehat{U}^{(n)}, \mu^{(n)}). \tag{5.8}$$

Thus, each $\widehat{\Psi}^\alpha_J$ is a particular linear combination of Maurer–Cartan forms whose coefficients are polynomials in the lifted coordinates $\widehat{U}^\alpha_K$ for $1 \le \#K \le \#J$. More generally, the lift of a differential form whose coefficients are linear combinations of vector field coefficient jets, i.e., a section of $\bigwedge^k T^*\mathrm{J}^n \otimes (\mathrm{J}^n \widetilde{TM})^* \to \mathrm{J}^n$, is defined as

$$\boldsymbol{\lambda}\left(\sum_{b=1}^{m} \sum_{\#A \le n} \zeta^b_A\, \omega^b_A \right) = \sum_{b=1}^{m} \sum_{\#A \le n} \mu^b_A \wedge \boldsymbol{\lambda}(\omega^b_A), \tag{5.9}$$

which is a differential form in $\boldsymbol{\Omega}^{k,1}$. With the above conventions, we can compactly write the group differential of a lifted form as follows.

Proposition 5.4. *Let $\Omega = \boldsymbol{\lambda}(\omega)$ be a lifted form. Then its group differential is the lift of its Lie derivative with respect to the prolonged vector field $\mathbf{v}^{(\infty)}$, so $d_G\, \Omega = \boldsymbol{\lambda}\big[\, \mathbf{v}^{(\infty)}(\omega) \,\big]$.*

Combining Propositions 5.3 and 5.4, we arrive at the fundamental formula for the differential of a lifted form.

Theorem 5.5. *Let $\Omega = \boldsymbol{\lambda}(\omega)$ be a lifted differential form on $\mathcal{E}^{(\infty)}$. Then*

$$d\Omega = d\boldsymbol{\lambda}(\omega) = \boldsymbol{\lambda}\big[\, d\omega + \mathbf{v}^{(\infty)}(\omega) \,\big]. \tag{5.10}$$

Before presenting the proof of Proposition 5.4, let us look closely at our running planar example.

Example 5.6. We continue analyzing the pseudo-group action in Example 5.2. According to (5.6), the lifts of the derivatives of the coefficients of a planar vector field, cf. (4.7), are the Maurer–Cartan forms (2.38–39):

$$\begin{gathered} \boldsymbol{\lambda}(\xi) = \mu, \qquad \boldsymbol{\lambda}(\varphi) = \nu, \qquad \boldsymbol{\lambda}(\xi_x) = \mu_X, \qquad \boldsymbol{\lambda}(\xi_u) = \mu_U, \\ \boldsymbol{\lambda}(\varphi_x) = \nu_X, \qquad \boldsymbol{\lambda}(\varphi_u) = \nu_U, \qquad \boldsymbol{\lambda}(\xi_{xx}) = \mu_{XX}, \qquad \ldots\,. \end{gathered}$$

Thus, by (5.8), the lifts of the prolonged vector field coefficients (4.8) are the following linear combinations of Maurer–Cartan forms:

$$\begin{aligned} \Psi &= \boldsymbol{\lambda}(\varphi) = \nu, \\ \widehat{\Psi}^X &= \boldsymbol{\lambda}(\widehat{\varphi}^{\,x}) = \nu_X + \widehat{U}_X(\nu_U - \mu_X) - \widehat{U}^2_X \mu_U, \\ \widehat{\Psi}^{XX} &= \boldsymbol{\lambda}(\widehat{\varphi}^{\,xx}) = \nu_{XX} + \widehat{U}_X(2\,\nu_{XU} - \mu_{XX}) + \widehat{U}^2_X(\nu_{UU} - 2\,\mu_{XU}) - \widehat{U}^3_X\, \mu_{UU} \\ &\qquad\qquad + \widehat{U}_{XX}(\nu_U - 2\,\mu_X) - 3\,\widehat{U}_X\, \widehat{U}_{XX}\, \mu_U, \end{aligned} \tag{5.11}$$

and so on.

With these in hand, and recalling (5.5), we apply our key formula (5.10) to determine the differentials of the implicit differentiation formulae (2.6):

$$
\begin{aligned}
dX &= d\boldsymbol{\lambda}(x) = \boldsymbol{\lambda}\big[\, dx + \mathbf{v}^{(\infty)}(x) \,\big] = \boldsymbol{\lambda}(dx + \xi) = \Omega + \mu, \\
dU &= d\boldsymbol{\lambda}(u) = \boldsymbol{\lambda}\big[\, du + \mathbf{v}^{(\infty)}(u) \,\big] = \boldsymbol{\lambda}(u_x\, dx + \theta + \varphi) = \widehat{U}_X\, \Omega + \Theta + \nu, \\
d\widehat{U}_X &= d\boldsymbol{\lambda}(u_x) = \boldsymbol{\lambda}\big[\, du_x + \mathbf{v}^{(\infty)}(u_x) \,\big] = \boldsymbol{\lambda}(u_{xx}\, dx + \theta_x + \widehat{\varphi}^{\,x}) = \widehat{U}_{XX}\, \Omega + \Theta_X + \widehat{\Psi}^X, \\
d\widehat{U}_{XX} &= d\boldsymbol{\lambda}(u_{xx}) = \boldsymbol{\lambda}\big[\, du_{xx} + \mathbf{v}^{(\infty)}(u_{xx}) \,\big] = \boldsymbol{\lambda}(u_{xxx}\, dx + \theta_{xx} + \widehat{\varphi}^{\,xx}) \\
&= \widehat{U}_{XXX}\, \Omega + \Theta_{XX} + \widehat{\Psi}^{XX},
\end{aligned}
$$

and so on. In each case, the group differential is the final term, while the preceding one or two terms are the jet differential; for instance,

$$
d_J\, \widehat{U}_X = \widehat{U}_{XX}\, \Omega + \Theta_X, \qquad \text{while} \qquad d_G\, \widehat{U}_X = \widehat{\Psi}^X.
$$

By the same reasoning, and recalling the prolongation formulae (4.8), the differentials of the basis lifted forms (5.5) are

$$
\begin{aligned}
d\Omega &= d\boldsymbol{\lambda}(dx) = \boldsymbol{\lambda}\big[\, d(dx) + \mathbf{v}^{(\infty)}(dx) \,\big] = \boldsymbol{\lambda}(d\xi) = \boldsymbol{\lambda}\big[\, (\xi_x + u_x\, \xi_u)\, dx + \xi_u\, \theta \,\big] \\
&= (\mu_X + \widehat{U}_X\, \mu_U) \wedge \Omega + \mu_U \wedge \Theta, \\
d\Theta &= d\boldsymbol{\lambda}(\theta) = \boldsymbol{\lambda}\big[\, d\theta + \mathbf{v}^{(\infty)}(\theta) \,\big] = \boldsymbol{\lambda}\big[-\theta_x \wedge dx + d\varphi - u_x\, d\xi - \widehat{\varphi}^{\,x}\, dx \,\big] \\
&= \boldsymbol{\lambda}\big[-\theta_x \wedge dx + (\varphi_u - u_x\, \xi_u)\theta \,\big] = -\,\Theta_X \wedge \Omega + \big(\, \nu_U - \widehat{U}_X\, \mu_U \,\big) \wedge \Theta, \\
d\Theta_X &= d\boldsymbol{\lambda}(\theta_x) = \boldsymbol{\lambda}\big[\, d\theta_x + \mathbf{v}^{(\infty)}(\theta_x) \,\big] = \boldsymbol{\lambda}\big[-\theta_{xx} \wedge dx + d\widehat{\varphi}^{\,x} - u_{xx}\, d\xi - \widehat{\varphi}^{\,xx}\, dx \,\big] \\
&= -\,\Theta_{XX} \wedge \Omega + \big[\, \nu_{XU} + \widehat{U}_X(\nu_{UU} - \mu_{XU}) - \widehat{U}_X^2\, \mu_{UU} - \widehat{U}_{XX}\, \mu_U \,\big] \wedge \Theta \\
&\qquad + \big[\, \nu_U - \mu_X - 2\, \widehat{U}_X\, \mu_U \,\big] \wedge \Theta_X.
\end{aligned}
$$

The higher-order formulae are similarly established. The direct verification of these formulae is a tedious, but instructive computation.

Proof of Proposition 5.4: As noted in [**53**], associated to each vector field

$$
\mathbf{v} \;=\; \sum_{a=1}^{m} \zeta^a(z)\, \frac{\partial}{\partial z^a} \;\in\; \mathcal{X}(M)
$$

with prolongation $\mathbf{v}^{(\infty)} \in \mathcal{X}(\mathrm{J}^\infty)$, there is a unique diffeomorphism invariant vector field $\widehat{\mathbf{V}}^{(\infty)} \in \mathcal{X}(\mathcal{E}^{(\infty)})$ on the groupoid which is tangent to the source fibers. We note that $\widehat{\mathbf{V}}^{(\infty)}$ and $\mathbf{v}^{(\infty)}$ are $\widetilde{\boldsymbol{\tau}}^{(\infty)}$–related vector fields, that is,

$$
d\widetilde{\boldsymbol{\tau}}^{(\infty)} \left(\widehat{\mathbf{V}}^{(\infty)}|_{\mathbf{g}^{(\infty)}} \right) = \mathbf{v}^{(\infty)}|_{\widetilde{\boldsymbol{\tau}}^{(\infty)}(\mathbf{g}^{(\infty)})}.
$$

Therefore, the Lie derivatives of $(\widetilde{\boldsymbol{\tau}}^{(\infty)})^*\omega$ and ω are similarly related:

$$
\widehat{\mathbf{V}}^{(\infty)}((\widetilde{\boldsymbol{\tau}}^{(\infty)})^*\omega) = (\widetilde{\boldsymbol{\tau}}^{(\infty)})^*\big[\, \mathbf{v}^{(\infty)}(\omega) \,\big].
$$

Noting that the Lie derivative with respect to $\widehat{\mathbf{V}}^{(\infty)}$ preserves the spaces of jet forms and group forms, an application of the jet projection π_J to both sides of this identity and using (5.1) results in the equation

$$\widehat{\mathbf{V}}^{(\infty)}(\Omega) = \boldsymbol{\lambda}\big[\,\mathbf{v}^{(\infty)}(\omega)\,\big] \tag{5.12}$$

for the lift $\Omega = \boldsymbol{\lambda}(\omega)$ of ω.

