Direct Reduction and Differential Constraints

Peter J. Olver†
School of Mathematics
University of Minnesota
Minneapolis, MN 55455
olver@math.umn.edu
http://www.math.umn.edu/∼olver

Abstract. Direct reductions of partial differential equations to systems of ordinary differential equations are in one-to-one correspondence with compatible differential constraints. The differential constraint method is applied to prove that a parabolic evolution equation admits infinitely many characteristic second order reductions, but admits a noncharacteristic second order reduction if and only if it is linearizable.

1. Introduction.

One of the most useful methods for determining particular explicit solutions to partial differential equations is to reduce them to ordinary differential equations (which are presumably easier to solve) through a suitably inspired ansatz. The classical Lie method for finding group-invariant solutions, first described in full generality in [**7**], generalizes and includes well-known methods for finding similarity solutions, travelling wave solutions, and other basic reduction methods. For example, the solutions which are invariant under a one-parameter symmetry group of a partial differential equation in two independent variables can be found by solving a reduced ordinary differential equation. In [**1**], Bluman and Cole extended Lie's reduction method to include nonclassical symmetry groups, where the invariance of the partial differential equation is only required on its intersection with the invariant surface condition characterizing the group-invariant functions. An alternative

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direct reduction method was proposed by Clarkson and Kruskal, [**2**], where a systematic approach for determining ansätze which reduce the partial differential equation to a single ordinary differential equation was developed and subsequently applied to many partial differential equations arising in a wide variety of physical systems. It was then realized that the direct approach is included in the nonclassical method, [**6**], although the latter slightly more general owing to a restriction on the type of ansatz allowed, [**12**]. More recently, in a study of blow-up of solutions to nonlinear diffusion equations, Galaktionov, [**3**], proposed a generalization of the direct method, which he called the method of "nonlinear separation", in which the ansatz involves two different functions of the similarity variable, and the partial differential equation reduces to a system of ordinary differential equations for the two unknown functions. One can readily envision extending Galaktionov's method to even more unknown functions, although at present I am unaware of any significant examples.

Earlier, Philip Rosenau and I, [**14**], [**15**], showed how (almost) all known reduction methods, including the classical and nonclassical methods, partial invariance, separation of variables, etc., can be placed into a general framework. The general formulation requires that the original system of partial differential equations be enlarged by appending additional differential constraints (called "side conditions" in our work), such that the resulting overdetermined system of partial differential equations satisfy some form of compatibility condition. Special cases of this approach appear in the earlier work of Yanenko, [**20**], and Meleshko, [**9**], and more recent extensions appear in the work of Vorob'ev, [**18**], [**19**]. For example, in the classical and nonclassical methods, the differential constraint is the invariant surface condition, and its compatibility with the partial differential equation implies that the ansatz based on similarity variables (group invariants) reduces the equation to an ordinary differential equation. Methods based on the Riquier–Ritt theory of overdetermined systems of partial differential equations, [**16**], and differential Gröbner bases, [**8**], can then, at least in principle, be effectively used to analyze these systems, and thereby determine classes of useful, compatible differential constraints.

This paper is devoted to a systematic study of the relationship between the higher order direct method of Galaktionov, and the method of differential constraints. The main result is that a partial differential equation admits a direct reduction to a system of k ordinary differential equations for k unknown functions of a single "similarity variable" if and only if an associated k^{th} order differential constraint is compatible with the equation. The constraint itself is not a general k^{th} order partial differential equation, but, rather, a parametrized k^{th} order ordinary differential equation written in a more general coordinate system. Thus, one can analyze reductions of the partial differential equation to (systems of) ordinary differential equations using either a direct analysis based on the Clarkson–Kruskal approach, or an analysis of the compatibility of the differential constraints. In answer to the obvious question of which of the two methods is more effective to use in practice, I have found that, based on computational experience, the Clarkson–Kruskal direct method appears to be easier to implement than the constraint method for first order reductions. However, second and higher order reductions appear to be more readily determined using the differential constraint method. In section 3, we illustrate how to effectively implement the constraint method by proving the striking result that a parabolic evolution equation of the particular form $u_t = u_{xx} + P(x, u, u_x)$, which includes Galaktionov's original example, admits a second order direct reduction with similarity variable *not equal to* the time t if and only if the equation is equivalent, via a change of variables, to either a linear equation, or to a forced Burgers' equation, which in turn is linearizable through a transformation of Hopf–Cole type. Thus, the Galaktionov reduction example, which uses t as the similarity variable, cannot, except for these very special equations, be extended to more general similarity variables. On the other hand, any second order evolution equation admits infinitely many second order direct redutions whose similarity variable is t. The differential constraint method used to prove these two results is easily applicable to a wide variety of equations, the required calculations being very reminiscent of those needed to solve the determining equations for the symmetry groups of the system of partial differential equations, [**13**].

2. Compatibility and Reduction.

For the most part, we will concentrate on the simple case of a single second order partial differential equations in two independent and one dependent variables, but our results can be straightforwardly generalized to arbitrary higher order systems of partial differential equations. Consider a partial differential equation of the form

$$\Delta(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = 0, \tag{1}$$

where x,t are independent variables and $u = f(x,t)$ the dependent variable. The differential equation admits a *direct reduction* if there exist functions $z = \zeta(x,t)$, $u = U(x,t,w)$, such that the Clarkson–Kruskal ansatz

$$u(x,t) = U(x,t,w(z)) = U\big(x,t,w(\zeta(x,t))\big), \tag{2}$$

reduces (1) to a single ordinary differential equation for $w = w(z)$. The ansatz (2) includes as a special case the classical method of Lie, [**7**], for finding group-invariant solutions to the partial differential equation (1) under a one-parameter symmetry group, in which case $\zeta(x,t)$ is the *similarity variable*. Note that U is not uniquely determined since we can incorporate any function of the similarity variable ζ into w.

