On the commutator of $\mathcal{C}^\infty$-symmetries and the reduction of Euler–Lagrange equations

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Abstract. A novel procedure to reduce by four the order of Euler–Lagrange equations associated to n-th order variational problems involving single variable integrals is presented. In preparation, a new formula for the commutator of two $\mathcal{C}^\infty$-symmetries is established. The method is based on a pair of variational $\mathcal{C}^\infty$-symmetries whose commutators satisfy a certain solvability condition. It allows one to recover a $(2n-2)$-parameter family of solutions for the original $2n$-th order Euler–Lagrange equation by solving two successive first order ordinary differential equations from the solution of the reduced Euler–Lagrange equation. The procedure is illustrated by two different examples.

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1. Introduction

The relevance of symmetry groups for reducing and solving ordinary differential equations dates back to the classical work of Sophus Lie, and has been intensively studied in recent decades [13, 14, 16, 17]. Lie also applied his theory to problems arising in the calculus of variations by introducing the concept of variational symmetry groups [13, 14]. Later, in 1918, Noether published her celebrated theorem that associates every one-parameter variational symmetry group with a conservation law of the Euler–Lagrange equations [12].

Throughout this paper we will restrict our attention to a nondegenerate n-th order scalar variational problem involving a single variable integral, whose associated Euler–Lagrange equation is thus a $2n$-th order ordinary differential equation. The main result in this regard states that if an n-th order variational problem admits a one-parameter variational symmetry group, then the order of the associated Euler–Lagrange equation can be reduced by two [13, 14, 18]. In addition, the general solution of the original equation is recovered by a single quadrature from the general solution of the reduced Euler–Lagrange equation. In this sense, variational symmetry groups double the power of standard symmetry groups. It is also well known that if an n-th order ordinary differential equation admits a solvable r-parameter symmetry group G, then the order of the equation can be stepwise reduced by r. However, this result cannot be extrapolated to the case of variational symmetry groups: the order of the associated Euler–Lagrange equation cannot be reduced by $2r$, unless G is abelian. This fact is closely related to the Marsden–Weinstein reduction of Hamiltonian systems [6]. For the non-abelian case, the Hamiltonian framework enables one to determine the maximum degree of reduction by means of the residual symmetry group [13]. An integrability condition for the non-abelian case in terms of solvable structures [1] was presented in [7].

On the other hand, there exist integration techniques which cannot be explained by the classical symmetry analysis and generalizations of the Lie approach are required. To this end, inspired by examples developed in [13], particularly the method of nonlocal exponential symmetries in Exercises 2.31–32, the authors of [8] introduced the concept of a $\mathcal{C}^\infty$-symmetry (or λ-symmetry) based on a new way of prolonging vector fields, for which it is still possible to calculate differential invariants by derivation of lower-order invariants. This led to new reduction methods for ordinary differential equations based on the existence of $\mathcal{C}^\infty$-symmetries. Furthermore, the authors established in [9] the concept of variational $\mathcal{C}^\infty$-symmetry by considering this modified prolongation formula. They showed that variational $\mathcal{C}^\infty$-symmetries also provide a reduction method for Euler–Lagrange equations, although such reduction of order is somehow partial, meaning that the reconstructed solutions depend upon one fewer parameters than the general solution to the original Euler–Lagrange equations; see [9, p. 174] for details. The extension of the concept of $\mathcal{C}^\infty$-symmetries to partial differential equations led to the notion of μ-symmetry [5, 3], whose application to the variational framework has also been studied in the recent literature [4, 11].

The aim of this paper is to show how one can use what we call a *solvable pair* of variational $\mathcal{C}^\infty$-symmetries — meaning that they satisfy condition (20) below — to reduce the order of the Euler–Lagrange equation associated to an n-th order variational problem by four. The solvability condition ensures that one of the symmetries is inherited as variational $\mathcal{C}^\infty$-symmetry of the corresponding reduced variational problem. It should be noted that such a result cannot hold in the case of two standard, noncommuting variational symmetries. The method presented here allows one to recover a $(2n-2)$-parameter family of solutions for the original Euler–Lagrange equation by solving two successive first order ordinary differential equations. We leave the extension of our method to higher order reductions associated with more than two $\mathcal{C}^\infty$-symmetries to future investigations, and only remark that it is not, to the best of our knowledge, a straightforward extension of the results in this paper.

The paper is organized as follows. In Section 2, with the aim of being self-contained, we briefly introduce the basics concerning variational problems as well as the concept of variational symmetry and variational $\mathcal{C}^\infty$-symmetry. The solvability condition, which is crucial in the application of our method, requires the determination of the Lie bracket of two $\mathcal{C}^\infty$-prolonged (or λ-prolonged) vector fields. However, in contrast to the ordinary prolongation of vector fields, a convenient characterization for such commutator was not known. With this aim, we address in Section 3 the problem of determining the commutator of two $\mathcal{C}^\infty$-prolonged vector fields in evolutionary form, leading to a formula involving a new type of symmetry that remains to be investigated in detail.

In Section 4, we focus on providing an operative characterization of the solvability condition by using the results obtained in the previous section. We first study the case in which the vector fields are given in evolutionary form. The characterization obtained for evolutionary vector fields allows us to address the general case and obtain explicit necessary and sufficient conditions for solvability.

In Section 5, our procedure is described step by step. Firstly, we reduce the order of the original $2n$-th order Euler–Lagrange equation by two by means of the first variational $\mathcal{C}^\infty$-symmetry. After that, assuming that the solvability condition (20) is verified, we prove that the second variational $\mathcal{C}^\infty$-symmetry is inherited for the reduced variational problem and therefore it can be used to reduce the order of the reduced Euler–Lagrange equation again by two. As a result of the double reduction, we obtain a $(2n-4)$-th order Euler–Lagrange equation whose general solution can be used to reconstruct a $(2n-2)$-parameter family of solutions to the original equation by solving two first order ordinary differential equations. The loss of two parameters in the resulting family of solutions is due to the partial reduction associated to each variational $\mathcal{C}^\infty$-symmetry.

Finally, in Sections 6 and 7, the method is applied to two different examples corresponding to two second order variational problems. In the case of Example I, a two-parameter family of solutions to the associated fourth order Euler–Lagrange equation is obtained in terms of elementary functions. It should be remarked that the variational problem considered in Example I does not admit standard variational symmetries. On the other hand, Example II corresponds to a family of variational problems that admit a commuting (abelian) pair of $\mathcal{C}^\infty$-symmetries. The application of our method allows to obtain a two-parameter family of solutions to the Euler–Lagrange equation expressed in terms of the solutions to a Schrödinger-type equation. For a particular case of the family that does not admit standard variational symmetries such solution is explicitly expressed in terms of elementary functions.

2. Preliminaries

We refer the reader to [13, 14, 17] for the basics concerning jet spaces, symmetry groups of differential equations, and variational problems.

2.1. Previous results

We work with scalar variational problems, and so let $x \in X = \mathbb{R}$ denote the independent variable and $u \in U = \mathbb{R}$ the dependent variable. Consider an n-th order variational problem

$$\mathcal{L}[u] = \int_\Omega L(x, u^{(n)})dx, \tag{1}$$

where the Lagrangian $L(x, u^{(n)})$ is defined on (an open subset of) the n-th order jet space $M^{(n)}$, for some open set M of the space of independent and dependent variables $X \times U$, and $\Omega \subset X$ is connected, open, and contained in the projection of $M \to X$. The coordinates on $M^{(n)}$ are denoted by $(x, u^{(n)}) = (x, u, u_1, \ldots, u_n)$, with u_i corresponding to the i-th order derivative of u with respect to x. In general, a smooth real-valued function on $M^{(n)}$, depending on x, u, and finitely many derivatives of u, is called a *differential function* of *order* n.

The *Euler operator* or variational derivative is given by

$$E_u = \sum_{i=0}^{n} (-D_x)^i \frac{\partial}{\partial u_i}, \tag{2}$$

where $D_x = \partial_x + u_1\partial_u + u_2\partial_{u_1} + \cdots$ is the usual total derivative operator with respect to x. Thus,

$$E_u[L] = 0 \tag{3}$$

is the Euler–Lagrange equation associated to (1).