Next we expand the left hand side of (5.12) using Cartan's formula relating Lie derivatives and interior products, cf. [**47**; (1.65)]:

$$\widehat{\mathbf{V}}^{(\infty)}(\Omega) = \widehat{\mathbf{V}}^{(\infty)} \,\lrcorner\, d\Omega + d(\widehat{\mathbf{V}}^{(\infty)} \,\lrcorner\, \Omega). \tag{5.13}$$

Since $\widehat{\mathbf{V}}^{(\infty)}$ is tangent to the source fibers, $\widehat{\mathbf{V}}^{(\infty)} \,\lrcorner\, \Omega = 0$ for any jet form $\Omega \in \mathbf{\Omega}_J^*$. Thus, decomposing $d\Omega = d_J\,\Omega + d_G\,\Omega$, the only nonzero term on the right hand side of (5.13) is $\widehat{\mathbf{V}}^{(\infty)} \,\lrcorner\, d_G\,\Omega$. Hence, substituting back into (5.12), we deduce the identity

$$\widehat{\mathbf{V}}^{(\infty)} \,\lrcorner\, d_G\,\Omega = \boldsymbol{\lambda}\big[\,\mathbf{v}^{(\infty)}(\omega)\,\big]. \tag{5.14}$$

In local coordinates, since Ω is a jet form, its group differential is a finite sum

$$d_G\,\Omega = \sum_{b=1}^{m} \sum_{\#A\le n} \mu_A^b \wedge \Omega_A^b, \tag{5.15}$$

involving wedge products of the Maurer–Cartan forms μ_A^b, with certain jet forms $\Omega_A^b \in \mathbf{\Omega}_J^*$. Thus, the left hand side of (5.14) is

$$\widehat{\mathbf{V}}^{(\infty)} \,\lrcorner\, d_G\,\Omega = \sum_{b=1}^{m} \sum_{\#A\le n} \frac{\partial^{\#A}\zeta^b}{\partial z^A}(Z)\,\Omega_A^b, \tag{5.16}$$

where we use the fact that the left hand side is invariant on a source fiber, and hence can simply be evaluated at the identity jet, which is easily done in local coordinates. On the other hand, the right hand side of (5.14) is

$$\boldsymbol{\lambda}\big[\,\mathbf{v}^{(\infty)}(\omega)\,\big] = \boldsymbol{\lambda}\left(\sum_{b=1}^{m} \sum_{\#A\le n} \frac{\partial^{\#A}\zeta^b}{\partial z^A}(z)\,\omega_A^b \right) = \sum_{b=1}^{m} \sum_{\#A\le n} \frac{\partial^{\#A}\zeta^b}{\partial z^A}(Z)\,\boldsymbol{\lambda}(\omega_A^b), \tag{5.17}$$

Since (5.16) and (5.17) must be equal for any vector field $\mathbf{v} \in \mathcal{X}(M)$, we deduce that $\boldsymbol{\lambda}(\omega_A^b) = \Omega_A^b$. Substituting this relation back into (5.15) and recalling (5.9) completes the proof of Proposition 5.4. *Q.E.D.*

Next, given a pseudo-group $\mathcal{G}$, we restrict the invariant differential forms to the associated subgroupoid $\mathcal{H}^{(\infty)} \subset \mathcal{E}^{(\infty)}$. Clearly, the restricted[12] Maurer–Cartan forms μ_A^b will no longer be linearly independent. The remarkable fact, proved in [**53**; Theorem 6.1], is

[12] For simplicity, we do not explicitly indicate the pull-back map when restricting the forms to $\mathcal{H}^{(\infty)}$.

that the linear constraints among the restricted Maurer–Cartan forms are precisely given by the lifts, cf. (5.7), of the linearized determining equations (4.9):

$$L^{(n)}(Z,\mu^{(n)}) = \boldsymbol{\lambda}\big[\, L^{(n)}(z,\zeta^{(n)})\,\big] = 0, \qquad n \geq 0. \tag{5.18}$$

Subject to these constraints, the preceding constructs can be effectively used in determining recurrence relations for pseudogroup actions, as we illustrate in the subsequent examples.

6. Invariant Differential Forms.

We now use the moving frame to construct the invariant differential forms corresponding to the prolonged action of the pseudo-group on the submanifold jet bundles J^n. The general invariantization procedure introduced in [**30, 31**] in the finite-dimensional case adapts straightforwardly — provided the prolonged pseudo-group actions are eventually free and hence admits a complete moving frame on (an open subset of) J^∞. Invariantization of a differential function or form on J^∞ is implemented by first lifting it to the bundle $\mathcal{E}^{(\infty)}$ as in the preceding section, and then pulling back the lifted function or form with the moving frame map.

Definition 6.1. Let[13] $\rho^{(\infty)}\colon \mathrm{J}^\infty \to \mathcal{H}^{(\infty)}$ be a complete moving frame. If Ω is any differential form on J^∞, then its *invariantization* is the invariant differential form

$$\iota(\Omega) = (\rho^{(\infty)})^*\,\big[\,\boldsymbol{\lambda}(\Omega)\,\big]. \tag{6.1}$$

Lemma 6.2. *The invariantization of an arbitrary differential form is an invariant differential form. Moreover, if Ω is already invariant, then $\iota(\Omega) = \Omega$ on their common domains of definition.*

Thus, in view of (5.2), invariantization defines an exterior algebra morphism,

$$\iota(\Omega+\Theta) = \iota(\Omega)+\iota(\Theta), \qquad \iota(\Omega\wedge\Theta) = \iota(\Omega)\wedge\iota(\Theta), \tag{6.2}$$

that projects the spaces of ordinary functions and forms to the spaces of invariant functions and forms. The proof of this result follows the finite-dimensional version in [**31**]. Indeed, the invariantization of a differential function/form is the unique invariant differential function/form that has the same values when restricted to the cross-section defining the moving frame.

In particular, invariantizing the coordinate functions on J^∞ leads to the *normalized differential invariants*

$$H^i = \iota(x^i), \qquad i=1,\ldots,p, \qquad I^\alpha_J = \iota(u^\alpha_J), \qquad \alpha=1,\ldots,q, \qquad \#J \geq 0, \tag{6.3}$$

which are the individual components of $I^{(\infty)}$ described earlier in (3.9). Secondly, invariantization of the basis horizontal one-forms leads to the invariant horizontal one-forms

$$\varpi^i = \iota(dx^i) = \omega^i + \kappa^i, \qquad i=1,\ldots,p, \tag{6.4}$$

---

[13] As usual, functions and forms may only be defined on an open subset of their domain space.

where ω^i, κ^i, are, respectively, the horizontal and vertical (contact) components. If the pseudo-group acts projectably, then the contact components vanish: $\kappa^i = 0$. Otherwise, the two components are not individually invariant, although the horizontal forms $\omega^1, \ldots, \omega^p$ are, in the language of [**48**], a contact-invariant coframe on J^∞.

The dual invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ are uniquely defined by the formula

$$dF = \sum_{i=1}^{p} (\mathcal{D}_i F)\, \varpi^i + \ \cdots\ , \tag{6.5}$$

valid for any differential function F, where we omit the invariant contact components (although these do play an important role in the study of invariant variational problems, cf. [**30, 31**]). The invariant differential operators do not, in general, commute, but are subject to linear commutation relations of the form

$$[\, \mathcal{D}_i, \mathcal{D}_j \,] = \sum_{k=1}^{p} Y_{i,j}^k\, \mathcal{D}_k, \qquad i,j = 1, \ldots, p, \tag{6.6}$$

where the coefficients $Y_{i,j}^k$ are certain differential invariants that must also be determined. Finally, invariantizing the basis contact one-forms

$$\vartheta_K^\alpha = \iota(\theta_K^\alpha), \qquad \alpha = 1, \ldots, q, \qquad \#K \geq 0, \tag{6.7}$$

provide a complete system of invariant contact one-forms. They are contact forms because both the lift map and the moving frame pull-back preserve the relevant contact ideals.

Theorem 6.3. *The invariant horizontal and contact one-forms* (6.4), (6.7) *form an invariant coframe on a dense open subset of the domain of definition of the moving frame.*

From now on, we restrict the domain of definition of our complete moving frame $\rho^{(\infty)}$, which we continue to denote by $\mathcal{V}^\infty \subset \mathrm{J}^\infty$, to the subset where the one-forms (6.4), (6.7) form an invariant coframe. The exceptional points correspond to jets $z^{(\infty)} = \mathrm{j}_\infty S$ of submanifolds that become tangent to the vertical fibers under the action of the groupoid element $\rho^{(\infty)}(z^{(\infty)}) \in \mathcal{H}^{(\infty)}$ in the chosen coordinate system; see (2.29). In particular, if the pseudo-group acts projectably, the one-forms (6.4), (6.7) prescribe an invariant coframe on the entire domain of definition of $\rho^{(\infty)}$.

On $\mathcal{V}^\infty$, the invariant horizontal and contact forms induce an invariant splitting of $T^*\mathrm{J}^\infty$. The contact component remains as in the standard, non-invariant splitting, while the invariant horizontal component agrees with the usual horizontal component if and only if the pseudo-group acts projectably. As a result, the invariant coframe serves to define the invariant variational quasi-tricomplex for the pseudo-group. See [**30, 31**] for further developments in the finite-dimensional case, all of which carry over to infinite-dimensional pseudo-group actions. Analysis of the resulting pseudo-group-invariant characteristic cohomology, cf. [**4, 27**], is left to a future project.

Example 6.4. Let us use the moving frame to derive the invariant differential forms for the pseudo-group of Examples 3.8, 5.2 and 5.6. First, invariantization of the horizontal forms leads to the invariant horizontal coframe elements (6.4), namely

$$\varpi^1 = \omega^1 = \iota(dx) = u\,dx, \qquad\qquad \varpi^2 = \omega^2 = \iota(dy) = dy.$$

Since $\mathcal{G}$ acts projectably, there are no "contact corrections", so $\kappa^1 = \kappa^2 = 0$. The dual invariant differential operators are, as before, $\mathcal{D}_1 = (1/u)\mathrm{D}_x,\ \mathcal{D}_2 = \mathrm{D}_y$.

To obtain the order zero invariant contact form, we apply the invariantization map to $\theta = du - u_x\,dx - u_y\,dy$. First, in view of the prolonged pseudo-group formulae (3.13) coupled with (5.3), the lifted contact form is

$$\Theta = \boldsymbol{\lambda}(\theta) = \pi_J(dU - U_X\,dX - U_Y\,dY) = \bigl(\,U_u - U_X\,X_u - U_Y\,Y_u\,\bigr)\,\theta = \frac{\theta}{f_x}\,.$$

Second, we use the moving frame normalizations (3.15) to pull back Θ, and so the invariantized zero$^{\text{th}}$ order contact form is

$$\vartheta = \iota(\theta) = \frac{\theta}{u} = \frac{du - u_x\,dx - u_y\,dy}{u}\,. \tag{6.8}$$

Higher order invariant contact forms are obtained by similarly invariantizing the higher-order contact forms, e.g.,

$$\vartheta_1 = \iota(\theta_x) = \frac{\theta_x}{u^2} - \frac{u_x\,\theta}{u^3}\,, \qquad\qquad \vartheta_2 = \iota(\theta_y) = \frac{\theta_y}{u}\,.$$

Alternatively, we can generate higher-order invariant contact forms by invariant (Lie) differentiation; a direct computation shows that

$$\mathcal{D}_1\vartheta = \frac{\theta_x}{u^2} - \frac{u_x\,\theta}{u^3} = \vartheta_1, \qquad\qquad \mathcal{D}_2\vartheta = \frac{\theta_y}{u} - \frac{u_y\,\theta}{u^2} = \vartheta_2 - J\,\vartheta\,. \tag{6.9}$$

The recurrence relations, to be derived shortly, can be used to establish all of the formulae connecting the differentiated and invariantized forms.