As was shown by Levi and Winternitz, [**6**], and Nucci and Clarkson, [**12**], there is a simple correspondence between direct reductions and nonclassical symmetries of the Bluman and Cole type, [**1**]. Let $\mathbf{v} = \tau(x,t)\partial_t + \xi(x,t)\partial_x$ be any vector field such that $\mathbf{v}(\zeta) = 0$, i.e., $\zeta(x,t)$ is the unique (up to functions thereof) invariant of the one-parameter group generated by $\mathbf{v}$. (Note that $\mathbf{v}$ is only determined up to an overall functional multiple — for example, if $\tau \neq 0$ we can divide through by τ and replace $\mathbf{v}$ by the simpler vector field $\widehat{\mathbf{v}} = \partial_t + \sigma(x,t)\partial_x$.) Applying $\mathbf{v}$ to the ansatz (2), we find

$$\tau u_t + \xi u_x = \tau U_t + \xi U_x \equiv V(x,t,w). \tag{3}$$

On the other hand, assuming $U_w \neq 0$, we can solve (2) for $w = W(x,t,u)$ using the Implicit Function Theorem. (We avoid singular points, and note that if $U_w \equiv 0$, the ansatz would not explicitly depend on w.) Substituting this into the right hand side of (3), we find that if u has the form (2), then it satisfies a first order quasi-linear partial differential equation of the form

$$\mathbf{v}(u) = \tau(x,t)u_t + \xi(x,t)u_x = \varphi(x,t,u). \tag{4}$$

Conversely, suppose u satisfies an equation of the form (4). Assuming $\mathbf{v} \neq 0$, we let $z = \zeta(x,t)$ be the unique (local) invariant of $\mathbf{v}$, and define $y = \eta(x,t)$ so that $\mathbf{v}(\zeta) = 0$, $\mathbf{v}(\eta) = 1$. Thus, the (y,z) coordinates serve to rectify the vector field, $\mathbf{v} = \partial_y$, and (4) reduces to a parametrized first order ordinary differential equation

$$u_y = \psi(y,z,u). \tag{5}$$

Fixing y_0, the general solution to the initial value problem for (5) with $u(y_0,z) = w(z)$, say, has the form $u = \widehat{U}(y,z,w(z))$. Replacing y and z by their formulas in terms of x,t leads us to the conclusion that u satisfies a direct reduction type ansatz. We have therefore proved:

Proposition 1. *There is a one-to-one correspondence between ansätze of the direct reduction form* (1) *with $U_w \neq 0$ and quasi-linear first order differential constraints* (4).

Solutions $u = f(x,t)$ to (4) are just the functions which are invariant under the one-parameter group generated by the vector field

$$\mathbf{w} = \tau(x,t)\partial_t + \xi(x,t)\partial_x + \varphi(x,t,u)\partial_u. \tag{6}$$

Note in particular that $\mathbf{w}$ generates a group of "fiber-preserving transformations", meaning that the transformations in x and t do not depend on the coordinate u.

In the direct method, one requires that the ansatz (2) reduces the partial differential equation (1) to an ordinary differential equation. In the nonclassical method of Bluman and Cole, one requires that the differential constraint (4) which requires the solution to be invariant under the group generated by $\mathbf{w}$ be compatible with the original partial differential equation (1), in the sense that the overdetermined system of partial differential equations defined by (1), (4), has no integrability conditions.

Theorem 2. *The ansatz* (2) *will reduce the partial differential equation* (1) *to a single ordinary differential equation for $w(z)$ if and only if the overdetermined system of partial differential equations defined by* (1), (4), *is compatible.*

Proof: The proof is based on the method introduced in [**14**] for studying nonclassical and more general reductions of partial differential equations. Let $y = \eta(x,t), z = \zeta(x,t)$, and $v = \omega(x,t,u)$ be rectifying coordinates for the vector field $\mathbf{w} = \partial_y$. Then the differential constraint takes the simple form $v_y = 0$ in these coordinates, leading to the simplified ansatz $v = v(z)$, which is just (2) rewritten in the (y,z,v) coordinates. Rewrite the differential equation (1) in these coordinates,

$$\widehat{\Delta}(y,z,v,v_y,v_z,v_{yy},v_{yz},v_{zz}) = 0.$$

Now, if v satisfies the constraint $v_y = 0$, then the equation reduces to

$$\widehat{\Delta}(y,z,v,0,v_z,0,0,v_{zz}) = 0,$$

which will be an ordinary differential equation for v as a function of z if and only if, apart for an overall factor, it is independent of y. To remove this ambiguity, we solve for v_{zz},

$$v_{zz} = F(y,z,v,v_y,v_z,v_{yy},v_{yz}). \tag{7}$$

The reduction

$$v_{zz} = F(y,z,v,0,v_z,0,0),$$

is equivalent to a single ordinary differential equation if and only if $F(y,z,v,0,v_z,0,0)$ is independent of y.