Roughly speaking, by a variational symmetry group of (1) we mean a local group of transformations G that leaves $\mathcal{L}[u]$ unchanged (modulo boundary terms) when evaluated on functions $u = f(x)$ whose graph is transformed by the action of the group on M. A connected group of transformations G forms a variational symmetry group if and only if, for every infinitesimal generator

$$\mathbf{v} = \xi(x, u)\partial_x + \eta(x, u)\partial_u \tag{4}$$

of G, the following infinitesimal symmetry condition holds:

$$\mathbf{v}^{(n)}(L) + LD_x(\xi) = 0, \tag{5}$$

where $\mathbf{v}^{(n)}$ stands for the standard n-th order prolongation of the vector field $\mathbf{v}$, [13]. In what follows, by a *variational symmetry* we will simply mean a smooth non-zero vector field (4) locally defined on M which satisfies (5).

It is well known that one variational symmetry of (1) allows to reduce the order of the associated Euler–Lagrange equation (3) by two. Furthermore, the solution of the original Euler–Lagrange equation can be recovered by a single quadrature from the solution of this reduced equation; see [13] and references therein. However, the existence of a two-parameter variational symmetry group does not assure the existence of a reduction in order by four for the Euler–Lagrange equation.

2.2. $\mathcal{C}^\infty$-symmetries and variational $\mathcal{C}^\infty$-symmetries

The goal of this paper is to study the problem of the double reduction of order from the point of view of the variational $\mathcal{C}^\infty$-symmetries, first introduced in [9]. The construction is based on a new way of prolonging vector fields. For a given smooth vector field (4) defined on M and an arbitrary first order differential function $\lambda(x,u,u_1) \in \mathcal{C}^\infty(M^{(1)})$, the (*infinite*) *$\lambda$-prolongation* of $\mathbf{v}$ is the vector field

$$\mathbf{v}^\lambda = \xi(x,u)\partial_x + \sum_{i=0}^{\infty} \eta^{[\lambda,(i)]}(x,u^{(i)})\partial_{u_i}, \tag{6}$$

defined on $M^{(\infty)}$, where $u_0 = u$, $\eta^{[\lambda,(0)]}(x,u) = \eta(x,u)$ and, for $i \geq 1$,

$$\begin{aligned} \eta^{[\lambda,(i)]}(x,u^{(i)}) = D_x\left(\eta^{[\lambda,(i-1)]}(x,u^{(i-1)})\right) &- D_x(\xi(x,u))u_i \\ &+ \lambda\left(\eta^{[\lambda,(i-1)]}(x,u^{(i-1)}) - \xi(x,u)u_i\right). \end{aligned} \tag{7}$$

A vector field of the form (6) will be called a $\mathcal{C}^\infty$-prolonged (or λ-prolonged) vector field. When $\lambda = 0$, this new prolongation formula reduces to the ordinary prolongation of the vector field. Observe that when we evaluate $\mathbf{v}^\lambda(P)$ on any differential function P, only finitely many terms in the sum are needed. It can be proved, [8, Theorem 2.1], that $\mathbf{v}^\lambda$ is the unique vector field which satisfies the following commutation relation:

$$[\ \mathbf{v}^\lambda, D_x\ ] = \lambda\mathbf{v}^\lambda + \rho D_x, \tag{8}$$

where $\rho = -(D_x + \lambda)\xi$.

Let us next present the definition of a variational $\mathcal{C}^\infty$-symmetry. The λ-prolongation formula motivates the following, cf. [9, Definition 2.1]:

Definition 2.1. *Given a smooth vector field* $\mathbf{v}$ *defined on* M*, as in* (4)*, and* $\lambda \in \mathcal{C}^\infty(M^{(1)})$*, the pair* $(\mathbf{v},\lambda)$ *is a* variational $\mathcal{C}^\infty$-symmetry (*or variational λ-symmetry*) *of the functional* (1) *with Lagrangian L provided the following infinitesimal invariance condition holds:*

$$\mathbf{v}^\lambda(L) + L(D_x + \lambda)\xi = 0. \tag{9}$$

In particular, when $\lambda = 0$, condition (9) reduces to the standard infinitesimal variational symmetry condition (5).

The importance of variational $\mathcal{C}^\infty$-symmetries is that they provide new reduction procedures for Euler–Lagrange equations, as spelled out in the following result, proved in [9, Theorem 1]:

Theorem 2.2. *Let $\mathcal{L}[u] = \int L(x, u^{(n)})dx$ be an n-th order variational problem with Euler–Lagrange equation $E_u[L] = 0$, of order $2n$. Let $(\mathbf{v}, \lambda)$ be a variational $\mathcal{C}^\infty$-symmetry of the problem. Then there exists a variational problem $\widetilde{\mathcal{L}}[w] = \int \widetilde{L}(y, w^{(n-1)})dy$, of order $n-1$, with Euler–Lagrange equation $E_w[\widetilde{L}] = 0$ of order $2n-2$, such that a $(2n-1)$-parameter family of solutions of $E_u[L] = 0$ can be found by solving a first order equation from the solutions of the reduced Euler–Lagrange equation $E_w[\widetilde{L}] = 0$.*

An outline of how the reduction method associated to a variational $\mathcal{C}^\infty$-symmetry works is described in Subsections 5.1 and 5.3.

3. The commutator of $\mathcal{C}^\infty$-prolonged vector fields in evolutionary form

For ordinary infinitesimal symmetries, the commutator of two vector fields determines the commutator of their corresponding prolongations. However, this result does not hold for the case of $\mathcal{C}^\infty$-prolonged vector fields associated with two different first order differential functions λ and μ. The key to the reduction procedure presented in this paper is to consider a pair of variational $\mathcal{C}^\infty$-symmetries satisfying a certain solvability condition involving their commutators — see (20) below. Thus, in this section, we address the outstanding problem of determining the commutator of two $\mathcal{C}^\infty$-prolonged vector fields.

Consider the $\mathcal{C}^\infty$-prolonged vector field

$$\mathbf{v}_Q^\lambda = \sum_{n=0}^{\infty} Q^{[\lambda,(n)]} \frac{\partial}{\partial u_n} \tag{10}$$

which is the λ-prolongation of an evolutionary vector field $\mathbf{v}_Q = Q\,\partial_u$, so that

$$Q^{[\lambda,(0)]} = Q \quad \text{and} \quad Q^{[\lambda,(n)]} = (D_x + \lambda)^n Q, \quad n \geq 1.$$

Note that

$$\mathbf{v}_Q^\lambda\, D_x = (D_x + \lambda)\mathbf{v}_Q^\lambda. \tag{11}$$

Let $\mathbf{v}_R^\mu$ be another $\mathcal{C}^\infty$-prolonged vector field, the μ-prolongation of $\mathbf{v}_R = R\,\partial_u$ and assume that $\mathbf{v}_Q^\lambda$ and $\mathbf{v}_R^\mu$ are pointwise linearly independent. From this point on, in order to streamline the notation, we denote $Q^{(n)} = Q^{[\lambda,(n)]}$ and $R^{(n)} = R^{[\mu,(n)]}$, for $n \geq 1$. We are interested in their commutator

$$\mathbf{v}_S^* = [\,\mathbf{v}_Q^\lambda, \mathbf{v}_R^\mu\,] = \sum_{n=0}^{\infty} S_n \frac{\partial}{\partial u_n}. \tag{12}$$

When $\lambda \neq \mu$ this is almost never a $\mathcal{C}^\infty$-prolonged vector field. However, we can establish a recurrence formula for its coefficients S_n as follows. First, the coefficient of ∂_u is

$$S = S_0 = \mathbf{v}_Q^\lambda(R) - \mathbf{v}_R^\mu(Q) = \mathbf{v}_Q(R) - \mathbf{v}_R(Q). \tag{13}$$

We next compute, using (11) and the analogous equation for $\mathbf{v}_R^\mu$,

$$\begin{aligned} S_1 &= \mathbf{v}_Q^\lambda(R^{(1)}) - \mathbf{v}_R^\mu(Q^{(1)}) \\ &= \mathbf{v}_Q^\lambda[\,(D_x+\mu)R\,] - \mathbf{v}_R^\mu[\,(D_x+\lambda)Q\,] \\ &= (D_x+\lambda+\mu)[\,\mathbf{v}_Q^\lambda(R) - \mathbf{v}_R^\mu(Q)\,] + \mathbf{v}_Q^\lambda(\mu)R - \mathbf{v}_R^\mu(\lambda)Q \\ &= (D_x+\lambda+\mu)S + \mathbf{v}_Q^\lambda(\mu)R - \mathbf{v}_R^\mu(\lambda)Q. \end{aligned}$$