7. Recurrence Formulae.

The recurrence formulae, cf. [**21**, **30**, **31**], relate the differentiated invariants and invariant forms to their normalized counterparts. These formulae are fundamental, since they completely determine the structure of the algebra of differential invariants, and thereby enable the systematic classification of generating differential invariants and their syzygies (differential identities). They also underly the intrinsic calculus of invariant variational problems and, indeed, the local structure of the entire invariant variational bicomplex. As in the finite-dimensional setting, the recurrence formulae are established using purely infinitesimal information, requiring only linear algebra and differentiation. In particular, they do *not* require the explicit formulae for either the Maurer–Cartan forms, or the normalized differential invariants and invariant forms, or the invariant differential operators, or even the moving frame itself! Beyond the standard formulae for the prolonged infinitesimal generators, the only information required is the specification of the moving frame cross-section.

Under the moving frame map, the pulled-back Maurer–Cartan forms will be denoted

$$\nu^{(\infty)} = (\rho^{(\infty)})^* \, \mu^{(\infty)}$$

with individual components

$$\nu_A^b = (\rho^{(\infty)})^* \, (\mu_A^b), \qquad b = 1, \ldots, m, \qquad \#A \geq 0. \tag{7.1}$$

As such, they are invariant one-forms, and so are certain invariant linear combinations of our invariant coframe elements (6.4), (6.7). Fortunately, the precise formulas need not be established a priori, as they will be a direct consequence of the recurrence formulas for the phantom differential invariants. In accordance with our interpretation of the invariantization process as the composition of the lift map followed by the moving frame pull-back, we identify the pulled-back Maurer–Cartan forms as the *invariantizations of the vector field coefficient jet coordinates* (4.2):

$$\iota(\zeta_A^b) = \nu_A^b, \qquad b = 1, \ldots, m, \qquad \#A \geq 0. \tag{7.2}$$

As with the lift map (5.9), we extend the invariantization process to differential functions or forms whose coefficients are linear combinations of vector field coefficient jets in the evident manner:

$$\iota\left(\sum_{b=1}^{m} \sum_{\#A \leq n} \zeta_A^b \, \omega_A^b \right) = \sum_{b=1}^{m} \sum_{\#A \leq n} \nu_A^b \wedge \iota(\omega_A^b). \tag{7.3}$$

If ω_A^b are k–forms on J^∞, then the result is an invariant differential $(k+1)$-form on J^∞.

Applying $(\rho^{(\infty)})^*$ to (5.18), we find that the pulled-back Maurer–Cartan forms ν_A^b are subject to the linear relations

$$L^{(n)}(I^{(0)}, \nu^{(n)}) = \iota\big[\, L^{(n)}(z, \zeta^{(n)}) \,\big] = 0, \qquad n \geq 0, \tag{7.4}$$

obtained by invariantizing the original linear determining equations (4.9). Here,

$$I^{(0)} = \iota(z) = \iota(x, u) = (H, I)$$

are the differential invariants in (6.3) obtained by invariantizing the coordinates on M. Further, the invariantizations of the prolonged infinitesimal generator coefficients (4.6),

$$\eta^i = (\rho^{(\infty)})^* \, \Xi^{\,i} = \iota(\xi^i) = \nu^i, \qquad \widehat{\psi}{}^{\,\alpha}_J = (\rho^{(\infty)})^* \, \widehat{\Psi}{}^{\alpha}_J = \iota(\widehat{\varphi}{}^{\,\alpha}_J) = \Phi^\alpha_J(I^{(n)}, \nu^{(n)}), \tag{7.5}$$

are certain linear combinations of the pulled-back Maurer–Cartan forms (7.2), whose coefficients are polynomials in the normalized differential invariants I_K^β for $1 \leq \#K \leq \#J$.

With all these in hand, the desired *universal recurrence formula* is immediately obtained by applying $(\rho^{(\infty)})^*$ to (5.10), using (6.1) and the fact that the exterior derivative commutes with any pull-back map.

Theorem 7.1. *If ω is any differential form on J^∞, then*

$$d\,\iota(\omega) = \iota\big[\, d\omega + \mathbf{v}^{(\infty)}(\omega) \,\big]. \tag{7.6}$$

We now specialize the universal formula (7.6) to establish the complete system of recurrence formulae for the normalized differential invariants and invariant one-forms. As first noted in [**21**; (13.7)] and [**31**; (5.21)], each recurrence formula equates an invariant exterior derivative of an invariantized function or form to the invariantization of its derivative plus a certain *correction term*, arising from $\iota\big[\,\mathbf{v}^{(\infty)}(\omega)\,\big]$, which is an invariant linear combination of the pulled-back Maurer–Cartan forms $\nu^{(\infty)}$. The latter are uniquely prescribed by the recurrence formulae for the phantom differential invariants. The resulting correction terms can be interpreted as a kind of "moving frame connection". We defer any development of a geometry of moving frame connections to a future research project.

First, taking ω in (7.6) to be one of the coordinate functions x^i, u^α_J yields recurrence formulae for the normalized differential invariants (6.3),

$$
\begin{aligned}
dH^i &= \iota\big(\,dx^i + \xi^i\,\big) = \varpi^i + \eta^i,\\
dI^\alpha_J &= \iota\big(\,du^\alpha_J + \widehat{\varphi}\,^\alpha_J\,\big) = \iota\left(\,\sum_{i=1}^p\, u^\alpha_{J,i}\;dx^i + \theta^\alpha_J + \widehat{\varphi}\,^\alpha_J\right) = \sum_{i=1}^p\, I^\alpha_{J,i}\,\varpi^i + \vartheta^\alpha_J + \widehat{\psi}\,^\alpha_J,
\end{aligned}
\tag{7.7}
$$

where the correction terms are the invariantized prolonged vector field coefficients (7.5), each of which is a certain invariant linear combination of pulled-back Maurer–Cartan forms ν^b_A, which are subject to the linear constraints (7.4). Each phantom differential invariant is, by definition, normalized to a constant value, and hence has zero differential. Therefore, the phantom recurrence formulae in (7.7) form a system of linear equations for the pulled-back Maurer–Cartan forms. If the pseudo-group acts locally freely on J^n, then, as we shall prove in [**54**], these equations can be uniquely solved for the Maurer–Cartan forms of order $\leq n$ as invariant linear combinations of the invariant horizontal and contact one-forms $\varpi^i, \vartheta^\alpha_J$. Substituting the resulting formulae into the remaining, non-phantom recurrence formulae in (7.7) leads to a complete system of recurrence relations, for both the vertical and horizontal differentials of all the normalized differential invariants.

Next, if we let ω in (7.6) be a one-form in the coordinate coframe dx^i, θ^α_J, and use the previously derived expressions for the pulled-back Maurer–Cartan forms, we are led to the corresponding recurrence formulae for differentials of the invariant coframe $\varpi^i, \vartheta^\alpha_J$. In particular, the formulae for the differentials of the invariant horizontal forms,

$$
d\varpi^k = -\sum_{i<j} Y^k_{i,j}\,\varpi^i\wedge\varpi^j + \;\cdots\;, \tag{7.8}
$$

where we only display the terms that do not involve invariant contact forms, prescribe the differential invariant coefficients $Y^k_{i,j}$ in the commutation relations (6.6) among the invariant differential operators. (As in [**21**], this follows from writing out the non-contact components in $d^2F = 0$, using formula (6.5), for a differential function F.) The full justification of these claims and more substantial illustrative examples will appear in the forthcoming papers [**14**, **54**].

Let us see how this all works in our running pseudo-group example.

Example 7.2. Consider the pseudo-group (3.10). For the particular moving frame constructed in Example 3.8, the normalized differential invariants are obtained by invariantizing the jet coordinates:

$$\iota(x) = H = 0, \qquad \iota(y) = y, \qquad \iota(u) = I_{00} = 1, \qquad \iota(u_x) = I_{10} = 0, \qquad \iota(u_y) = I_{01} = J,$$
$$\iota(u_{xx}) = I_{20} = 0, \qquad \iota(u_{xy}) = I_{11} = J_1, \qquad \iota(u_{yy}) = I_{02} = J_2,$$
$$\iota(u_{xxx}) = I_{30} = 0, \qquad \iota(u_{xxy}) = I_{21} = J_3, \qquad \iota(u_{xyy}) = I_{12} = J_4, \qquad \iota(u_{yyy}) = I_{03} = J_5,$$

and so on, where J, J_1, J_2 are the differential invariants (3.16), while the formulae for J_4, J_5, J_6 remain to be determined. According to Example 6.4, the invariant coframe on J^∞ consists of the invariantized horizontal forms

$$\varpi^1 = \iota(dx) = u\,dx, \qquad\qquad \varpi^2 = \iota(dy) = dy,$$

along with the invariantized contact forms

$$\vartheta = \iota(\theta) = \frac{\theta}{u}, \quad \vartheta_1 = \iota(\theta_x), \quad \vartheta_2 = \iota(\theta_y), \quad \vartheta_3 = \iota(\theta_{xx}), \quad \vartheta_4 = \iota(\theta_{xy}), \quad \vartheta_5 = \iota(\theta_{yy}), \quad \ldots\,.$$

The dual invariant differential operators are

$$\mathcal{D}_1 = \frac{1}{u}\,\mathrm{D}_x, \qquad\qquad \mathcal{D}_2 = \mathrm{D}_y.$$