On the other hand, the compatibility condition between (7) and the constraint $v_y = 0$ is found by a simple cross differentiation:

$$0 = v_{yzz} = D_y F = F_y + v_y F_v + v_{yy} F_{v_y} + v_{yz} F_{v_z} + v_{yyy} F_{v_{yy}} + v_{yyz} F_{v_{yz}}.$$

This will be satisfied, and hence the overdetermined system will be compatible, if and only if it vanishes as a differential consequence of the constraint $v_y = 0$, which means $F_y(y,z,v,0,v_z,0,0) = 0$. The equivalence of the two conditions is now clear, and the theorem is proven. *Q.E.D.*

Thus, there is a one-to-one correspondence between direct reductions of the partial differential equation (1) and compatible first order quasi-linear differential constraints. The general nonclassical method, which allows arbitrary point transformation symmetry groups, so that ξ and τ in (6) can also depend on u, is similarly equivalent to the more general, (but much harder to deal with) ansatz

$$u = U(x,t,w(z)) = U\big(x,t,w(\zeta(x,t,u))\big). \tag{8}$$

We have thus rederived the well-known connection between the Bluman-Cole nonclassical method and the Clarkson–Kruskal direct method, [**6**], [**12**].

In Galaktionov's generalization of the direct method, one makes a more general ansatz

$$u(x,t) = U(x,t,w_1(z),w_2(z)) = U\big(x,t,w_1(\zeta(x,t)),w_2(\zeta(x,t))\big), \tag{9}$$

depending on two unknown functions w_1, w_2, depending on the *similarity variable* $z = \zeta(x,t)$, and requires that it reduce the partial differential equation (1) to a coupled system of ordinary differential equations for $w_1(z)$ and $w_2(z)$. The generalization of the ansatz (9) to more than two functions is clear, but, to keep the exposition simple, we will usually restrict our attention to a second order direct reduction here. In general, we define the *order* of a direct reduction ansatz to be the number of (independent) functions of the similarity variable, so that the original Clarkson–Kruskal ansatz has order one. The second order analogue of the equivalent first order constraint (4) is obtained as follows. As above, (9), let $\mathbf{v} = \tau\partial_t + \xi\partial_x$ be a vector field with $\zeta(x,t)$ as its invariant function. If we differentiate the ansatz (9) twice with respect to $\mathbf{v}$ we find equations of the form

$$\mathbf{v}(u) = V(x,t,w_1,w_2), \tag{10}$$

$$\mathbf{v}^2(u) = \widehat{V}(x,t,w_1,w_2), \tag{11}$$

where

$$\begin{aligned} \mathbf{v}(u) &= \tau u_t + \xi u_x, \\ \mathbf{v}^2(u) &= \mathbf{v}(\mathbf{v}(u)) = \tau^2 u_{tt} + 2\tau\xi u_{xt} + \xi^2 u_{xx} + (\tau\tau_t + \xi\tau_x)u_t + (\tau\xi_t + \xi\xi_x)u_x. \end{aligned} \tag{12}$$

Assuming that the 2×2 Jacobian determinant of U, V with respect to w_1, w_2 is nonzero, we can use the Implicit Function Theorem to solve (9), (11), for

$$w_1 = W_1(x, t, u, \mathbf{v}(u)), \qquad w_2 = W_2(x, t, u, \mathbf{v}(u)). \tag{13}$$

(If the Jacobian determinant is identically zero, then the constraint will be of the form $u = \widetilde{U}(x, t, F(x, t, w_1, w_2))$, and so is actually a first order constraint for the combination $w = F(x, t, w_1, w_2)$.) Substituting (13) into (11), we conclude that u satisfies a second order differential equation of the form

$$\mathbf{v}^2(u) = \Phi(x, t, u, \mathbf{v}(u)). \tag{14}$$

If we substitute the explicit formulae (12) for the derivatives of u with respect to $\mathbf{v}$, we see that the constraint (14) looks like a second order quasi-linear partial differential equation, but it is really just an ordinary differential equation in disguise. This fact becomes clear in the rectifying (y, z) coordinates for $\mathbf{v} = \partial_y$, in terms of which (14) reduces to a parametrized second order ordinary differential equation

$$u_{yy} = \Psi(y, z, u, u_y). \tag{15}$$

Thus, the ansätze of the form (9) correspond to a particular types of quasi-linear second order differential constraints.

Conversely, the general solution to a differential constraint of the form (14), or, in the rectifying coordinates, (15), is found by solving the initial value problem for (15) with initial data $u(y_0, z) = w_1(z)$, $u_y(y_0, z) = w_2(z)$. The solution has the form $u = \widehat{U}(y, w_1(z), w_2(z))$. Replacing y and z by their formulas in terms of x, t leads us to the conclusion that u satisfies an ansatz of the form (8). We have therefore proved the analogue of Proposition 1 in the second order case. The n^{th} order generalization of this result is proved by analogous methods.

Proposition 3. *There is a one-to-one correspondence between direct reduction ansätze of order* n,

$$u(x, t) = U(x, t, w_1(z), \ldots, w_n(z)), \qquad z = \zeta(x, t), \tag{16}$$

and n^{th} *order differential constraints of the form*

$$\mathbf{v}^n(u) = \Phi(x, t, u, \mathbf{v}(u), \ldots, \mathbf{v}^{n-1}(u)). \tag{17}$$

The analogue of Theorem 2 can now be easily guessed. The proof, though, is more complicated, since, in contrast to the first order case, we cannot automatically reduce a second order constraint to the simple form $u_{yy} = 0$ through a clever change of coordinates. (This is because not every second order ordinary differential equation can be linearized by a point transformation; see [**17**] and [**5**] for explicit necessary and sufficient conditions for effecting such a linearization.)