In general,

$$\begin{aligned} S_n &= \mathbf{v}_Q^\lambda(R^{(n)}) - \mathbf{v}_R^\mu(Q^{(n)}) \\ &= \mathbf{v}_Q^\lambda[\,(D_x+\mu)R^{(n-1)}\,] - \mathbf{v}_R^\mu[\,(D_x+\lambda)Q^{(n-1)}\,] \\ &= (D_x+\lambda+\mu)[\,\mathbf{v}_Q^\lambda(R^{(n-1)}) - \mathbf{v}_R^\mu(Q^{(n-1)})\,] + \mathbf{v}_Q^\lambda(\mu)R^{(n-1)} - \mathbf{v}_R^\mu(\lambda)Q^{(n-1)} \\ &= (D_x+\lambda+\mu)S_{n-1} + \widetilde{S}_n, \end{aligned} \tag{14}$$

where we set

$$\widetilde{S}_0 = S = \mathbf{v}_Q^\lambda(R) - \mathbf{v}_R^\mu(Q), \qquad \widetilde{S}_i = \mathbf{v}_Q^\lambda(\mu)R^{(i-1)} - \mathbf{v}_R^\mu(\lambda)Q^{(i-1)}, \quad i \geq 1. \tag{15}$$

The recursive formula for generating the coefficients of the commutator (12) can thus be explicitly written as

$$S_n = \sum_{i=0}^{n} (D_x+\lambda+\mu)^i\,\widetilde{S}_{n-i}. \tag{16}$$

Consequently, we can write the commutator (12) in the following compelling form:

$$\mathbf{v}_S^* = [\,\mathbf{v}_Q^\lambda, \mathbf{v}_R^\mu\,] = \sum_{k=0}^{\infty} \widetilde{\mathbf{v}}_k^{\lambda+\mu}, \tag{17}$$

where

$$\widetilde{\mathbf{v}}_k^{\lambda+\mu} = \sum_{n=k}^{\infty} [\,(D_x+\lambda+\mu)^{n-k}\widetilde{S}_k\,]\frac{\partial}{\partial u_n} \tag{18}$$

can, formally, be identified with the $\lambda+\mu$ prolongation of the k-th order vector field

$$\widetilde{\mathbf{v}}_k = \widetilde{S}_k\frac{\partial}{\partial u_k}. \tag{19}$$

In particular, the first summand $\widetilde{\mathbf{v}}_0^{\lambda+\mu} = \mathbf{v}_S^{\lambda+\mu}$ in the commutator formula (17) is the $\lambda+\mu$ prolongation of the evolutionary vector field $\widetilde{\mathbf{v}}_0 = \mathbf{v}_S$. However, the additional terms tell us that $\mathbf{v}_S^*$ is not (usually) a $\mathcal{C}^\infty$-prolonged vector field. Moreover, one *cannot* compute these terms just from the expressions of S, λ, μ; one also needs to know Q and R. The interesting question is whether these kinds of vector fields lead to order reductions of ordinary differential equations, even though they project onto null vector fields on the base manifold. As far as we know, this question has not been considered in the literature to date, and so it remains an open problem which will be investigated in future research.

4. The solvability condition

Suppose that the functional (1) admits two variational $\mathcal{C}^\infty$-symmetries $(\mathbf{v}_1, \lambda_1)$ and $(\mathbf{v}_2, \lambda_2)$, where $\mathbf{v}_1, \mathbf{v}_2$ are smooth vector fields on M, and $\lambda_1, \lambda_2 \in \mathcal{C}^\infty(M^{(1)})$. In order to apply our reduction procedure, we will assume that the vector fields $\mathbf{v}_1^{\lambda_1}$ and $\mathbf{v}_2^{\lambda_2}$ are pointwise linearly independent on $M^{(\infty)}$, and that they satisfy the additional solvability condition

$$\left[\, \mathbf{v}_1^{\lambda_1}, \mathbf{v}_2^{\lambda_2} \right] \; = h\, \mathbf{v}_1^{\lambda_1}, \tag{20}$$

for some function $h \in \mathcal{C}^\infty(M)$. The solvability condition (20) is analogous to the usual condition for a two-dimensional solvable Lie algebra of ordinary symmetries. We will call such vector fields a *solvable pair* of variational $\mathcal{C}^\infty$-symmetries.

Remark: The linear independence condition on $\mathbf{v}_1^{\lambda_1}$ and $\mathbf{v}_2^{\lambda_2}$ does *not* require that the original vector fields $\mathbf{v}_1, \mathbf{v}_2$ be pointwise linearly independent on M. Indeed, in our second example, they are the same vector field on M, but have different functions λ_1, λ_2 prescribing their linearly independent prolongations to $M^{(\infty)}$.

At first sight, condition (20) has to be checked by using induction on the order of the λ-prolongation in order to assure that the corresponding variational $\mathcal{C}^\infty$-symmetries form a solvable pair. This procedure seems to be difficult in practice, therefore we focus on establishing a characterization of the solvability condition (20). We consider first the case in which the vector fields are given in evolutionary form by using the results obtained in Section 3.

4.1. The solvability condition for $\mathcal{C}^\infty$-prolonged vector fields in evolutionary form

According to the previous notation, we consider the case in which $\mathbf{v}_1 = \mathbf{v}_Q = Q\,\partial_u$ and $\mathbf{v}_2 = \mathbf{v}_R = R\,\partial_u$. We also set $\lambda_1 = \lambda$ and $\lambda_2 = \mu$. Now let us investigate a solvability condition in the form

$$\mathbf{v}_S^* = [\mathbf{v}_Q^\lambda, \mathbf{v}_R^\mu] = h\, \mathbf{v}_Q^\lambda. \tag{21}$$

This requires

$$S_n = h\, Q^{(n)} = h\, (D_x + \lambda)^n Q. \tag{22}$$

Assuming $Q \neq 0$, we can always write

$$S = h\, Q \qquad \text{with} \qquad h = S/Q, \tag{23}$$

so there are no conditions at order 0. Working by induction, suppose we know (22) holds at order $n-1$. Using (14) and the induction hypothesis, we compute

$$\begin{aligned} S_n &= (D_x+\lambda+\mu)S_{n-1} + \mathbf{v}_Q^\lambda(\mu)R^{(n-1)} - \mathbf{v}_R^\mu(\lambda)Q^{(n-1)} \\ &= (D_x+\lambda+\mu)[h\,Q^{(n-1)}] + \mathbf{v}_Q^\lambda(\mu)R^{(n-1)} - \mathbf{v}_R^\mu(\lambda)Q^{(n-1)} \\ &= h\,(D_x+\lambda)Q^{(n-1)} + [\,(D_x+\mu)h - \mathbf{v}_R^\mu(\lambda)\,]Q^{(n-1)} + \mathbf{v}_Q^\lambda(\mu)R^{(n-1)} \\ &= h\,Q^{(n)} + [\,(D_x+\mu)h - \mathbf{v}_R^\mu(\lambda)\,]Q^{(n-1)} + \mathbf{v}_Q^\lambda(\mu)R^{(n-1)}. \end{aligned}$$

Thus, as $\mathbf{v}_Q^\lambda$ and $\mathbf{v}_R^\mu$ are pointwise linearly independent, the induction step is valid provided

$$\mathbf{v}_Q^\lambda(\mu) = 0, \qquad \text{and} \qquad \mathbf{v}_R^\mu(\lambda) = (D_x+\mu)h. \tag{24}$$

As a consequence of the previous discussion, the following proposition has been proved:

Proposition 4.1. *Let $\mathbf{v}_Q = Q\,\partial_u$ and $\mathbf{v}_R = R\,\partial_u$ be two evolutionary vector fields and consider two different first order differential functions λ and μ such that $\mathbf{v}_Q^\lambda$ and $\mathbf{v}_R^\mu$ are pointwise linearly independent. Then we have that $[\mathbf{v}_Q^\lambda, \mathbf{v}_R^\mu] = h\,\mathbf{v}_Q^\lambda$ if and only if*

$$\mathbf{v}_Q^\lambda(\mu) = 0, \qquad \text{and} \qquad \mathbf{v}_R^\mu(\lambda) = (D_x+\mu)h. \tag{25}$$

In such a case the function h is given by $h = (\mathbf{v}_Q(R) - \mathbf{v}_R(Q))/Q$.