The recurrence formulae for the invariantly differentiated invariant functions and forms all follow from our fundamental identity (7.6). The first task is to compute the coefficients

$$\mathbf{v}^{(\infty)} = \xi\,\frac{\partial}{\partial x} + \eta\,\frac{\partial}{\partial y} + \varphi\,\frac{\partial}{\partial u} + \widehat{\varphi}^{\,x}\,\frac{\partial}{\partial u_x} + \widehat{\varphi}^{\,y}\,\frac{\partial}{\partial u_y} + \widehat{\varphi}^{\,xx}\,\frac{\partial}{\partial u_{xx}} + \;\cdots$$

of the prolonged infinitesimal generator $\mathbf{v} = a(x)\partial_x - a'(x)\,u\,\partial_u$. Invoking the standard prolongation formula (4.5), we find

$$\begin{aligned}
\xi &= a, \\
\eta &= 0, \\
\varphi &= -\,u\,a_x, \\
\widehat{\varphi}^{\,x} &= D_x(-\,u\,a_x - u_x a) + u_{xx}a = -\,u\,a_{xx} - 2\,u_x a_x, \\
\widehat{\varphi}^{\,y} &= D_y(-\,u\,a_x - u_x a) + u_{xy}a = -\,u_y a_x, \\
\widehat{\varphi}^{\,xx} &= D_x^2(-\,u\,a_x - u_x a) + u_{xxx}a = -\,u\,a_{xxx} - 3\,u_x\,a_{xx} - 3\,u_{xx}a_x, \\
\widehat{\varphi}^{\,xy} &= D_x D_y(-\,u\,a_x - u_x a) + u_{xxy}a = -\,u_y\,a_{xx} - 2\,u_{xy}a_x, \\
\widehat{\varphi}^{\,yy} &= D_y^2(-\,u\,a_x - u_x a) + u_{xyy}a = -\,u_{yy}a_x,
\end{aligned} \tag{7.9}$$

and so on. According to (7.5), their invariantizations are linear combinations of pulled-back Maurer–Cartan forms, which are subject to the invariantized determining equations. Thus, a basis is provided by the one-forms

$$\alpha_k = \iota(a_k) = \iota(\mathrm{D}_x^k a) \tag{7.10}$$

obtained by invariantizing the jet coordinates (derivatives) of the function $a(x)$. We do not need to compute the one-forms α_k directly, since the required formulas will shortly follow from the recurrence formulae for the phantom differential invariants.

We now apply (7.7) to obtain the differentials of the phantom invariants

$$
\begin{aligned}
0 = dH &= \iota(dx+\xi) = \iota(dx+a) = \varpi^1 + \alpha, \\
0 = dI_{00} &= \iota(du+\varphi) = \iota(u_x\,dx + u_y\,dy + \theta - u\,a_x) \\
&= I_{10}\,\varpi^1 + I_{01}\,\varpi^2 + \vartheta - I_{00}\,\alpha_1 = J\,\varpi^2 + \vartheta - \alpha_1, \\
0 = dI_{10} &= \iota(du_x + \widehat{\varphi}^{\,x}) = \iota(u_{xx}\,dx + u_{xy}\,dy + \theta_x - u\,a_{xx} - 2\,u_x a_x) \\
&= I_{20}\,\varpi^1 + I_{11}\,\varpi^2 + \vartheta_{10} - I_{00}\,\alpha_2 - 2\,I_{10}\,\alpha_1 = J_1\,\varpi^2 + \vartheta_1 - \alpha_2, \\
0 = dI_{20} &= \iota(du_{xx} + \widehat{\varphi}^{\,xx}) = \iota(u_{xxx}\,dx + u_{xxy}\,dy + \theta_{xx} - u\,a_{xxx} - 3\,u_x\,a_{xx} - 3\,u_{xx}a_x) \\
&= I_{30}\,\varpi^1 + I_{21}\,\varpi^2 + \vartheta_{20} - I_{00}\,\alpha_3 - 3\,I_{10}\,\alpha_2 - 3\,I_{20}\,\alpha_1 = J_3\,\varpi^2 + \vartheta_3 - \alpha_3,
\end{aligned}
$$

etc. Solving the resulting linear system produces the formulae for the pulled-back Maurer–Cartan forms:

$$
\alpha = -\,\varpi^1, \qquad \alpha_1 = J\,\varpi^2 + \vartheta, \qquad \alpha_2 = J_1\,\varpi^2 + \vartheta_1, \qquad \alpha_3 = J_3\,\varpi^2 + \vartheta_3, \qquad \ldots\,.
$$

Observe that, to deduce these formulae for the pulled-back Maurer–Cartan forms, we did not require any of our explicit formulas for either the moving frame map or the original Maurer–Cartan forms.

Substituting these expressions into the differentials of the non-constant differential invariants, we deduce

$$
\begin{aligned}
dy &= \iota(dy+\eta) = \varpi^2, \\
dJ = dI_{01} &= \iota(du_y + \widehat{\varphi}^{\,y}) = \iota(u_{xy}\,dx + u_{yy}\,dy + \theta_y - u_y a_x) \\
&= I_{11}\,\varpi^1 + I_{02}\,\varpi^2 + \vartheta_{01} - I_{01}\,\alpha_1 \\
&= J_1\,\varpi^1 + (J_2 - J^2)\,\varpi^2 + \vartheta_2 - J\,\vartheta, \\
dJ_1 = dI_{11} &= \iota(du_{xy} + \widehat{\varphi}^{\,xy}) = \iota(u_{xxy}\,dx + u_{xyy}\,dy + \theta_{xy} - u_y\,a_{xx} - 2\,u_{xy}a_x) \\
&= I_{21}\,\varpi^1 + I_{12}\,\varpi^2 + \vartheta_{11} - I_{01}\,\alpha_2 - 2\,I_{11}\,\alpha_1 \\
&= J_3\,\varpi^1 + (J_4 - 3\,J\,J_1)\,\varpi^2 + \vartheta_4 - J\,\vartheta_1 - 2\,J_1\,\vartheta, \\
dJ_2 = dI_{02} &= \iota(du_{yy} + \widehat{\varphi}^{\,yy}) = \iota(u_{xyy}\,dx + u_{yyy}\,dy + \theta_{yy} - u_{yy}a_x) \\
&= I_{12}\,\varpi^1 + I_{03}\,\varpi^2 + \vartheta_{02} - I_{02}\,\alpha_1 \\
&= J_4\,\varpi^1 + (J_5 - J\,J_2)\,\varpi^2 + \vartheta_5 - J_2\,\vartheta.
\end{aligned}
$$

Breaking these formulae up into horizontal and vertical[14] components yields the explicit

[14] Since the pseudo-group acts projectably, the invariant horizontal forms contain no contact components, and hence the invariant vertical differential coincides with the usual vertical differential. Non-projectable actions are slightly more complicated; see [**31**] for details.

recurrence formulae for the differential invariants,

$$\begin{aligned}
&\mathcal{D}_1 J = J_1, &\qquad &\mathcal{D}_2 J = J_2 - J^2, &\qquad &d_V\, J = \vartheta_2 - J\,\vartheta,\\
&\mathcal{D}_1 J_1 = J_3, &\qquad &\mathcal{D}_2 J_1 = J_4 - 3\,J\,J_1, &\qquad &d_V\, J_1 = \vartheta_4 - J\,\vartheta_1 - 2\,J_1\,\vartheta,\\
&\mathcal{D}_1 J_2 = J_4, &\qquad &\mathcal{D}_2 J_2 = J_5 - J\,J_2, &\qquad &d_V\, J_2 = \vartheta_5 - J_2\,\vartheta,
\end{aligned}$$

the first couple of which we earlier produced by direct calculation. Proceeding by induction (or, more directly, by (8.32) below), we easily verify that all higher-order differential invariants are obtained by successively applying the invariant total derivative operators to the fundamental invariant $J = I_{01}$:

$$\begin{gathered}
J_1 = \mathcal{D}_1 J, \qquad J_2 = \mathcal{D}_2 J + J^2, \qquad J_3 = \mathcal{D}_1^2 J,\\
J_4 = \mathcal{D}_1 \mathcal{D}_2 J + 2\,J\,\mathcal{D}_1 J = \mathcal{D}_2 \mathcal{D}_1 J + 3\,J\,\mathcal{D}_1 J, \qquad J_5 = \mathcal{D}_2^2 J + 3\,J\,\mathcal{D}_2 J + J^3, \qquad \ldots .
\end{gathered}$$

Similarly, we can determine the differentials of the basic invariant horizontal and contact forms. Taking ω to be dx or dy in (7.6), we find

$$\begin{aligned}
d\varpi^1 &= d\iota(dx) = \iota\big[\, d(dx) + \mathbf{v}^{(\infty)}(dx)\,\big] = \iota(da) = \iota(a_x\, dx)\\
&= \alpha_1 \wedge \varpi^1 = -J\varpi^1 \wedge \varpi^2 + \vartheta \wedge \varpi^1,\\
d\varpi^2 &= d\iota(dy) = \iota\big[\, d(dy) + \mathbf{v}^{(\infty)}(dy)\,\big] = 0.
\end{aligned}$$

In view of (7.8), we deduce the basic commutation formula

$$[\,\mathcal{D}_1, \mathcal{D}_2\,] = J\,\mathcal{D}_1 \tag{7.11}$$

for the invariant differential operators. Finally, taking $\omega = \theta$ to be the order 0 contact form, we deduce

$$\begin{aligned}
d\vartheta &= d\iota(\theta) = \iota\big[\, d\theta + \mathbf{v}^{(\infty)}(\theta)\,\big] = \iota\big[\; dx \wedge \theta_x + \, dy \wedge \theta_y - a_x\,\theta\,\big]\\
&= \varpi^1 \wedge \vartheta_1 + \varpi^2 \wedge \vartheta_2 - \alpha_1 \wedge \vartheta = \varpi^1 \wedge \vartheta_1 + \varpi^2 \wedge \big(\,\vartheta_2 - J\,\vartheta\,\big).
\end{aligned}$$

Therefore,

$$\mathcal{D}_1 \vartheta = \vartheta_1, \qquad \mathcal{D}_2 \vartheta = \vartheta_2 - J\,\vartheta, \tag{7.12}$$

which reproduces (6.9). The recurrence formulae for the higher order contact forms are similarly constructed.

Our second pseudo-group Example 3.9 can be handled by analogous manipulations. But we prefer to wait for the alternative, more powerful computational approach based on power series expansions, which will be presented next.

8. Power Series.

A practical disadvantage of the computational algorithms developed above is that they must be implemented order by order, and so may require an excessive amount of computing. In [**53**], we showed how formal power series expansions can be used to concisely formulate the structure equations for general pseudo-groups. In this section, we explain how power series can streamline the computation of moving frame normalizations and resulting recurrence formulae. Throughout, the symbols h, k, H and K will be used to denote formal parameters in power series. Keep in mind that these parameters are *not* affected by any pseudo-group transformations.