Theorem 4. *The ansatz* (16) *will reduce the partial differential equation* (1) *to a coupled system of* n *distinct ordinary differential equations for* $w_1(z), \ldots, w_n(z)$, *if and only if the overdetermined system of partial differential equations defined by* (1), (17) *is compatible.*

Proof: For simplicity, we only prove Theorem 4 for second order reductions/constraints. As in the proof of Theorem 2, we begin by introducing rectifying coordinates $y = \eta(x,t)$, $z = \zeta(x,t)$, the vector field $\mathbf{v}$, so the differential constraint takes the form (15). Rewrite the original differential equation (1) in these coordinates,

$$\widehat{\Delta}(y, z, u, u_y, u_z, u_{yy}, u_{yz}, u_{zz}) = 0. \tag{18}$$

Substitution of the ansatz (9), which is now $u = U(y, z, w_1(z), w_2(z))$, into the partial differential equation (18) leads to an equation of the form

$$H(y, z, w_1, w_2, w_1', w_2', w_1'', w_2'') = 0. \tag{19}$$

Lemma 5. *Equation* (19) *reduces to a pair of ordinary differential equations for* $w_1(z)$, $w_2(z)$, *if and only if*

$$H_{yy} = AH + BH_y, \tag{20}$$

for functions A, B *depending on* y, z, *and* w_1, w_2 *and their derivatives.*

Remark: The two ordinary differential equations for w_1, w_2 will be independent provided $H_y \neq CH$ for some function C. However, this does not guarantee that the two equations form a normal system for w_1, w_2; for example, they might define an overdetermined system of ordinary differential equations for some combination $W(z, w_1, w_2)$. Such technical complications can be avoided by assuming that a particular 2×2 Wronskian matrix is nonsingular. For example, if

$$\det \begin{vmatrix} H_{w_1''} & H_{w_2''} \\ H_{yw_1''} & H_{yw_2''} \end{vmatrix} \neq 0,$$

then the system is a normal second order system for w_1, w_2. Similarly, if $H_{w_1''} \equiv H_{w_2''} \equiv 0$, while

$$\det \begin{vmatrix} H_{w_1'} & H_{w_2'} \\ H_{yw_1'} & H_{yw_2'} \end{vmatrix} \neq 0,$$

then the equation reduces to a normal first order system of ordinary differential equations for w_1, w_2. Thus, by analyzing the resulting system, or the function H itself, one can ensure that the system of ordinary differential equations for w_1, w_2 is normal, and so we will not comment further on this technical complication.

Proof: For each fixed value of y, (19) will determine a ordinary differential equation (usually of second order) for $w_1(z), w_2(z)$. Moreover, every y derivative of H, i.e., $H_y = 0$, $H_{yy} = 0$, etc., will have the same property. Now, if w_1 and w_2 are to satisfy only two ordinary differential equations, then exactly two of these infinitely many conditions can be independent. This implies, in particular, that H_{yy} must vanish as a direct consequence of the conditions $H = 0$ and $H_y = 0$, which implies that an identity of the form (20) holds.

Conversely, suppose that H satisfies (20), which we now regard as a linear, homogeneous second order ordinary differential equation for H as a function of y, with z, w_1, w_2, etc. entering as parameters. Choose y_0 so that the two equations

$$\begin{aligned} F_0(z, w_1, w_2, w_1', w_2', w_1'', w_2'') &\equiv H(y_0, z, w_1, w_2, w_1', w_2', w_1'', w_2'') = 0, \\ F_1(z, w_1, w_2, w_1', w_2', w_1'', w_2'') &\equiv H_y(y_0, z, w_1, w_2, w_1', w_2', w_1'', w_2'') = 0, \end{aligned} \tag{21}$$

are independent, and hence define a pair of ordinary differential equations for $w_1(z), w_2(z)$. Assuming the reduction is of order two, this will always be possible, since otherwise we would have an identity of the form $H_y = AH$, which, by the method used below to analyze (20), will imply that the equation reduces to a single ordinary differential equation for some combination $w = W(y, z, w_1, w_2)$, which implies that the original ansatz was really a first order reduction for the single unknown w. By linearity, the solution to (20) then has the form

$$H = F_0 H^{(1)} + F_1 H^{(2)},$$

where $H^{(1)}, H^{(2)}$ are functions of $y, z, w_1, w_2, w_1', w_2', w_1'', w_2''$, and form a fundamental set of solutions with initial conditions $H^{(1)}|_{y_0} = 1$, $H_y^{(1)}|_{y_0} = 0$, $H^{(2)}|_{y_0} = 0$, $H_y^{(2)}|_{y_0} = 1$. Therefore, if w_1, w_2 solve the system (21), then $H \equiv 0$ for all y, as desired. *Q.E.D.*

Using the definition of H, we find (20) has the form

$$D_y^2 \Delta = \widehat{A}\,\Delta + \widehat{B}\,D_y \Delta, \qquad \text{whenever} \qquad u = U(y, z, w_1, w_2), \tag{22}$$

where $\widehat{A}, \widehat{B}$ are obtained by replacing w_1, w_2, and their derivatives according to (13), which, in the y, z coordinates, takes the form

$$w_1 = W_1(y, z, u, u_y), \qquad w_2 = W_2(y, z, u, u_y),$$

and its derivatives. On the other hand, the partial differential equation (18) and the differential constraint (15) will be compatible if and only if

$$D_x^2 \Delta = \widetilde{A}\,\Delta + \widetilde{B}\,D_x \Delta, \qquad \text{whenever} \qquad u_{yy} = \Psi, \tag{23}$$

for functions $\widetilde{A}, \widetilde{B}$. The equivalence of conditions (22), (23) relies on a simple lemma, whose proof is left to the reader.