4.2. The solvability condition for the general case

The characterization for the solvability condition obtained for the case of $\mathcal{C}^\infty$-prolonged vector fields in evolutionary form allows us to address the general case. Assuming $\mathbf{v}_1$ does not vanish in a neighborhood of a point of M, we can locally choose rectifying coordinates (x,u) for the vector field $\mathbf{v}_1$ so that

$$\mathbf{v}_1 = \partial_u, \qquad \mathbf{v}_2 = \xi(x,u)\partial_x + \eta(x,u)\partial_u, \tag{26}$$

where, to streamline the notation, we omit the 2 subscripts on the coefficients of the second infinitesimal generator. Throughout this subsection we will use the following commutation relations, which hold by (8):

$$[\mathbf{v}_1^{\lambda_1}, D_x] = \lambda_1\mathbf{v}_1^{\lambda_1}, \qquad [\mathbf{v}_2^{\lambda_2}, D_x] = \lambda_2\mathbf{v}_2^{\lambda_2} + \rho_2 D_x, \tag{27}$$

where

$$\rho_2 = -(D_x+\lambda_2)\,\xi \in \mathcal{C}^\infty(M^{(1)}).$$

Theorem 4.2. *Let $(\mathbf{v}_1,\lambda_1),(\mathbf{v}_2,\lambda_2)$ be variational $\mathcal{C}^\infty$-symmetries of the form (26). Then they form a solvable pair, i.e, (20) is satisfied for some function $h(x,u)$, if and only if $h = \partial\eta/\partial u$ and the functions ξ,λ_1,λ_2 and $\rho_2 = -(D_x+\lambda_2)\,\xi$ satisfy the four conditions*

$$\textit{(i)}\ \frac{\partial\xi}{\partial u} = 0, \quad \textit{(ii)}\ \mathbf{v}_1^{\lambda_1}(\lambda_2) = 0, \quad \textit{(iii)}\ \mathbf{v}_1^{\lambda_1}(\rho_2) = 0, \quad \textit{(iv)}\ \mathbf{v}_2^{\lambda_2}(\lambda_1) = (D_x+\lambda_2)h + \rho_2\lambda_1.$$

Proof. Let us assume that $(\mathbf{v}_1,\lambda_1)$ and $(\mathbf{v}_2,\lambda_2)$ form a solvable pair. Evaluating both members of (20) on the coordinate function u shows that $h=\partial\eta/\partial u$. On the other hand, if we do the same for the coordinate function x then we find that $\mathbf{v}_1^{\lambda_1}(\xi)=\partial\xi/\partial u=0$, which proves condition *(i)*.

Let $\mathbf{v}_R=R\,\partial_u$ be the evolutionary form of the vector field $\mathbf{v}_2=\xi(x,u)\partial_x+\eta(x,u)\partial_u$, where $R=\eta-u_1\xi$. It can be checked that

$$\mathbf{v}_2^{\lambda_2}=\mathbf{v}_R^{\lambda_2}+\xi\,D_x. \tag{28}$$

By using (28), $\mathbf{v}_1^{\lambda_1}(\xi)=0$ and the first formula in (27), we have that

$$[\mathbf{v}_1^{\lambda_1},\mathbf{v}_2^{\lambda_2}]=[\mathbf{v}_1^{\lambda_1},\mathbf{v}_R^{\lambda_2}+\xi\,D_x]=[\mathbf{v}_1^{\lambda_1},\mathbf{v}_R^{\lambda_2}]+\xi\,[\mathbf{v}_1^{\lambda_1},D_x]=[\mathbf{v}_1^{\lambda_1},\mathbf{v}_R^{\lambda_2}]+\xi\,\lambda_1\,\mathbf{v}_1^{\lambda_1}. \tag{29}$$

As we are assuming that the variational $\mathcal{C}^\infty$-symmetries form a solvable pair, it follows from (29) that

$$[\mathbf{v}_1^{\lambda_1},\mathbf{v}_R^{\lambda_2}]=\widetilde{h}\,\mathbf{v}_1^{\lambda_1},\quad \text{where}\quad \widetilde{h}=h-\xi\lambda_1. \tag{30}$$

Observe that (30) corresponds to the solvability condition (21) for $\mathcal{C}^\infty$-prolonged vector fields in evolutionary form for the particular case of $Q=1$. In consequence, by Proposition 4.1, we deduce that

$$\mathbf{v}_1^{\lambda_1}(\lambda_2)=0,\quad \mathbf{v}_R^{\lambda_2}(\lambda_1)=(D_x+\lambda_2)\,\widetilde{h}. \tag{31}$$

The first condition in (31) proves *(ii)*. By using (28) and $\widetilde{h}=h-\xi\lambda_1$, the second condition in (31) yields *(iv)*.

In order to prove *(iii)*, observe that by the first commutation relation in (27) and by condition *(ii)*, we have that for any differential function f the following relation holds:

$$\begin{aligned}\mathbf{v}_1^{\lambda_1}\left(\,(D_x+\lambda_2)\,f\,\right) &= \mathbf{v}_1^{\lambda_1}(D_x(f))+\lambda_2\mathbf{v}_1^{\lambda_1}(f)\\ &= \lambda_1\mathbf{v}_1^{\lambda_1}(f)+D_x(\mathbf{v}_1^{\lambda_1}(f))+\lambda_2\mathbf{v}_1^{\lambda_1}(f)\\ &= (D_x+\lambda_1+\lambda_2)\,\mathbf{v}_1^{\lambda_1}(f).\end{aligned} \tag{32}$$

In particular, for $f=-\xi$,

$$\mathbf{v}_1^{\lambda_1}(\rho_2)=\mathbf{v}_1^{\lambda_1}(-(D_x+\lambda_2)\,\xi)=-(D_x+\lambda_1+\lambda_2)\mathbf{v}_1^{\lambda_1}(\xi)=0. \tag{33}$$

Vice versa, if *(i-iv)* hold, then both conditions in (25) are satisfied for the evolutionary vector fields $\mathbf{v}_1$ and $\mathbf{v}_R$ and the differential functions λ_1 and λ_2. Therefore, by Proposition 4.1, we have that $[\mathbf{v}_1^{\lambda_1},\mathbf{v}_R^{\lambda_2}]=\widetilde{h}\,\mathbf{v}_1^{\lambda_1}$, for $\widetilde{h}=h-\xi\,\lambda_1$. By (29) we deduce that

$$[\mathbf{v}_1^{\lambda_1},\mathbf{v}_2^{\lambda_2}]=h\,\mathbf{v}_1^{\lambda_1},$$

hence the variational $\mathcal{C}^\infty$-symmetries form a solvable pair and the theorem is proved.

□

As a consequence of Theorem 4.2 we obtain the following corollary, which provides an alternative characterization to the solvability condition (20):

Corollary 4.3. *Two variational $\mathcal{C}^\infty$-symmetries* $(\mathbf{v}_1, \lambda_1), (\mathbf{v}_2, \lambda_2)$ *of the form* (26) *are a solvable pair if and only if*

a) $[\mathbf{v}_1^{[\lambda_1,(1)]}, \mathbf{v}_2^{[\lambda_2,(1)]}] = h\, \mathbf{v}_1^{[\lambda_1,(1)]}$, *where* $h = \partial\eta/\partial u$,

b) $\mathbf{v}_1^{\lambda_1}(\lambda_2) = 0.$

Proof. For the necessary condition, *a)* follows immediately from (20) and *b)* is condition *(ii)* in Theorem 4.2.