Given a local diffeomorphism of M mapping the source variables $z = (x, u)$ to the target variables $Z = (X, U)$, we introduce the formal power series

$$X^i[\![h, k]\!] = \sum_{\#I, \#J \geq 0} \frac{1}{I!\, J!}\, X^i_{IJ}\, h^I k^J, \qquad U^\alpha[\![h, k]\!] = \sum_{\#I, \#J \geq 0} \frac{1}{I!\, J!}\, U^\alpha_{IJ}\, h^I k^J, \qquad \text{in} \qquad \begin{aligned} h &= (h^1, \ldots, h^p), \\ k &= (k^1, \ldots, k^q), \end{aligned} \tag{8.1}$$

to represent its infinite jet or, equivalently, Taylor expansion at the source point. The groupoid structure of $\mathcal{D}^{(\infty)}$ is recovered by formal composition and inversion of power series, making sure that the target of the initial series matches the source of its successor. Similarly, the infinite jet of a submanifold S at a point $z = (x, u) \in S$ is represented by the power series

$$u^\alpha[\![h]\!] = \sum_{\#J \geq 0} \frac{1}{J!}\, u^\alpha_J\, h^J \qquad \text{in} \qquad h = (h^1, \ldots, h^p). \tag{8.2}$$

Given a diffeomorphism represented by (8.1), we let

$$\widehat{U}^\alpha[\![H]\!] = \sum_{\#J \geq 0} \frac{1}{J!}\, \widehat{U}^\alpha_J\, H^J, \qquad \text{where} \qquad H = (H^1, \ldots, H^p), \tag{8.3}$$

denote the corresponding Taylor expansion of the transformed submanifold[15] at the target point $Z = (X, U)$. The transformed power series (8.3) can be explicitly determined by eliminating h from the composite power series

$$\widehat{U}[\![H]\!] = U[\![h, u[\![h]\!] - u[\![0]\!]]\!], \qquad \text{when} \qquad H = X[\![h, u[\![h]\!] - u[\![0]\!]]\!] - X[\![0, 0]\!]. \tag{8.4}$$

In other words one inverts the second equation to rewrite the parameters $h = F[\![H]\!]$ as power series in H, and then substitutes these expressions into the first power series to produce (8.3). The individual coefficients of the resulting power series yield the implicit differentiation formulae (2.32).

Example 8.1. Consider the planar case, $M = \mathbb{R}^2$, with a single independent variable x and a single dependent variable u. The Taylor expansion for a plane curve $C \subset \mathbb{R}^2$ at a point $(x, u) \in C$ has the form

$$u[\![h]\!] = u + u_x\, h + \tfrac{1}{2}\, u_{xx}\, h^2 + \cdots .$$

[15] As before, we assume that the transformed submanifold continues to be represented locally as the graph of a function.

Let

$$X[\![h,k]\!] = X + X_x\, h + X_u\, k + \tfrac{1}{2}\, X_{xx}\, h^2 + X_{xu}\, h\, k + \tfrac{1}{2}\, X_{uu}\, k^2 + \cdots ,$$
$$U[\![h,k]\!] = U + U_x\, h + U_u\, k + \tfrac{1}{2}\, U_{xx}\, h^2 + U_{xu}\, h\, k + \tfrac{1}{2}\, U_{uu}\, k^2 + \cdots ,$$

be the Taylor expansion of a general local diffeomorphism of $\mathbb{R}^2$. According to (8.4), to obtain the Taylor series

$$\widehat{U}[\![H]\!] = U + \widehat{U}_X\, H + \tfrac{1}{2}\, \widehat{U}_{XX} H^2 + \cdots \tag{8.5}$$

for the transformed curve, we first invert the power series

$$H = X[\![h, u[\![h]\!] - u[\![0]\!]]\!] - X[\![0,0]\!] = \sum_{k=1}^{\infty} \frac{1}{k!}\, \mathrm{D}_x^k X\, h^k$$
$$= (X_x + u_x\, X_u)\, h + \tfrac{1}{2}\, (X_{xx} + 2\, X_{xu}\, u_x + X_{uu}\, u_x^2 + X_u u_{xx}) h^2 + \cdots ,$$

to produce the expansion

$$h = \frac{1}{X_x + u_x\, X_u}\, H - \frac{1}{2}\, \frac{X_{xx} + 2\, X_{xu}\, u_x + X_{uu}\, u_x^2 + X_u u_{xx}}{(X_x + u_x\, X_u)^3}\, H^2 + \cdots .$$

Substituting this series into

$$U[\![h, u[\![h]\!] - u[\![0]\!]]\!] \;=\; \sum_{k=0}^{\infty} \frac{1}{k!}\, \mathrm{D}_x^k U\, h^k$$
$$= U + (U_x + u_x\, U_u)\, h + \tfrac{1}{2}\, (U_{xx} + 2\, U_{xu}\, u_x + U_{uu}\, u_x^2 + U_u u_{xx}) h^2 + \cdots$$

leads to the power series

$$\widehat{U}[\![H]\!] = U + \widehat{U}_X\, H + \tfrac{1}{2}\, \widehat{U}_{XX}\, H^2 + \cdots ,$$

whose coefficients $U, \widehat{U}_X, \widehat{U}_{XX}, \ldots$ are precisely the implicit differentiation formulae (2.6).

Given a pseudo-group $\mathcal{G}$, we will identify the infinite jets of its transformations with their Taylor series at the source point. The induced action of $\mathcal{G}^{(\infty)}$ on the submanifold jet bundle J^∞ is obtained by restricting the general prolonged action (8.3) to the pseudo-group jets, as constrained by the determining equations. A complete coordinate cross-section $K^\infty \subset \mathrm{J}^\infty$ is specified by normalizing an appropriate subset of the Taylor coefficients in $\widehat{U}[\![H]\!]$ to suitably prescribed constants. Solving the normalization equations for the pseudo-group jet parameters yields a complete moving frame $\rho^{(\infty)}\colon \mathrm{J}^\infty \to \mathcal{H}^{(\infty)}$, now expressed in power series form. Moreover, substituting the induced moving frame formulae back into the series $\widehat{U}[\![H]\!]$ leads to a (vector-valued) power series

$$I[\![H]\!] = (\rho^{(\infty)})^* \left(\widehat{U}[\![H]\!] \right), \tag{8.6}$$

whose coefficients I_J^α are the normalized differential invariants (3.9).

Definition 8.2. The *invariantization* of a formal power series

$$F[\![\,h\,]\!] = \sum_{\#J\geq 0} F_J\, h^J,$$

whose coefficients F_J are differential functions, or, more generally, differential forms, is the formal power series

$$\iota\big(\,F[\![\,h\,]\!]\,\big) = \sum_{\#J\geq 0} \iota(F_J) H^J, \tag{8.7}$$

obtained by invariantizing the individual coefficients. For clarity, we will consistently distinguish the formal parameters h in the original series from the formal parameters H in its invariantization, i.e., formally, $\iota(h) = H$.

In particular, the invariantization of the dependent variable series (8.2) is the normalized differential invariant series (8.6),

$$I[\![\,H\,]\!] = \iota\big(\,u[\![\,h\,]\!]\,\big). \tag{8.8}$$

The power series moving frame method is best assimilated by working through an explicit example.

Example 8.3. The transformations of the pseudo-group (3.10) can be written in power series form

$$X = f[\![\,h\,]\!], \qquad\qquad Y = y + k, \qquad\qquad U = \frac{u[\![\,h,k\,]\!]}{f'[\![\,h\,]\!]}\,, \tag{8.9}$$

where

$$f[\![\,h\,]\!] = f + f_x\, h + \tfrac{1}{2}\, f_{xx}\, h^2 + \tfrac{1}{6}\, f_{xxx}\, h^3 + \ \cdots\ ,$$

while

$$f'[\![\,h\,]\!] = f_x[\![\,h\,]\!] = \frac{\partial f}{\partial h}\,[\![\,h\,]\!] = f_x + f_{xx}\, h + \tfrac{1}{2}\, f_{xxx}\, h^2 + \ \cdots$$

indicates the differentiated series. The prolonged pseudo-group action on the surface jet space $\mathrm{J}^\infty = \mathrm{J}^\infty(\mathbb{R}^3, 2)$ is found by inverting the power series

$$H = \widetilde{f}\,[\![\,h\,]\!] \equiv f[\![\,h\,]\!] - f[\![\,0\,]\!] = f_x\, h + \tfrac{1}{2}\, f_{xx}\, h^2 + \ \cdots\ , \qquad\qquad K = k. \tag{8.10}$$

Substituting the resulting expressions for $h = \widetilde{f}^{-1}[\![\,H\,]\!]$, $K = k$, into the series (8.9) for U leads to

$$\widehat{U}[\![\,H,K\,]\!] = \sum_{m,n\geq 0} \frac{1}{m!\,n!}\,\widehat{U}_{m,n}\, H^m\, K^n = \frac{u[\![\,\widetilde{f}^{-1}[\![\,H\,]\!],K\,]\!]}{f'[\![\,\widetilde{f}^{-1}[\![\,H\,]\!]\,]\!]}\,, \tag{8.11}$$

whose coefficients $\widehat{U}_{m,n} = \mathrm{D}_X^m \mathrm{D}_Y^n U$ are the prolonged pseudo-group transformations (3.13).

Let us employ this formulation to construct a power series expansion for the moving frame. The normalizations chosen in Example 3.8 are equivalent to setting

$$\widehat{U}[\![\,H,0\,]\!] = 1, \qquad \text{so that} \qquad \widehat{U}_{0,0} = 1, \qquad \widehat{U}_{m,0} = 0, \qquad m \geq 1, \tag{8.12}$$

or, expressed in another way, setting

$$\widehat{U}[\![H,K]\!] = 1 + K\, V[\![H,K]\!], \tag{8.13}$$

for some power series $V[\![H,K]\!]$. We solve the normalization equations for the derivative parameters $f_m = \partial_x^m f$, or, equivalently, the power series $f[\![h]\!]$. Using (8.10–11), the normalization equations (8.12) can be written in the series form

$$1 = \widehat{U}[\![H,0]\!] = \frac{u[\![h,0]\!]}{f'[\![h]\!]} \qquad \text{and hence} \qquad f'[\![h]\!] = u[\![h,0]\!]. \tag{8.14}$$

The result is equivalent to the individual normalizations $f_m = u_{m-1,0}$, $m \geq 1$, the first few of which were found, much more laboriously, in Example 3.8.