Lemma 6. *Consider the ansatz $u = U(y, z, w_1(z), w_2(z))$, and the equivalent second order differential constraint $u_{yy} = \Psi(y, z, u, u_y)$. A function P depending on y, z, u, and derivatives of u vanishes when $u = U$ if and only if it vanishes when we substitute the constraint $u_{yy} = \Psi$ and all its derivatives.*

This completes the proof of our basic Theorem 4. *Q.E.D.*

Example 7. Galaktionov's simplest example, [**3**], is the parabolic equation

$$u_t = u_{xx} + u_x^2 + u^2, \tag{24}$$

which arises in the study of heat propagation. He proved that the ansatz

$$u(x, t) = w_1(t) + w_2(t) \cos x, \tag{25}$$

reduces (24) to the first order planar dynamical system

$$w_1' = w_1^2 + w_2^2, \qquad w_2' = 2 w_1 w_2 - w_2. \tag{26}$$

The resulting solutions are of great value in studying the blow up of general solutions to (24), where the standard methods based on similarity solutions (of which there are none) fail.

The similarity variable is t, and the differential constraint corresponding to (25) is

$$u_{xx} - (\cot x) u_x = 0, \tag{27}$$

which has the equivalent first order form

$$u_t - u_x^2 - u^2 - (\cot x) u_x = 0. \tag{28}$$

The compatibility condition for the overdetermined system (24), (28), is found by differentiating (24) with respect to t and (28) twice with respect to x and eliminating the third derivative u_{xxt}. The result is easily found to vanish as a direct consequence of the two equations, and so, as guaranteed by Theorem 4, the overdetermined system is compatible.

Remark: In Galaktionov's approach, one begins by making the more general nonlinear separation ansatz $u = w_1(t) + w_2(t) v(x)$, and then finds that only very special functions $v(x)$ — namely translates of the cosine — will render the resulting equation solvable. Another nonlinear separation ansatz, namely $u = U(w_1(z_1) + w_2(z_2))$ where w_1 and w_2 are functions of *different* similarity variables $z_1 = \zeta_1(x,t)$, $z_2 = \zeta_2(x,t)$, was considered by Miller and Rubel, [**11**], in a recent analysis of Laplace–Beltrami equations on Riemannian surfaces. These and similar reductions are governed by particular types of third order differential constraints.

Finally, we investigate whether one might reasonably allow more general types of differential constraints. The problem here is that, if one allows completely general constraints, there are always infinitely many, although their determination is problematic. This statement is not as surprising as it might appear at first if one considers compatible zero$^{\text{th}}$ order constraints. A zero$^{\text{th}}$ order differential constraint takes the form $u = f(x,t)$, and is compatible if and only if f is a solution to the partial differential equation (1), so that any differential equation trivially admits infinitely many compatible first order constraints; however, they cannot all be found without having already solved the equation! Therefore it is not so surprising that the equation admits infinitely many differential constraints of arbitrary order. However, the constraints are not necessarily of the required form (17) for the direct reduction approach, and so may be of little help for finding explicit solutions to the partial differential equation. We outline a proof of this result in the case of first order differential constraints.

Theorem 8. *A second order partial differential equation admits infinitely many compatible first order differential constraints*

$$Q(x, t, u, u_x, u_t) = 0. \tag{29}$$

Proof: Consider the case when the equation is an evolution equation

$$u_t = K(x, t, u, u_x, u_{xx})$$

first, which we rewrite in the more convenient form

$$u_{xx} = P(x, t, u, u_x, u_t). \tag{30}$$

We also assume that the constraint (29) can be solved for

$$u_t = F(x, t, u, u_x). \tag{31}$$

The integrability condition between (30) and (31) is found by comparing the expressions for u_{xxt}. Differentiating (31) and using the equation (30), we find

$$u_{xt} = X(x, t, u, u_x, F^{(1)}) = \widehat{D}_x F, \qquad u_{tt} = Y(x, t, u, u_x, F^{(1)}) = \widehat{D}_t F,$$

where we use the modified total derivative operators

$$\widehat{D}_x = \partial_x + u_x \partial_u + \widehat{L} \partial_{u_x}, \qquad \widehat{D}_t = \partial_t + \widehat{P} \partial_u + X \partial_{u_x},$$

and where $\widehat{P} = P(x, t, u, u_x, F)$ is obtained by replacing u_t in P according to (31). Therefore,

$$u_{xxt} = A(x, t, u, u_x, F^{(2)}) = \widehat{D}_x X, \tag{32}$$

is a function depending on the variables x, t, u, u_x and second order derivatives of F with respect to these variables, whose explicit form could be readily computed if desired. On the other hand, differentiating (30) gives an alternative expression

$$u_{xxt} = B(x, t, u, u_x, F^{(2)}) = \widehat{D}_t \widehat{P}. \tag{33}$$

Equating the two formulas for u_{xxt}, we conclude that the differential constraint (31) will be compatible with the equation (30) if and only if the function $F(x, t, u, p)$ satisfies a single (complicated) second order partial differential equation, namely $A = B$. Generically, we can solve for F_{pp}, say, and so, if the equation is analytic, the Cauchy-Kovalevskaya Existence Theorem will guarantee the existence of infinitely many solutions.