Conversely, we first observe that applying the commutator in condition *a)* to the local coordinate function x yields

$$\mathbf{v}_1^{\lambda_1}(\xi) = \frac{\partial \xi}{\partial u} = 0, \tag{34}$$

which corresponds to condition *(i)* in Theorem 4.2. By the first commutation relation in (27) and condition *b)*, it can be checked, as in the proof of Theorem 4.2, that relation (32) holds for any differential function f. In particular, for $f = -\xi$ we get

$$\mathbf{v}_1^{\lambda_1}(\rho_2) = \mathbf{v}_1^{\lambda_1}\Big(-(D_x + \lambda_2)\, \xi\Big) = 0, \tag{35}$$

and for $f = \eta$:

$$\mathbf{v}_1^{\lambda_1}\Big((D_x + \lambda_2)\, \eta\Big) = (D_x + \lambda_1 + \lambda_2)\, \mathbf{v}_1^{\lambda_1}(\eta) = (D_x + \lambda_1 + \lambda_2)\, h, \quad \text{where} \quad h = \frac{\partial \eta}{\partial u}. \tag{36}$$

On the other hand, if we apply the commutator in condition *a)* to the local coordinate u_1 then we obtain:

$$\mathbf{v}_1^{\lambda_1}\Big((D_x + \lambda_2)\, \eta - u_1 (D_x + \lambda_2)\, \xi\Big) - \mathbf{v}_2^{\lambda_2}(\lambda_1) = h\, \lambda_1, \tag{37}$$

which, by (35), (36) and a straightforward computation, yields

$$\mathbf{v}_2^{\lambda_2}(\lambda_1) = (D_x + \lambda_2)\, h + \rho_2\, \lambda_1. \tag{38}$$

Observe that (34), (35), (38) and condtion *b)* prove that conditions *(i–iv)* in Theorem 4.2 are satisfied and in consequence the variational $\mathcal{C}^\infty$-symmetries form a solvable pair. □

The results obtained in this section prove that the solvability condition (20), which involves infinite λ-prolongations, can be checked through relations defined only on the first order jet space. In particular, the characterization given in Corollary 4.3 allows one to check the solvability condition by just considering the first order λ-prolongations and verifying that the function λ_2 is a first order invariant of the first variational $\mathcal{C}^\infty$-symmetry.

5. Reduction of order by use of two variational $\mathcal{C}^\infty$-symmetries

We present now the double reduction of order associated to a pair of variational $\mathcal{C}^\infty$-symmetries $(\mathbf{v}_1, \lambda_1)$ and $(\mathbf{v}_2, \lambda_2)$ that form a solvable pair and where $\mathbf{v}_1$ and $\mathbf{v}_2$ are of the form of (26). According to (9), the following conditions hold:

$$\mathbf{v}_1^{\lambda_1}(L) = 0, \qquad \mathbf{v}_2^{\lambda_2}(L) + L(D_x + \lambda_2)\xi = 0. \tag{39}$$

5.1. Reduction associated with the variational $\mathcal{C}^\infty$-symmetry (∂_u, λ_1)

In this subsection, we focus on the use of the variational $\mathcal{C}^\infty$-symmetry $(\mathbf{v}_1, \lambda_1)$ to reduce the order of the original Euler–Lagrange equation $E_u[L] = 0$ by two, as spelled out in Theorem 2.2. For that purpose, let us introduce a first order invariant of $\mathbf{v}_1^{\lambda_1}$, i.e, a function $w = w(x, u, u_1)$ such that

$$\mathbf{v}_1^{\lambda_1}(w) = \frac{\partial w}{\partial u} + \lambda_1 \frac{\partial w}{\partial u_1} = 0. \tag{40}$$

Observe that, by condition *(ii)* in Theorem 4.2, the general solution to (40) has the form $w = \varphi(x, \lambda_2)$. In applications, we will choose w to simply be a convenient multiple of λ_2. The differential functions

$$x, \quad w_i = \frac{d^i w}{dx^i}, \qquad i = 1, \ldots, n-1,$$

form a complete system of invariants of order $\leq n$ of the vector field $\mathbf{v}_1^{\lambda_1}$. As a consequence of the first formula in (39), the original Lagrangian $L = L(x, u^{(n)})$ can be locally expressed as a reduced Lagrangian

$$\widetilde{L} = \widetilde{L}(x, w^{(n-1)}), \tag{41}$$

defined on the reduced jet space $M_1^{(n-1)}$, where M_1 is the domain of definition of the coordinates (x, w).

As in [9, formula (28)], the effect of the transformation $\{x = x, w = w(x, u, u_1)\}$ on the Euler–Lagrange equation is

$$E_u[L] = \frac{\partial w}{\partial u} E_w[\widetilde{L}] - D_x \left(\frac{\partial w}{\partial u_1} E_w[\widetilde{L}] \right), \tag{42}$$

which by (40) yields

$$E_u[L] = -\left(D_x + \lambda_1\right) \left(\frac{\partial w}{\partial u_1} E_w[\widetilde{L}] \right). \tag{43}$$

Formula (43) will be used later in the reconstruction of solutions after the reduction of order: if $w = h(x)$ is a solution of the reduced Euler–Lagrange equation $E_w[\widetilde{L}] = 0$, then any solution $u = f(x)$ to the first order auxiliary ordinary differential equation $w(x, u, u_1) = h(x)$ satisfies the original Euler–Lagrange equation $E_u[L] = 0$.

We recall that in the standard symmetry reduction procedure, the function λ_1 does not appear in formula (43) and therefore any solution to the reduced Euler-Lagrange equation $E_w[\tilde{L}] = c$, where $c \in \mathbb{R}$, produces a solution to the original Euler-Lagrange equation $E_u[L] = 0$. However, according to (43), the reduced equation associated to a variational $\mathcal{C}^\infty$-symmetry is $E_w[\tilde{L}] = 0$ (i.e. $c = 0$), which produces the reduction by one in the number of parameters after reconstructing the corresponding family of solutions to the original Euler-Lagrange equation.

5.2. The variational $\mathcal{C}^\infty$-symmetry inherited from $(\mathbf{v}_2, \lambda_2)$

In this subsection we will prove that $(\mathbf{v}_2, \lambda_2)$ provides a variational $\mathcal{C}^\infty$-symmetry of the variational problem associated to the reduced Lagrangian (41). For that purpose, let us observe first that:

(i) By conditions *(ii)* and *(iii)* in Theorem 4.2, both ρ_2 and λ_2 can be written in terms of the set of invariants x, w, and we use $\widetilde{\rho}_2$ and $\widetilde{\lambda}_2$ to indicate the resulting expressions.

(ii) Condition (20) implies that the vector field $\mathbf{v}_2^{\lambda_2}$, once it has been expressed in terms of the local variables x, u, w_i, for $i \geq 0$, can be projected [15] onto a well-defined vector field on the space $M_1^{(\infty)}$, denoted

$$\mathbf{X} = \xi(x)\partial_x + \sum_{i=1}^{\infty} \tau_{i-1}(x, w^{(i-1)})\partial_{w_{i-1}}; \tag{44}$$

see [10, p. 480] for further details.

(iii) The total derivative operator D_x also projects onto the reduced total differential operator

$$\widetilde{D}_x = \partial_x + \sum_{i=0}^{\infty} w_{i+1}\partial_{w_i}, \tag{45}$$

which acts on the differential functions depending on x, w and derivatives of w.

The remarks (i), (ii), and (iii) imply that both members of the second commutation relation in (27), once expressed in terms of the local variables x, u, w_i, can be projected to $M_1^{(\infty)}$, where the following relation holds:

$$[\mathbf{X}, \widetilde{D}_x] = \widetilde{\lambda}_2\mathbf{X} + \widetilde{\rho}_2\widetilde{D}_x. \tag{46}$$

Evaluating both sides of (46) on the coordinate function x, we deduce that

$$\widetilde{\rho}_2 = -(\widetilde{D}_x + \widetilde{\lambda}_2)\,\xi.$$

On the other hand, the λ_2 prolongation $\mathbf{Y}^{\widetilde{\lambda}_2}$ of the vector field

$$\mathbf{Y} = \xi(x)\partial_x + \tau(x, w)\partial_w \tag{47}$$

also satisfies the commutation relation (46). Since $\mathbf{Y}$ has the same the order zero components of (44), as in [8], we conclude that they must agree:

$$\mathbf{X} = \mathbf{Y}^{\widetilde{\lambda}_2}. \tag{48}$$

Since $\widetilde{L}$ and ξ do not depend on u, the second formula in (39) can be written in terms of the local variables x, w_i, which, by (48) , yields

$$\mathbf{Y}^{\widetilde{\lambda}_2}(\widetilde{L}) + \widetilde{L}(\widetilde{D}_x + \widetilde{\lambda}_2)\,\xi = 0. \tag{49}$$