We substitute the moving frame formulae (8.14) into the lifted series (8.11), resulting in

$$\widehat{U}[\![H,K]\!] \quad \longmapsto \quad I[\![H,K]\!] = 1 + K\, J[\![H,K]\!], \tag{8.15}$$

where the coefficients of $J[\![H,K]\!]$ are the independent (non-phantom) normalized differential invariants. We use (8.10) and (8.14) to write

$$J[\![H,K]\!] = \frac{u[\![h,k]\!] - u[\![h,0]\!]}{k\, u[\![h,0]\!]}\,, \tag{8.16}$$

where the first parameter

$$H = \widetilde{f}\,[\![h]\!] = \int_0^h f'[\![\eta]\!]\, d\eta = \int_0^h u[\![\eta,0]\!]\, d\eta = u\, h + \tfrac{1}{2}\, u_x\, h^2 + \tfrac{1}{6}\, u_{xx}\, h^3 + \ \cdots$$

is obtained by term-by-term integration. Explicitly inverting the power series:

$$h = \widetilde{f}^{-1}[\![H]\!] = \frac{1}{u}\, H - \frac{u_x}{2\,u^3}\, H^2 - \frac{u u_{xx} - 3\,u_x^2}{6\,u^5}\, H^3 - \ \cdots\,, \qquad k = K. \tag{8.17}$$

On the other hand,

$$\begin{aligned}\frac{u[\![h,k]\!] - u[\![h,0]\!]}{k\, u[\![h,0]\!]} &= \frac{u_y}{u} + \frac{u u_{xy} - u_x u_y}{u^2}\, h + \frac{u_{yy}}{2\,u}\, k + \\ &+ \frac{u^2 u_{xxy} - u\, u_y u_{xx} - 2\,u\, u_x u_{xy} + 2\, u_x^2 u_y}{2\,u^3}\, h^2 + \frac{u\, u_{xyy} - u_x u_{yy}}{2\,u^2}\, h\, k + \frac{u_{yyy}}{6\,u}\, k^2 + \ \cdots\,.\end{aligned}$$

Substituting (8.17) into this series produces the formulae

$$\begin{aligned}J[\![H,K]\!] &= \frac{u_y}{u} + \frac{u\, u_{xy} - u_x u_y}{u^3}\, H + \frac{u_{yy}}{2\,u}\, K + \\ &+ \frac{u^2 u_{xxy} - u\, u_y u_{xx} - 3\,u\, u_x u_{xy} + 3\, u_x^2 u_y}{2\,u^5}\, H^2 + \frac{u\, u_{xyy} - u_x u_{yy}}{2\,u^3}\, HK + \frac{u_{yyy}}{6\,u}\, K^2 + \ \cdots\,.\end{aligned} \tag{8.18}$$

The individual coefficients of (8.18) are the fundamental normalized differential invariants for our pseudo-group.

We can also use power series to re-express the prolongation formula for vector fields. Given a vector field $\mathbf{v}$ as in (4.1), let

$$\xi^i[\![h,k]\!] = \sum_{\#I,\#L\,\geq\,0} \xi^i_{IL}\,\frac{h^I\,k^L}{I!\,L!}\,, \qquad\qquad \varphi^\alpha[\![h,k]\!] = \sum_{\#I,\#L\,\geq\,0} \varphi^\alpha_{IL}\,\frac{h^I\,k^L}{I!\,L!}\,, \tag{8.19}$$

be the Taylor expansions representing its infinite jet at a point $z=(x,u)$. The coefficients in the composite series

$$\varphi^\alpha[\![h, u[\![h]\!] - u[\![0]\!]]\!] = \sum_{\#J\,\geq\,0} \mathrm{D}^J_x\varphi^\alpha\,\frac{h^J}{J!}$$

are the total derivatives of φ^α with respect to x. Define the vector-valued power series $\widehat{\varphi}\,[\![h]\!]$, whose components

$$\widehat{\varphi}^{\,\alpha}[\![h]\!] = \sum_{\#J\,\geq\,0} \widehat{\varphi}^{\,\alpha}_J\,\frac{h^J}{J!}\,, \qquad\qquad \alpha=1,\ldots,q, \tag{8.20}$$

provide the prolonged vector field coefficients. The prolongation formula (4.5) can then be written in vector-valued series form

$$\widehat{\varphi}\,[\![h]\!] = \varphi[\![h, u[\![h]\!] - u[\![0]\!]]\!] - \nabla_h u[\![h]\!]\,\big(\xi[\![h, u[\![h]\!]]\!] - \xi[\![0,0]\!]\big), \tag{8.21}$$

where $\nabla_h u[\![h]\!]$ is the matrix-valued power series obtained by forming the $q\times p$ Jacobian matrix of $u[\![h]\!]$ with respect to the formal parameters h.

Example 8.4. According to (7.9), the prolonged infinitesimal generator of the pseudo-group (3.10) has the form

$$\begin{aligned}\mathbf{v}^{(\infty)} &= a\,\partial_x - u\,a_x\,\partial_u - (\,u\,a_{xx} + 2\,u_x\,a_x\,)\,\partial_{u_x} - u_y\,a_x\,\partial_{u_y} - \\ &- (\,u\,a_{xxx} + 3\,u_x\,a_{xx} + 3\,u_{xx}\,a_x\,)\,\partial_{u_{xx}} - \big(\,u_y\,a_{xx} + 2\,u_{xy}\,a_x\,\big)\,\partial_{u_{xy}} - u_{yy}\,a_x\,\partial_{u_{yy}} - \ \cdots\ .\end{aligned} \tag{8.22}$$

In this case, the prolonged infinitesimal generator series (8.21) has the explicit form

$$\begin{aligned}\widehat{\varphi}\,[\![h,k]\!] &= -\,u[\![h,k]\!]\,a_h[\![h]\!] - u_h[\![h,k]\!]\,\big(a[\![h]\!] - a[\![0]\!]\big) \\ &= -\,\frac{\partial}{\partial h}\,\big\{\,u[\![h,k]\!]\,\big(\,a[\![h]\!] - a[\![0]\!]\,\big)\,\big\},\end{aligned} \tag{8.23}$$

where

$$a[\![h]\!] = a + a_x h + \tfrac{1}{2}\,a_{xx}h^2 + \ \cdots \tag{8.24}$$

is the Taylor series representing the infinite jet of the function $a(x)$.

Finally, we employ power series to establish a complete system of recurrence formulae for the normalized differential invariants. Let $\widehat{\psi}\,[\![H]\!]$ be the vector-valued power series whose coefficients are the invariant forms (7.5). Its components

$$\widehat{\psi}^{\,\alpha}[\![H]\!] = \sum_{\#J\,\geq\,0} \widehat{\psi}^{\,\alpha}_J\,\frac{H^J}{J!} = \iota\big(\,\widehat{\varphi}^{\,\alpha}[\![h]\!]\big), \qquad\qquad \alpha=1,\ldots,q, \tag{8.25}$$

are obtained by invariantizing the prolonged vector field series (8.20) as described in Definition 8.2. Our key recurrence formula (7.6), when evaluated on the differential invariant series $I[\![H]\!] = \iota(u[\![h]\!])$, takes the form

$$
\begin{aligned}
dI[\![H]\!] &= \iota\big(du[\![h]\!] + \widehat{\varphi}\,[\![h]\!] \big) = \iota\big(\nabla_h u[\![h]\!]\, dx + \theta[\![h]\!] + \widehat{\varphi}\,[\![h]\!] \big) \\
&= \nabla_H I[\![H]\!]\, \varpi + \vartheta[\![H]\!] + \widehat{\psi}\; [\![H]\!].
\end{aligned}
\tag{8.26}
$$

Applying (8.21), we obtain the explicit formulae for

$$
\widehat{\psi}\; [\![H]\!] = \psi[\![H, I[\![H]\!] - I[\![0]\!]]\!] - \nabla_H I[\![H]\!]\, \big(\eta[\![H, I[\![H]\!]]\!] - \eta[\![0,0]\!] \big),
\tag{8.27}
$$

in which

$$
\eta[\![H,K]\!] = \iota\big(\xi[\![h,k]\!] \big), \qquad\qquad \psi[\![H,K]\!] = \iota\big(\varphi[\![h,k]\!] \big),
\tag{8.28}
$$

are power series whose coefficients are the pulled-back Maurer–Cartan forms $\nu^{(\infty)}$, (7.1), or, equivalently, the invariantizations of the expansions (8.19). The phantom coefficients in $I[\![H]\!]$ are used to uniquely prescribe the pulled-back Maurer–Cartan forms $\nu^{(\infty)}$, and thus the correction terms in the recurrence formulae.

Example 8.5. Let us return to the pseudo-group in Example 8.3. Let

$$
\alpha[\![H]\!] = \alpha + \alpha_1\, H + \tfrac{1}{2}\, \alpha_2\, H^2 + \ \cdots \ = \iota\big(a[\![h]\!] \big)
\tag{8.29}
$$

be the series whose coefficients are the pulled-backed Maurer–Cartan forms (7.10), which we identify as the invariantization of the series (8.24). Then, according to formula (8.26),

$$
\begin{aligned}
dI[\![H,K]\!] = \frac{\partial I}{\partial H}\,[\![H,K]\!]\, \varpi^1 &+ \frac{\partial I}{\partial K}\,[\![H,K]\!]\, \varpi^2 + \vartheta[\![H,K]\!] \\
&- \frac{\partial}{\partial H}\, \big\{\, I[\![H,K]\!]\, \big(\alpha[\![H]\!] - \alpha[\![0]\!] \big)\, \big\},
\end{aligned}
\tag{8.30}
$$

where we used (8.23) to compute

$$
\begin{aligned}
\widehat{\psi}\,[\![H,K]\!] = \iota\big(\widehat{\varphi}\,[\![h,k]\!] \big) &= \iota \left(-\, \frac{\partial}{\partial h}\, \big\{\, u[\![h,k]\!]\, \big(a[\![h]\!] - a[\![0]\!] \big)\, \big\} \right) \\
&= -\, \frac{\partial}{\partial H}\, \big\{\, I[\![H,K]\!]\, \big(\alpha[\![H]\!] - \alpha[\![0]\!] \big)\, \big\}.
\end{aligned}
$$

Since we are normalizing $\widehat{U}[\![H,0]\!] = 1$, we have

$$
I[\![H,0]\!] = 1 \qquad \text{and hence} \qquad I_H[\![H,0]\!] = 0, \qquad dI[\![H,0]\!] = 0.
$$

Therefore, when we substitute $K = 0$ in (8.30), we can solve for the pulled-back Maurer–Cartan forms

$$
\alpha_H[\![H]\!] = I_K[\![H,0]\!]\, \varpi^2 + \vartheta[\![H,0]\!] = \sum_{j=0}^{\infty} \frac{H^j}{j!}\big(I_{j,1}\, \varpi^2 + \vartheta_{j,0} \big),
$$

and so, upon integrating with respect to H,

$$\alpha[\![\,H\,]\!] - \alpha[\![\,0\,]\!] = \int_0^H \bigl(\, I_K[\![\,\widehat H,0\,]\!]\,\varpi^2 + \vartheta[\![\,\widehat H,0\,]\!]\,\bigr)\, d\widehat H = \sum_{j=1}^{\infty} \frac{H^j}{j!}\,\bigl(\, I_{j-1,1}\,\varpi^2 + \vartheta_{j-1,0}\,\bigr). \tag{8.31}$$