The proof for a general second order partial differential equation (1) is similar. Indeed, expressing the constraint in the form (31) and differentiating, we can find formulas for u_{xt} and u_{tt} solely in terms of $x, t, u, u_x, u_t, u_{xx}$. Substituting these into (1) reduces it to an equation of the form (30) once we solve for u_{xx}. This completes the proof of the theorem. *Q.E.D.*

3. Evolution Equations.

In this section, we investigate the second order direct reduction method for the particular example of a second order evolution equation

$$u_t = K(x, t, u, u_x, u_{xx}). \tag{34}$$

Assume first that the similarity variable is *not* $z = t$, i.e., is not characteristic. Then the associated vector field can be taken in the reduced form $\mathbf{v} = \partial_t + \xi(x, t)\partial_x$, and the constraint (14) has the form

$$u_{tt} + 2\xi u_{xt} + \xi^2 u_{xx} + (\xi_t + \xi\xi_x) u_x = \Phi(x, t, u, u_t + \xi u_x). \tag{35}$$

The compatibility of (34) and (35) is most easily determined if we first solve (1) for the second derivative (assuming the equation is of truly second order)

$$u_{xx} = P(x,t,u,u_x,u_t). \tag{36}$$

Therefore, we can eliminate the variable u_{xx} from the constraint equation (35) by replacing it by P. In order to simplify the explicit formulas, it is helpful to generalize the constraint, and consider one of the more general form

$$u_{tt} = Q(x,t,u,u_x,u_{xx}) = \alpha(x,t)u_{xt} + B(x,t,u,u_x,u_t), \tag{37}$$

which, by the preceding remark, includes those leading to direct reductions of order 2. Cross-differentiating (36), (37) leads to the compatibility conditions

$$D_t^2 P = D_x^2(\alpha u_{xt} + B). \tag{38}$$

To analyze the compatibility condition (38), we replace the second and third order derivatives u_{xx}, u_{tt}, u_{xxx}, u_{xtt}, u_{xxt}, u_{ttt}, by their expressions obtained from (36), (37), and their derivatives, namely,

$$u_{xxx} = D_x P, \qquad u_{xxt} = D_t P, \qquad u_{xtt} = D_x(\alpha u_{xt} + B), \qquad u_{ttt} = D_t(\alpha u_{xt} + B), \tag{39}$$

each of which can, in turn, be rewritten in terms of the remaining variables, which are x,t,u,u_t,u_x, and u_{xt}. The result of substituting into (38) will be an equation which depends on x,t,u,u_t,u_x is a fairly arbitrary manner, but which is a polynomial in the second order derivative variable u_{xt}. Analysis of the coefficients of the powers of u_{xt} will provide powerful necessary conditions for the existence of such reductions. We will illustrate the method in a particular case below.

On the other hand, if the similarity variable is characteristic, so $\zeta(x,t) = t$, then the vector field can be taken to be $\mathbf{v} = \partial_x$, and hence the constraint (14) has the simpler form

$$u_{xx} = \Phi(x,t,u,u_x). \tag{40}$$

Now, using (36), we can eliminate u_{xx} from the constraint equation (40), and so such reductions are associated with a first order constraint

$$P(x,t,u,u_t,u_x) = \Phi(x,t,u,u_x). \tag{41}$$

Assuming $P_{u_t} \neq 0$, *any* first order constraint can be rewritten in the form (41). Therefore, as a consequence of Theorem 8, we find that there are infinitely many compatible constraints of this type, and so the equation (34) admits *an infinite number of second order direct reductions in which t is the similarity variable!* The analysis of these reductions is complicated, being governed by the solutions to the second order partial differential equation obtained by equating (32) and (33), and so one is forced to impose additional forms on the ansatz (9), such as that of Galaktionov, in order to pin down reductions which can be practically and explicitly analyzed.

The method of determining the compatibility of the system (34), (37) is very reminiscent of the methods for computing symmetry groups of differential equations. One uses conditions involving higher order derivatives which don't appear in the original functions (in this case u_{xt}) to deduce restrictions on the form these functions take in their dependence on some of the lower order derivatives. Substituting these expressions into the compatibility conditions leads to a new set of restrictions and so on. With enough computational stamina and/or ingenuity one can effectively determine the complete set of compatible second order differential constraints of the form (37). Here, for simplicity, we shall illustrate this technique with a particular class of equations, of the form

$$u_t = u_{xx} - H(x, t, u, u_x), \tag{42}$$

and prove the following remarkable result.

Theorem 9. *If the parabolic equation* (42) *is compatible with a differential constraint of the form* (37)*, then the equation is equivalent, under a change of variables* $v = \varphi(x, t, u)$*, to either a linear partial differential equation or to an equation of Burgers' type*

$$v_t = v_{xx} + v v_x + h(x, t). \tag{43}$$

Note that the Burgers' equation (43) can itself, be linearized by the Hopf-Cole transformation

$$v = 2(\log w)_x, \tag{44}$$

so that w satisfies the linear equation

$$w_t = w_{xx} - h(x, t) w. \tag{45}$$

Corollary 10. *If a parabolic equation of the form* (42) *admits a second order direct reduction with respect to a non-characteristic similarity variable, then the equation can be linearized using via a change of variables and/or a Hopf-Cole transformation.*

Remark: The linear and Burgers' type equations are also distinguished by the fact that any equation of the form (42) which possesses a generalized symmetry of order five or more, or even a formal symmetry of rank five or more (and, in particular, a recursion operator), is necessarily equivalent to such an equation; see [**10**]. I do not know whether there is a deep connection between the higher order direct reduction method and the existence of formal symmetries and/or recursion operators. For example, an interesting speculation is whether the only general second order evolution equations admitting a second order reduction are those possessing higher order symmetries, of which there are two further inequivalent canonical forms:

$$u_t = (u^{-2} u_x + cxu + du)_x, \qquad u_t = (u^{-2} u_x)_x + 1.$$

Proof of Theorem 8.