This serves to prove the following theorem:

Theorem 5.1. *The pair $(\mathbf{Y}, \widetilde{\lambda}_2)$ forms a variational $\mathcal{C}^\infty$-symmetry of the variational problem associated to the reduced Lagrangian $\widetilde{L} = \widetilde{L}(x, w^{(n-1)})$ obtained after the reduction process carried out with the variational $\mathcal{C}^\infty$-symmetry (∂_u, λ_1).*

5.3. Reduction associated with $(\mathbf{Y}, \widetilde{\lambda}_2)$

Assuming the vector field (47) does not vanish in a neighborhood of a point, we introduce the rectifying change of variables

$$y = y(x, w), \qquad \alpha = \alpha(x, w), \tag{50}$$

so that $\mathbf{Y} = \partial_\alpha$. By [9, Proposition 2.2] the pair $(\mathbf{Y}, \overline{\lambda}_2)$, where $\overline{\lambda}_2 = \widetilde{\lambda}_2 / D_x(y)$, remains a variational $\mathcal{C}^\infty$-symmetry of the transformed functional $\overline{\mathcal{L}}[\alpha] = \int \overline{L}(y, \alpha^{(n-1)})dy$. In these local coordinates, the infinitesimal symmetry condition (49) becomes:

$$\mathbf{Y}^{\overline{\lambda}_2}(\overline{L}) = 0. \tag{51}$$

Let us introduce a function $z = z(y, \alpha, \alpha_1)$ that satisfies

$$\mathbf{Y}^{\overline{\lambda}_2}(z) = \frac{\partial z}{\partial \alpha} + \overline{\lambda}_2 \frac{\partial z}{\partial \alpha_1} = 0. \tag{52}$$

Then, as above,

$$y, \; z_i = \frac{d^i z}{dy^i}, \quad i = 0, \ldots, n-2.$$

form a complete system of invariants of $\mathbf{Y}^{\overline{\lambda}_2}$. Thus, by (51), we can re-express the Lagrangian $\overline{L}(y, \alpha^{(n-1)})$ in the reduced form $\widehat{L}(y, z^{(n-2)})$, whose associated Euler–Lagrange equation, $E_z[\widehat{L}] = 0$, is a $(2n-4)$-th order ordinary differential equation.

Furthermore, by proceeding as in the last part of Subsection 3.1, we deduce that both Euler–Lagrange equations $E_\alpha[\overline{L}] = 0$ and $E_z[\widetilde{L}] = 0$ are related by means of the formula

$$E_\alpha[\overline{L}] = -\left(D_y + \overline{\lambda}_2\right) \left(\frac{\partial z}{\partial \alpha_1} E_z[\widehat{L}] \right) . \tag{53}$$

5.4. Reconstruction of solutions

Let us now discuss how to recover solutions to the original variational problem from solutions to the reduced Euler–Lagrange equation.

- *First reconstruction*
  Let $z = F(y; C_1, \cdots, C_{2n-4})$, where $C_i \in \mathbb{R}$ for $i = 1, \cdots, 2n-4$, be the general solution of the $(2n-4)$-th order ordinary differential equation $E_z[\widehat{L}] = 0$. This solution annihilates the right-hand side of (53), which implies that
  $$z(y, \alpha, \alpha_1) = F(y; C_1, \cdots, C_{2n-4}) \tag{54}$$
  satisfies the Euler–Lagrange equation $E_\alpha[\overline{L}] = 0$. On the other hand, the change of variables formula for Euler–Lagrange equations [13, Theorem 4.8] implies
  $$E_w[\widetilde{L}] = \left| \frac{\partial(y, \alpha)}{\partial(x, w)} \right| E_\alpha[\overline{L}].$$
  Therefore, the solution (54), once re-expressed in terms of the coordinates $\{x, w, w_1\}$, also satisfies the Euler–Lagrange equation $E_w[\widetilde{L}] = 0$. We conclude that a $(2n-3)$-parameter family of solutions of $E_w[\widetilde{L}] = 0$ can be found by solving the first order ordinary differential equation
  $$z(y(x, w), \alpha(x, w), \alpha_1(x, w, w_1)) = F(y(x, w); C_1, \cdots, C_{2n-4}). \tag{55}$$
- *Second reconstruction*
  Let $w = H(x; C_1, \cdots, C_{2n-3})$ be the general solution of equation (55). By (43), this solution also satisfies the original Euler–Lagrange equation $E_u[L] = 0$. Therefore a $(2n-2)$-parameter family of solutions of $E_u[L] = 0$ can be obtained by solving the first order ordinary differential equation
  $$w(x, u, u_1) = H(x; C_1, \cdots, C_{2n-3}). \tag{56}$$

As a consequence of the preceding discussion, we have established the key theorem.

Theorem 5.2. *Let $\mathcal{L}[u] = \int L(x, u^{(n)})dx$ be an n-th order variational problem with Euler–Lagrange equation $E_u[L] = 0$, of order $2n$. Let $(\mathbf{v}_1, \lambda_1), (\mathbf{v}_2, \lambda_2)$ be variational $\mathcal{C}^\infty$-symmetries that form a solvable pair, as in (20). Then there exists a variational problem $\widehat{\mathcal{L}}[z] = \int \widehat{L}(x, z^{(n-2)})dx$ of order $n-2$ such that a $(2n-2)$-parameter family of solutions of $E_u[L] = 0$ can be reconstructed from the solutions of the associated $(2n-4)$-th order Euler–Lagrange equation $E_z[\widehat{L}] = 0$ by solving two successive first order ordinary differential equations.*

6. Example I

Let us consider the second order variational problem
$$\mathcal{L}[u] = \int L(x, u, u_1, u_2)dx, \tag{57}$$

with Lagrangian

$$L(x,u,u_1,u_2)=\frac{x(2uu_2-u_1^2+4u_1u^2+u^4+1)^2}{4u^4}. \tag{58}$$

The Euler–Lagrange equation associated to (57) has the complicated form

$$\begin{aligned}-x-2u^3-5u_1u-2u^7+4u_3u^3-10u_1u^2u_2-5xu^5u_2-6xu^2u_2^2\\ +10xu_1^2-5u^5u_1+4u^4u_2-2u_1^2u^3+5u_1^3u+xu^8\\ -9xu_1^4-5xu_2u+2xu_4u^3+21xuu_1^2u_2-8xu_3u^2u_1=0.\end{aligned} \tag{59}$$

A straightforward computation shows that equation (59) does not admit any (infinitesimal) Lie point symmetries, which also implies that the variational problem (57) does not admit standard variational symmetries. We will apply the reduction procedure presented in this paper to obtain a two-parameter family of solutions to the Euler–Lagrange equation (59).

If we set

$$\mathbf{v}_1=\partial_u,\qquad \lambda_1=\frac{u_1}{u}-u+\frac{1}{u},\qquad \mathbf{v}_2=u^2\partial_u,\qquad \lambda_2=-\frac{u_1}{u}-u-\frac{1}{u}, \tag{60}$$

then

$$\mathbf{v}_1^{\lambda_1}(L)=0,\qquad \mathbf{v}_2^{\lambda_2}(L)=0, \tag{61}$$

and therefore $(\mathbf{v}_1,\lambda_1)$ and $(\mathbf{v}_2,\lambda_2)$ are variational $\mathcal{C}^\infty$-symmetries of (57). In addition, it can be easily checked that conditions *a)* and *b)* in Corollary 4.3 are satisfied, therefore the variational $\mathcal{C}^\infty$-symmetries satisfy

$$\left[\,\mathbf{v}_1^{\lambda_1},\mathbf{v}_2^{\lambda_2}\right]\ =2u\,\mathbf{v}_1^{\lambda_1},$$

and hence form a solvable pair with $h=2u$. Alternatively, it can be also checked that the explicit conditions *(i)-(iv)* in Theorem 4.2 hold.