Substituting into (8.30), we find that the horizontal recurrence formulae are given in power series form by

$$d_H\, I[\![\,H,K\,]\!] = I_H[\![\,H,K\,]\!]\,\varpi^1 + \left[\, I_K[\![\,H,K\,]\!] - \frac{\partial}{\partial H}\left(I[\![\,H,K\,]\!]\int_0^H I_K[\![\,\widehat H,0\,]\!]\, d\widehat H \right)\right]\varpi^2,$$

or, in components,

$$\mathcal{D}_1 I_{jk} = I_{j+1,k}, \qquad\qquad \mathcal{D}_2 I_{jk} = I_{j,k+1} - \sum_{i=0}^{j} \binom{j+1}{i}\, I_{ik}\, I_{j-i,1}. \tag{8.32}$$

Consequently, the lowest order differential invariant $J = I_{01}$ serves to generate the entire differential invariant algebra through invariant differentiation. Since the I_{jk} are functionally independent, there are no syzygies among the differentiated invariants $\mathcal{D}_1^j\mathcal{D}_2^k J$. Furthermore, the vertical component of (8.30) yields

$$d_V\, I[\![\,H,K\,]\!] = \vartheta[\![\,H,K\,]\!] - \frac{\partial}{\partial H}\left(I[\![\,H,K\,]\!]\int_0^H \vartheta[\![\,\widehat H,0\,]\!]\, d\widehat H \right),$$

with individual coefficients

$$d_V\, I_{jk} = \vartheta_{jk} - \sum_{i=0}^{j} \binom{j+1}{i} I_{ik}\, \vartheta_{j-i,0}. \tag{8.33}$$

The initial cases reproduce our earlier results found in Example 7.2.

Finally, using (8.23) and the fact that the group acts projectably, the differentials of the invariant contact forms are provided by the power series

$$\begin{aligned} d\vartheta[\![\,H,K\,]\!] &= \iota\,(\, d\theta[\![\,h,k\,]\!] + d_V\,\widehat\varphi\,[\![\,h,k\,]\!]\,) \\ &= \iota\left(dx \wedge \frac{\partial\theta}{\partial h}\,[\![\,h,k\,]\!] + dy \wedge \frac{\partial\theta}{\partial k}\,[\![\,h,k\,]\!] - \frac{\partial}{\partial h}\,\bigl\{\,(\,a[\![\,h\,]\!] - a[\![\,0\,]\!]\,) \wedge \theta[\![\,h,k\,]\!]\,\bigr\}\right) \\ &= \varpi^1 \wedge \frac{\partial\vartheta}{\partial H}\,[\![\,H,K\,]\!] + \varpi^2 \wedge \frac{\partial\vartheta}{\partial K}\,[\![\,H,K\,]\!] - \frac{\partial}{\partial H}\,\bigl\{\,(\,\alpha[\![\,H\,]\!] - \alpha[\![\,0\,]\!]\,) \wedge \vartheta[\![\,H,K\,]\!]\,\bigr\}. \end{aligned}$$

Substituting the formula (8.31) for the normalized Maurer–Cartan forms, we find

$$\begin{aligned} d\vartheta[\![\,H,K\,]\!] = \varpi^1 \wedge \frac{\partial\vartheta}{\partial H}\,[\![\,H,K\,]\!] - \frac{\partial}{\partial H}\left[\left(\int_0^H \vartheta[\![\,\widehat H,0\,]\!]\, d\widehat H\right) \wedge \vartheta[\![\,H,K\,]\!]\right] \\ + \varpi^2 \wedge \left\{ \frac{\partial\vartheta}{\partial K}\,[\![\,H,K\,]\!] - \frac{\partial}{\partial H}\left[\left(\int_0^H \frac{\partial I}{\partial K}\,[\![\,\widehat H,0\,]\!]\, d\widehat H\right)\, \vartheta[\![\,H,K\,]\!]\right]\right\}, \end{aligned}$$

which give both the horizontal and vertical recurrence formulae for the invariantized contact forms. In particular, the horizontal components yield

$$\mathcal{D}_1 \vartheta_{jk} = \vartheta_{j+1,k}, \qquad\qquad \mathcal{D}_2 \vartheta_{jk} = \vartheta_{j,k+1} - \sum_{i=0}^{j} \binom{j+1}{i} I_{j-i,1}\, \vartheta_{jk},$$

of which the case $j = k = 0$ appears in (6.9).

Example 8.6. As our ultimate illustrative example, we apply the power series moving frame method to analyze the action of the pseudo-group (3.20) on surfaces $S \subset \mathbb{R}^3$. We first write the power series expansions

$$\begin{aligned} X &= f[\![\, h \,]\!], \\ Y &= f'[\![\, h \,]\!]\,(y + k) + g[\![\, h \,]\!] = f'[\![\, 0 \,]\!]\, y + g[\![\, 0 \,]\!] + f'[\![\, h \,]\!]\big(\, k - a[\![\, h \,]\!] \,\big), \\ U &= u[\![\, h, k \,]\!] + \frac{f''[\![\, h \,]\!]\, k + g'[\![\, h \,]\!]}{f'[\![\, h \,]\!]} = u[\![\, h, k \,]\!] + \frac{f''[\![\, h \,]\!]}{f'[\![\, h \,]\!]}\big(\, k - a[\![\, h \,]\!] \,\big) - a'[\![\, h \,]\!]\,, \end{aligned} \tag{8.34}$$

for the pseudo-group transformations, where we have introduced the power series

$$a[\![\, h \,]\!] = -\,\frac{\big(\, f'[\![\, h \,]\!] - f'[\![\, 0 \,]\!] \,\big)\, y + g[\![\, h \,]\!] - g[\![\, 0 \,]\!]}{f'[\![\, h \,]\!]}$$

for later computational convenience. Inverting the power series

$$H = \widetilde{f}\,[\![\, h \,]\!] \equiv f[\![\, h \,]\!] - f[\![\, 0 \,]\!], \qquad\qquad K = f'[\![\, h \,]\!]\big(\, k - a[\![\, h \,]\!] \,\big), \tag{8.35}$$

and substituting the result into the series for U in (8.34), yields the power series $\widehat{U}[\![\, H, K \,]\!]$ for the prolonged action on the surface jet bundle $\mathrm{J}^\infty = \mathrm{J}^\infty(\mathbb{R}^3, 2)$, whose first few coefficients were given in (3.22).

The moving frame normalizations chosen in Example 3.9 are equivalent to setting

$$\widehat{U}[\![\, H, K \,]\!] = \tfrac{1}{2}\, K^2\, V[\![\, H, K \,]\!], \quad \text{where} \quad V[\![\, H, K \,]\!] = 1 + V_1\, H + V_2\, K + \;\cdots \tag{8.36}$$

is a power series whose constant term equals 1. When we substitute (8.35) into the normalization equations (8.36), the left hand side becomes the third power series in (8.34), while the right hand side becomes

$$\tfrac{1}{2}\, f'[\![\, h \,]\!]^2 \,\big(\, k - a[\![\, h \,]\!] \,\big)^2 v[\![\, h, k \,]\!],$$

where we set $v[\![\, h, k \,]\!] = V[\![\, H, K \,]\!]$ when their parameters are related by (8.35). The resulting power series equation,

$$u[\![\, h, k \,]\!] = a'[\![\, h \,]\!] - \frac{f''[\![\, h \,]\!]}{f'[\![\, h \,]\!]}\big(\, k - a[\![\, h \,]\!] \,\big) + \tfrac{1}{2}\, f'[\![\, h \,]\!]^2 \,\big(\, k - a[\![\, h \,]\!] \,\big)^2 v[\![\, h, k \,]\!], \tag{8.37}$$

will prescribe the complete moving frame formulae for the pseudo-group parameters in $f[\![\, h \,]\!], a[\![\, h \,]\!]$ as follows. First, setting $k = a[\![\, h \,]\!]$ in (8.37), we find

$$a'[\![\, h \,]\!] = u[\![\, h, a[\![\, h \,]\!] \,]\!]. \tag{8.38}$$

We can view (8.38) as the power series analog of the first order nonlinear ordinary differential equation

$$\frac{da}{dx} = u(x, a(x)) \qquad \text{with initial conditions} \qquad a(0) = 0,$$

reflecting the fact that the power series

$$a[\![h]\!] = a_x h + \tfrac{1}{2} a_{xx} h^2 + \cdots$$

has no constant term. The series solution to this ordinary differential equation has coefficients

$$a_x = u, \quad a_{xx} = u_x + a_x u_y = u_x + u\,u_y, \quad a_{xxx} = u_{xx} + 2\,u\,u_{xy} + u^2 u_{yy} + u_x u_y + u\,u_y^2,$$

and, in general,

$$a_j = (\mathrm{D}_x + u\,\mathrm{D}_y)^{j-1}\, u. \tag{8.39}$$

Second, differentiating (8.37) with respect to k and then setting $k = a[\![h]\!]$ yields

$$f''[\![h]\!] = -\, u_y[\![h, a[\![h]\!]]\!]\, f'[\![h]\!], \tag{8.40}$$

which is the power series form of the second order linear ordinary differential equation

$$\frac{d^2 f}{dx^2} = -\, u_y(x, a(x))\, \frac{df}{dx}\ .$$

The series solution, based upon (8.39), yields the normalization formulae

$$f_{xx} = -\, u_y f_x, \qquad f_{xxx} = -\,(u_{xy} + a_x u_{yy}) f_x - u_y f_{xx} = -\,(u_{xy} + u\,u_{yy} - u_y^2)\, f_x,$$

and, in general,

$$f_j = f_x\, (\mathrm{D}_x + u\,\mathrm{D}_y - u_y)\, \frac{1}{f_x}\, f_{j-1} = f_x\, (\mathrm{D}_x + u\,\mathrm{D}_y - u_y)^{j-1}(1)\,, \qquad j \geq 2. \tag{8.41}$$

To normalize the one remaining coefficient f_x, we differentiate (8.37) twice with respect to k and set $h = k = 0$, yielding

$$u_{yy} = f_x^2, \qquad \text{so that} \qquad f_x = \sqrt{u_{yy}}\ .$$

Thus, our pseudo-group normalization formulae (8.39), (8.41) become

$$f_j = \sqrt{u_{yy}}\, (\mathrm{D}_x + u\,\mathrm{D}_y - u_y)^{j-1}(1), \qquad a_j = (\mathrm{D}_x + u\,\mathrm{D}_y)^{j-1} u, \qquad j = 1, 2, \ldots\,. \tag{8.42}$$

Substituting these normalized values into the power series (8.36) produces the differential invariant power series

$$I[\![H, K]\!] = \tfrac{1}{2}\, K^2\, J[\![H, K]\!], \tag{8.43}$$

where the non-constant coefficients of

$$J[\![H, K]\!] = 1 + J_1\, H + \tfrac{1}{3}\, J_2\, K + \cdots = 1 + \frac{u_{xyy} + u u_{yyy} + 2\, u_y u_{yy}}{u_{yy}^{3/2}}\, H + \frac{u_{yyy}}{3\, u_{yy}^{3/2}}\, K + \cdots \tag{8.44}$$

form a complete system of normalized differential invariants. The first two terms recover our earlier formulae (3.24).