To analyze the compatibility conditions (38), it is useful to introduce the modified total derivative operators:

$$\begin{aligned} \widehat{D}_x &= \partial_x + u_x \partial_u + P \partial_{u_x} + u_{xt} \partial_{u_t} + X \partial_{u_{xt}}, \\ \widehat{D}_t &= \partial_t + u_t \partial_u + u_{xt} \partial_{u_x} + Q \partial_{u_t} + Y \partial_{u_{xt}}, \end{aligned} \tag{46}$$

whose coefficients are given as the values of the associated derivatives of u when evaluated on solutions to the overdetermined system (42), (37):

$$\begin{aligned} u_{xx} &= P = u_t + H(x,t,u,u_x), \\ u_{tt} &= Q = \alpha u_{xt} + B(x,t,u,u_x,u_t), \\ u_{xxt} &= X = \widehat{D}_t P = Q + H_t + u_t H_u + u_{xt} H_{u_x} \\ &= \left(\alpha + H_{u_x}\right) u_{xt} + C(x,t,u,u_x,u_t), \\ u_{xtt} &= Y = \widehat{D}_x Q = \alpha X + \alpha_x u_{xt} + B_x + u_x B_u + P B_{u_x} + u_{xt} B_{u_t} \\ &= \left(\alpha^2 + \alpha H_{u_x} + \alpha_x + B_{u_t}\right) u_{xt} + E(x,t,u,u_x,u_t). \end{aligned} \tag{47}$$

Note that if $F(x,t,u,u_x,u_t,u_{xt})$ is any function, then, as remarked earlier, on solutions to the overdetermined system, the total derivatives $D_x F$ and $D_t F$ reduce to functions depending only on x,t,u,u_x,u_t,u_{xt}, which are given explicitly as $\widehat{D}_x F$ and $\widehat{D}_t F$, respectively. In particular, the integrability condition (38) reduces to

$$Z = W, \qquad \text{where} \qquad Z = \widehat{D}_t X, \quad W = \widehat{D}_x Y. \tag{48}$$

We now analyze Z and W in (48) using the explicit formulas (46), (47). First, the coefficient of u_{xt}^2 in (48) is found to be

$$H_{u_x u_x} = B_{u_t u_t}.$$

Differentiating with respect to u_t, and recalling that H does not depend on this variable, we deduce that B is a quadratic function of u_t of the form

$$B = \tfrac{1}{2} H_{u_x u_x} u_t^2 + K u_t + L, \tag{49}$$

where K and L depend on x,t,u,u_x. Substituting (49) into (47), (48), and using the fact that the dependence of B on u_t is now fixed, we find that the coefficient of $u_t^2 u_{xt}$ in the resulting equation is

$$\tfrac{1}{2} H_{u_x u_x u_x} = \tfrac{5}{2} H_{u_x u_x u_x},$$

which implies that H is a quadratic function of u_x. Therefore, our preliminary analysis has demonstrated the following.

Lemma 11. *If the parabolic equation* (42) *is compatible with a differential constraint of the form* (37), *then it necessarily has the form*

$$u_t = u_{xx} - R u_x^2 - S u_x - T, \tag{50}$$

where R, S, T depend only on x, t, u.

At this point we could continue the direct analysis of the integrability conditions for equations of the form (50). However, the computations rapidly become extremely complicated, it becomes important to recognize that a straightforward change of variables can be profitably used to map the equation (50) to a simpler, more readily analyzed form.

Lemma 12. *Any equation of the form* (50) *can, through a change of variables of the form* $u \mapsto \varphi(x,t,u)$, *be mapped to one without the quadratic term in* u_x, *i.e., with* $R = 0$.

Proof: If $v = \varphi(x,t,u)$, then

$$v_t - v_{xx} = \varphi_u \left(u_t - u_{xx} - \frac{\varphi_{uu}}{\varphi_u} u_x^2 - 2\frac{\varphi_{xu}}{\varphi_u} u_x - \frac{\varphi_{xx}}{\varphi_u} \right),$$

Hence if we set $\varphi = \int \exp\left(-\int^u R(x,t,\bar{u})\,d\bar{u}\right)\,du$, $\widehat{S} = \frac{1}{2}\varphi_u S + \varphi_{xu}$, $\widehat{T} = \varphi_u T + \varphi_{xx}$, then

$$v_t - v_{xx} + \widehat{S} v_x + \widehat{T} = \varphi_u \left(u_t - u_{xx} + R u_x^2 + S u_x + T \right),$$

proving the lemma. *Q.E.D.*

Therefore, without loss of generality, we can assume that our evolution equation has the simplified quasi-linear form

$$u_t = u_{xx} - S(x,t,u) u_x - T(x,t,u). \tag{51}$$

Furthermore, according to (49), the differential constraint has the simplified form

$$u_{tt} = \alpha(x,t) u_{xt} + K(x,t,u,u_x) u_t + L(x,t,u,u_x). \tag{52}$$

The coefficients of the various independent derivatives of u in the compatibility condition (48) are now analyzed in order. First, the coefficient of $u_t u_{xt}$ implies that $K_{u_x} = S_u$, hence $K = S_u u_x + J(x,t,u)$. Using this, the coefficient of $u_x^2 u_{xt}$ implies $S_{uu} = 0$, so $S = \beta(x,t) u + \gamma(x,t)$. The analysis now splits into two subcases depending upon whether β is zero or not.