6.1. Reduction associated with the variational $\mathcal{C}^\infty$-symmmetry (∂_u,λ_1)

A first order invariant of the vector field $\mathbf{v}_1^{\lambda_1}$ is given by

$$w=\frac{u_1+1+u^2}{u}=-\lambda_2. \tag{62}$$

Consider the invariant obtained by derivation

$$w_1=\frac{dw}{dx}=\frac{u_2u-u_1^2-u_1+u_1u^2}{u^2}.$$

It can be checked that the Lagrangian (58) can be locally expressed in terms of the invariants $\{x,w,w_1\}$ as the following reduced Lagrangian:

$$\widetilde{L}(x,w,w_1)=\frac{x(w^2+2w_1-2)^2}{4}, \tag{63}$$

whose associated Euler–Lagrange equation is the second order ordinary differential equation

$$E_w[\widetilde{L}] = -2xw_2 - 2w_1 + (w^2 - 2)(xw - 1) = 0. \tag{64}$$

Equation (64) does not admit Lie point symmetries, the determination of its solutions appears to be a nontrivial task. We will use the variational $\mathcal{C}^\infty$-symmetry $(u^2\partial_u, \lambda_2)$ to reduce the order of equation (64) by two, as established in Theorem 5.2.

6.2. Inherited variational $\mathcal{C}^\infty$-symmetry from $(u^2\partial_u, \lambda_2)$

Changing variables to $x, w = -\lambda_2$ rectifies $\mathbf{v}_2$, so that it becomes the variational $\mathcal{C}^\infty$-symmetry $(\mathbf{Y}, \widetilde{\lambda}_2)$, where $\mathbf{Y} = -2\partial_w$, $\widetilde{\lambda}_2 = -w$, of the reduced Lagrangian (63):

$$\mathbf{Y}^{\widetilde{\lambda}_2}(\widetilde{L}) = 0.$$

6.3. Reduction associated with $(\mathbf{Y}, \widetilde{\lambda}_2)$

The function

$$z(x, w, w_1) = \frac{w^2}{2} + w_1 \tag{65}$$

satisfies $\mathbf{Y}^{\widetilde{\lambda}_2}(z) = 0$. Thus, in terms of the invariants $\{x, z\}$ the reduced Lagrangian obtained from (63) is of order 0,

$$\widehat{L}(x, z) = x(z - 1)^2, \tag{66}$$

whose Euler–Lagrange equation becomes

$$2x(z - 1) = 0. \tag{67}$$

The point $x = 0$ is a removable singularity, and the solution of the algebraic Euler–Lagrange equation (67) is given by $z = 1$.

6.4. Reconstruction of solutions

By (65), a one-parameter family of solutions of the Euler–Lagrange equation $E_w[\widetilde{L}] = 0$ can be found by solving the first order ordinary differential equation

$$w_1 + \frac{w^2}{2} = 1. \tag{68}$$

The general solution of equation (68) is given by

$$w(x; C_1) = \frac{\sqrt{2}\left(1 + C_1 e^{\sqrt{2}\,x}\right)}{C_1 e^{\sqrt{2}\,x} - 1}, \tag{69}$$

where $C_1 \in \mathbb{R}$. It can be checked that the solution (69) satisfies the second order ordinary differential equation (64).

By (62) and (69), a two-parameter family of solutions of the original Euler–Lagrange equation, $E_u[L]=0$, can be found by solving the first order ordinary differential equation

$$u_1+u^2+1=u\,\frac{\sqrt{2}\left(1+C_1e^{\sqrt{2}\,x}\right)}{C_1e^{\sqrt{2}\,x}-1}. \tag{70}$$

Equation (70) is of Riccati-type, so it can be converted into the following linear second order ordinary differential equation

$$\Psi''(x)=\frac{\sqrt{2}\left(1+C_1e^{\sqrt{2}\,x}\right)}{C_1e^{\sqrt{2}\,x}-1}\Psi'(x)-\Psi(x) \tag{71}$$

through the standard transformation $u(x)=\Psi'(x)/\Psi(x)$. A basis of the solution space to (71) is given by

$$\begin{aligned}\Psi_1(x;C_1) &= C_1e^{\frac{\sqrt{2}}{2}x}\sin\left(\frac{\sqrt{2}}{2}x\right)+e^{-\frac{\sqrt{2}}{2}x}\cos\left(\frac{\sqrt{2}}{2}x\right),\\ \Psi_2(x;C_1) &= -C_1e^{\frac{\sqrt{2}}{2}x}\cos\left(\frac{\sqrt{2}}{2}x\right)+e^{-\frac{\sqrt{2}}{2}x}\sin\left(\frac{\sqrt{2}}{2}x\right).\end{aligned} \tag{72}$$

Therefore, the solution of the Riccati equation (70) can be expressed as follows:

$$u(x;C_1,C_2)=\frac{C_2\Psi_1'(x;C_1)+\Psi_2'(x;C_1)}{C_2\Psi_1(x;C_1)+\Psi_2(x;C_1)}, \tag{73}$$

where $C_1,C_2\in\mathbb{R}$, which becomes

$$u(x;C_1,C_2)=\frac{\sqrt{2}\left(C_1e^{\frac{\sqrt{2}}{2}x}-e^{\frac{-\sqrt{2}}{2}x}\right)\left((C_2-1)\cos\left(\frac{\sqrt{2}}{2}x\right)+(C_2+1)\sin\left(\frac{\sqrt{2}}{2}x\right)\right)}{2\left(C_2e^{\frac{-\sqrt{2}}{2}x}-C_1e^{\frac{\sqrt{2}}{2}x}\right)\cos\left(\frac{\sqrt{2}}{2}x\right)+2\left(C_1C_2e^{\frac{\sqrt{2}}{2}x}+e^{-\frac{\sqrt{2}}{2}x}\right)\sin\left(\frac{\sqrt{2}}{2}x\right)}. \tag{74}$$

As a result of the procedure described, a two-parameter family of solutions (74) to the original fourth order Euler Lagrange equation (59) has been obtained.

7. Example II

Consider the family of second order variational problems

$$\mathcal{L}[u]=\int L(x,u,u_1,u_2)dx, \tag{75}$$

with Lagrangians of the form

$$L=L\left(x,\frac{2uu_2-u^4-3u_1^2}{2u^2}\right). \tag{76}$$

Let $E_u[L]=0$ be the associated Euler–Lagrange equations. It can be checked that the pairs (∂_u,λ_1) and (∂_u,λ_2) are variational $\mathcal{C}^\infty$-symmetries of (75) for the respective functions

$$\lambda_1=\frac{u_1}{u}-u \qquad \text{and} \qquad \lambda_2=\frac{u_1}{u}+u, \tag{77}$$

i.e,

$$\partial_u^{\lambda_1}(L) = \partial_u^{\lambda_2}(L) = 0. \tag{78}$$

It can be checked that conditions *a)* and *b)* in Corollary 4.3 are satisfied with $h = 0$, hence the variational $\mathcal{C}^\infty$-symmetries form a solvable (or abelian) pair:

$$\left[\partial_u^{\lambda_1}, \partial_u^{\lambda_2}\right] = 0.$$

In this particular example the order in which the variational $\mathcal{C}^\infty$-symmetries are used to reduce the order of the Euler–Lagrange equation associated to (76) is not relevant because the corresponding infinite λ-prolongations commute, so their roles can be exchanged.

7.1. Reduction associated with the variational $\mathcal{C}^\infty$-symmetry (∂_u, λ_1)

A first order invariant of $\partial_u^{\lambda_1}$ is given by

$$w = \frac{u_1 + u^2}{u} = \lambda_2. \tag{79}$$

By derivation, we obtain the following second order invariant:

$$w_1 = \frac{dw}{dx} = \frac{-u_1^2 + u_1 u^2 + u u_2}{u^2}.$$

It can be checked that the Lagrangian (76) is expressed in terms of the coordinates $\{x, w, w_1\}$ as the following reduced Lagrangian:

$$\widetilde{L} = \widetilde{L}\left(x, w_1 - \frac{1}{2}w^2\right), \tag{80}$$

whose associated Euler–Lagrange equation is a second order ordinary differential equation of the form

$$E_w[\widetilde{L}] = \left((w w_1 - w_2)\frac{\partial^2 \widetilde{L}(x,s)}{\partial s^2} - w\frac{\partial \widetilde{L}(x,s)}{\partial s} - \frac{\partial^2 \widetilde{L}(x,s)}{\partial x \partial s}\right)\Bigg|_{s=-\frac{1}{2}w^2+w_1} = 0. \tag{81}$$

7.2. Inherited variational $\mathcal{C}^\infty$-symmetry from (∂_u, λ_2)

In terms of $\{x, w\}$ the symmetry (∂_u, λ_2) becomes $(\mathbf{Y}, \widetilde{\lambda}_2) = (2\partial_w, w)$, satisfying the variational $\mathcal{C}^\infty$-symmetry condition $\mathbf{Y}^{\widetilde{\lambda}_2}(\widetilde{L}) = 0$ for the reduced Lagrangian (80).