The infinitesimal generators of this pseudo-group have the form

$$\mathbf{v} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \varphi \frac{\partial}{\partial u} = a(x) \frac{\partial}{\partial x} + \big[\, a'(x)\, y + b(x) \,\big] \frac{\partial}{\partial y} + \big[\, a''(x)\, y + b'(x) \,\big] \frac{\partial}{\partial u}\,, \tag{8.45}$$

where $a(x), b(x)$ are arbitrary scalar functions. The corresponding Taylor series are

$$\xi[\![\, h,k \,]\!] = a[\![\, h \,]\!], \quad \eta[\![\, h,k \,]\!] = a_h[\![\, h \,]\!]\,(y+k) + b[\![\, h \,]\!], \quad \varphi[\![\, h,k \,]\!] = a_{hh}[\![\, h \,]\!]\,(y+k) + b_h[\![\, h \,]\!],$$

and thus the prolonged infinitesimal generator coefficient series (8.21) is

$$\begin{aligned} \widehat{\varphi}\,[\![\, h,k \,]\!] = a_{hh}[\![\, h \,]\!]\,(y+k) + b_h[\![\, h \,]\!] - u_h[\![\, h,k \,]\!]\,(a[\![\, h \,]\!] - a[\![\, 0 \,]\!]) \\ - u_k[\![\, h,k \,]\!]\big(\, a_h[\![\, h \,]\!]\,(y+k) - a_h[\![\, 0 \,]\!]\, y + b[\![\, h \,]\!] - b[\![\, 0 \,]\!] \,\big). \end{aligned} \tag{8.46}$$

Invariantization results in

$$\begin{aligned} \widehat{\psi}\,[\![\, H,K \,]\!] = \alpha_{HH}[\![\, H \,]\!]\, K + \beta_H[\![\, H \,]\!] - I_H[\![\, H,K \,]\!]\,\big(\, \alpha[\![\, H \,]\!] - \alpha[\![\, 0 \,]\!] \,\big) \\ - I_K[\![\, H,K \,]\!]\big(\, \alpha_H[\![\, H \,]\!]\, K + \beta[\![\, H \,]\!] - \beta[\![\, 0 \,]\!] \,\big), \end{aligned}$$

where the coefficients of

$$\alpha[\![\, H \,]\!] = \alpha + \alpha_1 H + \tfrac{1}{2}\, \alpha_2 H^2 + \cdots, \qquad \beta[\![\, H \,]\!] = \beta + \beta_1 H + \tfrac{1}{2}\, \beta_2 H^2 + \cdots,$$

are the moving frame pull-backs of the independent Maurer–Cartan forms, so

$$\alpha_k = \iota(a_k), \qquad \beta_k = \iota(b_k).$$

According to (8.26)

$$dI[\![\, H,K \,]\!] = I_H[\![\, H,K \,]\!]\, \varpi^1 + I_K[\![\, H,K \,]\!]\, \varpi^2 + \vartheta[\![\, H,K \,]\!] + \widehat{\psi}\,[\![\, H,K \,]\!]. \tag{8.47}$$

The phantom components of this series identity are the terms in $H^j, H^j K$, and K^2. Substituting $K = 0$ yields

$$\beta_H[\![\, H \,]\!] = -\,\vartheta[\![\, H,0 \,]\!].$$

Differentiating with respect to K and then setting $K = 0$ yields

$$\alpha_{HH}[\![\, H \,]\!] = -\, I_{KK}[\![\, H,0 \,]\!]\big(\, \varpi^2 - \beta[\![\, H \,]\!] + \beta[\![\, 0 \,]\!] \,\big) - \vartheta_K[\![\, H,0 \,]\!].$$

Finally, the coefficient of K^2 yields

$$\alpha_1 = \tfrac{1}{2}\big(\, J_1\, \varpi^1 + J_2\, \varpi^2 + \vartheta_{0,2} \,\big).$$

Substituting these back into (8.47) yields a complete system of recurrence formulae for the differential invariants. In particular, the horizontal component is

$$\begin{aligned} d_H\, I[\![\, H,K \,]\!] &= \big(\, I_H[\![\, H,K \,]\!] - \tfrac{1}{2}\, J_1 \,\{\, H\, I_H[\![\, H,K \,]\!] + K\, I_K[\![\, H,K \,]\!] \,\}\,\big)\, \varpi^1 \\ &+ \Big(I_K[\![\, H,K \,]\!] - \tfrac{1}{2}\, J_2 \,\{\, H\, I_H[\![\, H,K \,]\!] + K\, I_K[\![\, H,K \,]\!] \,\} - K\, I_{KK}[\![\, H,0 \,]\!] \\ &+ K\, I_K[\![\, H,K \,]\!] \int_0^H I_{KK}[\![\, \widehat{H},0 \,]\!]\, d\widehat{H} + I_H[\![\, H,K \,]\!] \int_0^H \!\! \int_0^{\widehat{H}} I_{KK}[\![\, \widetilde{H},0 \,]\!]\, d\widetilde{H}\, d\widehat{H} \Big)\, \varpi^2. \end{aligned}$$

Expanding the series term by term will produce the complete system of recurrence relations among the differentiated and normalized invariants. In particular, we are able to conclude that J_1, J_2 generate all higher-order differential invariants by invariant differentiation. Similar manipulations will produce the recurrence formulae for the invariant differential forms.

Clearly, the computations can become quite intricate. Nevertheless, we hope that the reader is convinced that they are completely systematic and can, with sufficient computing resources, be straightforwardly implemented on a suitably powerful computer algebra system.

9. Further directions.

In this paper, we have succeeded in establishing a general, completely algorithmic moving frame calculus for Lie pseudo-group actions. A broad range of applications of these methods in geometry, physics and applied sciences is apparent.

(*a*) One immediate area of application is to the analysis of symmetry groups of differential equations, [**47**]. We now have a comprehensive and efficient algorithm that can be applied to the symmetry analysis of the differential equations of physical and mathematical significance, including gauge theory, [**5**, **27**], fluid mechanics and meteorology, [**47**, **59**], and many other systems of partial differential equations with infinite-dimensional symmetry groups. The first applications of these methods, to the Korteweg–deVries and KP equations, appear in [**12**, **13**, **14**].

(*b*) As we showed in [**53**], the moving frame method can produce the structure equations for the symmetry group directly from the determining system, providing an attractive alternative to the series expansion procedure advocated by Lisle, and Reid, [**37**, **38**, **57**]. Other methods, and some comparisons between them, can be found in the papers of Morozov, [**44**, **45**]. An advantage of our algorithm is that it enables one to also compute recurrence relations and syzygies, and thereby expose the structure of the algebra of differential invariants without having to solve the determining equations or explicitly compute the moving frame.

(*c*) Owing to the overall complexity of the computations, any serious implementation of the methods discussed here will, ultimately, rely on computer algebra. Thus, the development of appropriate software packages is a significant priority. Efficient implementation of the structure equations through some form of differential Gröbner basis methods would be crucial. Evelyne Hubert, [**25**], has implemented the finite-dimensional moving frame algorithms using the MAPLE package VESSIOT, [**2**], which can be adapted to the infinite-dimensional situation. A good source of interesting examples can be found in the classifications of Lie, [**35**], and Cartan, [**9**].

(*d*) As noted in [**16**], the symmetry groups of integrable soliton equations in more than one space dimension, including the KP, DKP, and Davey–Stewartson equations, exhibit a Kac–Moody Lie algebraic structure. This motivates developing in detail the connections between the structure theory of Lie pseudo-groups and Kac–Moody Lie algebras based on the underlying moving frame calculus.

(e) Symmetry classification methods developed by Lisle, Reid and Wittkopf, [**39**, **58**], rely on the invariant differential operators, and so can be effectively handled by our moving frame approach. Mansfield, [**41**], has already demonstrated their efficacy when the symmetry group is finite-dimensional.

(f) The group foliation method of Vessiot, [**62**, **28**], provides a powerful, but underdeveloped approach to the construction of explicit, non-invariant solutions to partial differential equations. Modern developments by Ovsiannikov, [**55**], and Martina, Nuktu, Sheftel, and Winternitz, [**42**, **46**], have underscored its potential for applications. Since the method relies on the classification of the differential invariants and their syzygies, our moving frame algorithms should play a key role in its further development. See also Anderson and Fels, [**3**], for a related method based on exterior differential systems.

(g) Adapting Kogan's recursive construction, [**29**], in the pseudo-group context would enable one to directly relate the differential invariants and invariant differential forms of smaller sub-pseudo-groups to those of a larger pseudo-groups. Such an algorithm would help resolve complicated pseudo-group actions by splitting them into simpler sub-pseudo-group actions.

(h) Applications to variational problems admitting infinite pseudo-groups of symmetries, cf. [**4**], are also immediate via a straightforward adaptation of the constructions in [**30**, **31**]. In particular, we can now construct the explicit formulas relating variational problems that admit an infinite-dimensional symmetry group with the differential invariant form of their Euler–Lagrange equations. Connections with Noether's Second Theorem, [**47**], should also be pursued.

(i) Computation of the characteristic cohomology of the invariant variational bicomplex was investigated by Anderson and Pohjanpelto in the projectable case, [**4**], and generalized to non-projectable actions by Itskov, [**27**]. Again, the moving frame calculus provides an ideal tool for further developments in cohomology theory and computations for general pseudo-group actions.

(j) Additional applications worth investigating include the classification of characteristic classes, [**6**], Gel'fand–Fuks cohomology, [**22**], and Chern–Moser invariants of real hypersurfaces, [**15**].

(k) The analysis of joint invariants and joint differential invariants for pseudo-groups can be based on an adaptation of the moving frame methods introduced in [**50**], and would be a worthwhile project, particularly in light of the applications in computer vision, geometric numerical integration, [**43**], and the design of symmetry-preserving numerical algorithms, [**51**].

(l) A longer range hope is that these constructions will help elucidate the incompletely developed foundations of the theory of Lie pseudo-groups. For instance, how are Cartan's notions of holohedric and merihedric equivalence, [**10**, **11**], reflected in our version of the structure equations?

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