Suppose first that $\beta \neq 0$. Then the change of variables $u \mapsto \beta u + \gamma + 2(\beta_x/\beta)$, will map equation (51), with $S = \beta u + \gamma$, to a simpler equation of the form

$$u_t = u_{xx} - u u_x - \widehat{T}(x,t,u). \tag{53}$$

Moreover, the differential constraint (52) has $K = u_x + J(x,t,u)$. The resulting coefficient of $u_x u_{xt}$ in (48) implies $J_u = -u - \frac{1}{2}\alpha$, so $J = -\frac{1}{2}u^2 - \frac{1}{2}\alpha u + \kappa(x,t)$. Finally, the remaining coefficient of u_{xt} requires that $\widehat{T} = h(x,t) = \frac{1}{2}\alpha_t - \frac{1}{2}\alpha_{xx} - \alpha\alpha_x - \kappa_x$ is a function of x,t only, and hence the equation reduces to the forced Burgers' equation (43).

On the other hand, if $\beta = 0$, then we can rescale $u \mapsto u \exp\left(\frac{1}{2}\int \gamma(x,t)\,dx\right)$ to make the coefficient of u_x in (51) go away, so the equation takes the form $u_t = u_{xx} - \widehat{T}(x,t,u)$, in which case the differential constraint (52) has $K = J(x,t,u)$. The coefficient of $u_x u_{xt}$ in the compatibility condition implies that J does not depend on u. The coefficient of u_t^2 implies $L = \frac{1}{2}\widehat{T}_{uu} u_x^2 + M u_x + N$, where M, N only depend on x,t,u. The coefficient of $u_x^2 u_t$ then requires that $\widehat{T} = \mu(x,t)u^2 + \nu(x,t)u + \rho(x,t)$ is a quadratic polynomial in u. The coefficient of $u_x u_t$ shows $M = -[2\mu_x + (\alpha+2)\mu]u + \sigma(x,t)$. Finally, the coefficient of $u^2 u_t$ is $0 = 4\mu^2$, hence $\widehat{T}$ is necessarily linear in u, and so the equation (51) must be linear. This completes the proof of Theorem 9. *Q.E.D.*

For linear evolution equations, the preceding methods will reveal that the equation always admits infinitely many compatible linear constraints. However, not every such constraint corresponds to a direct reduction using a non-characteristic similarity variable, and the equations governing the possible reductions are quite complicated.

Theorem 13. *A linear second order evolution equation admits infinitely many compatible second order constraints, all of which are necessarily linear.*

Proof: A complete analysis of the compatibility conditions (48) proves that every compatible second order constraint for a linear equation

$$u_t = u_{xx} + f(x,t)u_x + g(x,t)u + h(x,t), \tag{54}$$

must necessarily be linear

$$u_{tt} = \alpha(x,t)u_{xt} + \beta(x,t)u_t + \gamma(x,t)u_x + \delta(x,t)u + \varepsilon(x,t). \tag{55}$$

The resulting compatibility conditions are

$$\begin{aligned}
\alpha_{xx} &= \alpha_t - (2\alpha + f)\alpha_x - f_x\alpha - 2\beta_x + 2f_t,\\
\beta_{xx} &= \beta_t - 2(\beta + g)\alpha_x + f\beta_x - 2\gamma_x - (f_t + g_x)\alpha + 2g_t,\\
\gamma_{xx} &= \gamma_t - 2(\gamma + f_t)\alpha_x - f\gamma_x - 2\delta_x - (f_{xt} + ff_t + g_t)\alpha - f_t\beta - f_x\gamma + f_{tt},\\
\delta_{xx} &= \delta_t - 2(\delta + g_t)\alpha_x - 2g\gamma_x + f\delta_x - (g_{xt} + gf_t)\alpha - g_t\beta - g_x\gamma + g_{tt},\\
\varepsilon_{xx} &= \varepsilon_t - 2(\varepsilon + h_t)\alpha_x - 2h\gamma_x + f\varepsilon_x - (h_{xt} + hf_t)\alpha - h_t\beta - h_x\gamma - h\delta + g\varepsilon + h_{tt}.
\end{aligned} \tag{56}$$

The system (56) is of Kovalevskaya type for the coefficients $\alpha, \beta, \gamma, \delta, \varepsilon$, and therefore admits infinitely many analytic solutions, [**13**]. *Q.E.D.*

If the linear constraint (55) arises from a second order reduction with noncharacteristic similarity variable, then it must have the form (14),

$$\mathbf{v}^2(u) = A(x,t)\mathbf{v}(u) + B(x,t)u + C(x,t),$$

for some vector field $\mathbf{v} = \partial_t + \xi(x,t)\partial_x$, where we replace u_{xx} by its value given by (54). Therefore, the coefficients of (55) must have the form

$$\alpha = -2\xi, \quad \beta = A - \xi^2, \quad \gamma = A\xi - f\xi^2 - \xi_t - \xi\xi_x, \quad \delta = B - g\xi^2, \quad \varepsilon = C - h\xi^2. \tag{57}$$

Substituting (57) into the compatibility conditions lead to an overdetermined system of five partial differential equations for the four unknowns ζ, A, B, C. I have not been able to successfully analyze this system in general, although particular cases could be handled directly.

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