7.3. Reduction associated with $(\mathbf{Y}, \widetilde{\lambda}_2)$

A first order invariant of $\mathbf{Y}^{\widetilde{\lambda}_2}$ is given by

$$z = w_1 - \frac{1}{2}w^2, \tag{82}$$

therefore the Lagrangian (80) can be expressed as the reduced order 0 Lagrangian $\widehat{L} = \widehat{L}(x, z)$, whose Euler–Lagrange equation is simply the algebraic equation

$$\frac{\partial \widehat{L}(x,z)}{\partial z} = 0 \tag{83}$$

defining z implicitly as a function of x.

7.4. Reconstruction of solutions

Let $z = H(x)$ be a solution of the algebraic equation (83). By (82), a one-parameter family of solutions to the Euler–Lagrange equation (81) can be obtained by solving the first order ordinary differential equation

$$w_1 - \frac{1}{2}w^2 = H(x). \tag{84}$$

Equation (84) is of Riccati-type so it can be transformed into the Schrödinger-type equation

$$\psi''(x) = -\frac{1}{2}H(x)\psi(x) \tag{85}$$

by setting $w = -2\psi'(x)/\psi(x)$. Therefore, if ψ_1 and ψ_2 are two linearly independent solutions to equation (85), then the solution to the Riccati equation (84) is given by

$$w(x; C_1) = -2\,\frac{C_1\psi_1'(x) + \psi_2'(x)}{C_1\psi_1(x) + \psi_2(x)}, \qquad C_1 \in \mathbb{R}. \tag{86}$$

By (79), a two-parameter family of solutions to the original Euler–Lagrange equation $E_u[L] = 0$ can be found by solving the first order ordinary differential equation

$$\frac{u_1}{u} + u = -2\,\frac{C_1\psi_1'(x) + \psi_2'(x)}{C_1\psi_1(x) + \psi_2(x)}.$$

This is a Bernoulli equation, whose general solution is given by

$$u(x; C_1, C_2) = \frac{(C_1\psi_1(x) + \psi_2(x))^{-2}}{\displaystyle\int (C_1\psi_1(x) + \psi_2(x))^{-2}dx + C_2}, \qquad C_1, C_2 \in \mathbb{R}. \tag{87}$$

7.5. A particular case

Let us illustrate the preceding with a particular case in which the solution (87) can be expressed in terms of elementary functions. Consider the Lagrangian

$$L\left(x, \frac{2uu_2 - u^4 - 3u_1^2}{2u^2}\right) = x\left(k - \frac{2uu_2 - u^4 - 3u_1^2}{2u^2}\right)^2, \tag{88}$$

where $k \in \mathbb{R}$. The associated Euler–Lagrange equation is

$$\begin{aligned} -14u_1u_2u^2 + 2xu_4u^3 - 2xku^3u_2 + 2xku^2u_1^2 + 21xuu_2u_1^2 + xu^8 - 9xu_1^4 \\ - 5u_1u^5 + 9u_1^3u + 4u_3u^3 + 2xku^6 - 6xu^2u_2^2 - 5xu^5u_2 - 2ku_1u^3 - 8xu_1u_3u^2 = 0. \end{aligned} \tag{89}$$

Furthermore, when $k \neq 0$, (89) does not admit Lie point symmetries, and hence the variational problem associated to the Lagrangian (88) does not admit standard variational symmetries. The corresponding Euler–Lagrange equation (81) becomes

$$-2xw_2 - 2w_1 + (yw+1)(2k+w^2) = 0, \tag{90}$$

which does not admit Lie point symmetries for $k \neq 0$. Observe that this also implies again that the variational problem associated to the reduced Lagrangian $\widetilde{L} = x\,\left(\,k - w_1 + \frac{1}{2}w^2\right)$ does not have standard variational symmetries.

In this case, the reduced order 0 Lagrangian is

$$\widehat{L}(x,z) = x(k-z)^2,$$

with associated Euler–Lagrange equation

$$-2x(k-z) = 0. \tag{91}$$

The point $x = 0$ is a removable singularity, and the solution to the Euler–Lagrange equation (91) is $z = k$. In order to recover the solution of the original Euler–Lagrange equation, we distinguish three different cases:

- CASE 1 : If $k > 0$ then the solution to the Riccati equation (84) is given by

  $$w(x;C_1) = \sqrt{2k}\,\tan\left(\,\frac{\sqrt{2k}}{2}\,(x+C_1)\right)$$

  and the 2-parameter family of solutions (87) becomes:

  $$u(x;C_1,C_2) = \frac{\sqrt{k}\,\tan\Omega\,\left(1+\tan^2\alpha(x)\right)}{\left(\tan\Omega\,\tan\alpha(x)-1\right)\left(C_2\sqrt{k}\,\tan^2\Omega\tan\alpha(x) - C_2\sqrt{k}\,\tan\Omega - \sqrt{2}\,\right)},$$

  where

  $$\Omega = \frac{\sqrt{2k}}{2}C_1, \qquad \alpha(x) = \frac{\sqrt{2k}}{2}x, \qquad C_1, C_2 \in \mathbb{R}.$$

- CASE 2: For $k < 0$, the solution (86) becomes

  $$w(x;C_1) = -\sqrt{-2k}\,\tanh\left(\,\frac{\sqrt{-2k}}{2}\,(x+C_1)\right)$$

  and the 2-parameter family of solutions (87) of the original Euler–Lagrange equation (89) is

  $$u(x;C_1,C_2) = \frac{2\sqrt{-2k}}{\left(\cosh(2\alpha(x))+1\right)\left(2\tanh(\alpha(x)) + C_2\sqrt{-2k}\right)},$$

  where

  $$\alpha(x) = \frac{\sqrt{-2k}}{2}(x+C_1), \qquad C_1, C_2 \in \mathbb{R}.$$

- CASE 3: $k = 0$. For the particular case of $k = 0$ a solution to equation (84) is locally given by

$$w(x; C_1) = \frac{2}{2C_1 - x}$$

and the 2-parameter family (87) becomes

$$u(x; C_1, C_2) = \frac{1}{(2C_1 - x)(1 - C_2 x + 2C_1 C_2)}.$$

8. Concluding remarks

A method to reduce by four the order of Euler–Lagrange equations associated to n-th order variational problem involving single variable integrals has been developed. This is done by means of a combined used of two variational $\mathcal{C}^\infty$-symmetries admitted by the variational problem that form a solvable pair, in accordance with condition (20). Once the appropriate variational $\mathcal{C}^\infty$-symmetry has been used to reduce the order of the Euler–Lagrange equation by two, the solvability condition (20) ensures that the reduced variational problem admits a second variational $\mathcal{C}^\infty$-symmetry. Furthermore, a $(2n-2)$-parameter family of solutions to the original Euler–Lagrange equation can be obtained from the general solution of the $(2n-4)$-th order reduced equation obtained after the double reduction of order, by solving two auxiliary first order ordinary differential equations.

Besides, in order to find an operative characterization of the solvability condition (20), we have determined a formula – see expression (18) – for the commutator of two $\mathcal{C}^\infty$-prolonged vector fields in evolutionary form. A study on the possible use of either the commutator (17) or the vector fields of the form (18) to produce new order reductions for ordinary differential equations is still in progress.

Finally, we have considered two examples in which the method successes in providing two parameter family of solutions of Euler–Lagrange equations of order four even in the absence of standard variational symmetries. It would be worthwhile to investigate further extensions of our method to possible higher order reductions induced by more than two variational $\mathcal{C}^\infty$-symmetries and/or to systems of ordinary differential equations or even partial differential equations involving two or more independent variables. We expect that such extensions will help one to find new solutions of physically interesting problems.

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