Algebras of Differential Invariants

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G — Lie group (or Lie pseudo-group)
acting on a manifold M

$\mathcal{I}(G)$ — "algebra" (sheaf) of all differential invariants
for p-dimensional submanifolds $S \subset M$

Goal: Describe the structure of $\mathcal{I}(G)$ in as much detail as possible.

Classical Geometries

- **Euclidean:** $$G = \begin{cases} \mathrm{SE}(m) = \mathrm{SO}(m) \ltimes \mathbb{R}^m \\ \mathrm{E}(m) = \mathrm{O}(m) \ltimes \mathbb{R}^m \end{cases}$$

  $\boxed{z \longmapsto A \cdot z + b}$ $\quad A \in \mathrm{SO}(m)$ or $\mathrm{O}(m)$, $\quad b \in \mathbb{R}^m$, $\quad z \in \mathbb{R}^m$

  $\Longrightarrow$ isometries: rotations, translations, (reflections)

- **Equi-affine:** $G = \mathrm{SA}(m) = \mathrm{SL}(m) \ltimes \mathbb{R}^m$

  $A \in \mathrm{SL}(m)$ — volume-preserving

- **Affine:** $G = \mathrm{A}(m) = \mathrm{GL}(m) \ltimes \mathbb{R}^m$

  $A \in \mathrm{GL}(m)$

- **Projective:** $G = \mathrm{PSL}(m+1)$

  acting on $\mathbb{R}^m \subset \mathbb{RP}^m$

Invariants

Definition. If G is a group acting on M, then an invariant is a real-valued function $I : M \to \mathbb{R}$ that does not change under the action of G:

$$I(g \cdot z) = I(z) \qquad \text{for all} \qquad g \in G, \quad z \in M$$

★ If G acts transitively, there are no (non-constant) invariants.

Differential Invariants

Given a submanifold (curve, surface, . . .)

$$S \subset M$$

a differential invariant is an invariant of the prolonged action of G on its derivatives (jets):

$$\boxed{I(g \cdot z^{(k)}) \;=\; I(z^{(k)})}$$

Examples of Differential Invariants

Euclidean Group on $\mathbb{R}^3$

$$G = \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$$

$$\Longrightarrow \quad \text{group of rigid motions}$$

$$z \quad \longmapsto \quad R\,z + b \qquad\qquad R \in \mathrm{SO}(3)$$

- Induced action on curves and surfaces.

Euclidean Curves $\quad C \subset \mathbb{R}^3$

- κ — curvature: order $= 2$
- τ — torsion: order $= 3$
- $\kappa_s, \tau_s, \kappa_{ss}, \ldots$ — derivatives w.r.t. arc length ds

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Theorem. Every Euclidean differential invariant of a space curve $C \subset \mathbb{R}^3$ can be written

$$I = H(\kappa, \tau, \kappa_s, \tau_s, \kappa_{ss}, \ \ldots \ )$$

Thus, κ and τ *generate* the differential invariants of space curves under the Euclidean group.

Euclidean Surfaces $\quad S \subset \mathbb{R}^3$

- $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ — mean curvature: order $= 2$
- $K = \kappa_1 \, \kappa_2$ — Gauss curvature: order $= 2$
- $\mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ \ldots$ — derivatives with respect to the equivariant Frenet frame on S

Theorem. Every Euclidean differential invariant of a non-umbilic surface $S \subset \mathbb{R}^3$ can be written

$$I = \Phi(H, K, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ \ldots\ )$$

Thus, H, K *generate* the differential invariant algebra of (generic) Euclidean surfaces.

Equi-affine Group on $\mathbb{R}^3$

$$G = \mathrm{SA}(3) = \mathrm{SL}(3) \ltimes \mathbb{R}^3 \quad \text{— volume preserving}$$

$$z \longmapsto A\,z + b, \qquad \det A = 1$$

Curves in $\mathbb{R}^3$:

- κ — equi-affine curvature: order $= 4$
- τ — equi-affine torsion: order $= 5$
- $\kappa_s, \tau_s, \kappa_{ss}, \ldots$ — diff. w.r.t. equi-affine arc length

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Surfaces in $\mathbb{R}^3$:

- P — Pick invariant: order $= 3$
- $Q_0, Q_1, \ldots, Q_4$ — fourth order invariants
- $\mathcal{D}_1 P, \mathcal{D}_2 P, \mathcal{D}_1 Q_\nu, \ldots$ diff. w.r.t. the equi-affine frame

General Problems

Determine the structure of the algebra of differential invariants: generators, syzygies, commutators, etc.

Find a minimal system of generating differential invariants.

The Basis Theorem

Theorem. The differential invariant algebra $\mathcal{I}(G)$ is locally generated by a finite number of differential invariants

$$I_1, \ \ldots \ , I_\ell$$

and $p = \dim S$ invariant differential operators

$$\mathcal{D}_1, \ \ldots \ , \mathcal{D}_p$$

meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa .$$

★ Lie groups: *Lie, Ovsiannikov, Fels–PJO*

★ Lie pseudo-groups: *Tresse, Kumpera, Pohjanpelto–PJO, Kruglikov–Lychagin*

A Rational Basis Theorem

Theorem. (Kruglikov–Lychagin) If the Lie pseudo-group acts transitively and algebraically, then the differential invariant algebra $\mathcal{I}(G)$ is generated by a finite number of rational differential invariants

$$I_1, \ \ldots \ , I_\ell$$

and $p = \dim S$ rational invariant differential operators

$$\mathcal{D}_1, \ \ldots \ , \mathcal{D}_p$$

.

- Relies on an algebraic theorem due to Rosenlicht proving the existence of bases of rational invariants
- Not constructive.

Key Issues

- Minimal basis of generating invariants: $I_1, \ldots, I_\ell$
- Commutation formulae for

  the invariant differential operators:

$$[\mathcal{D}_j, \mathcal{D}_k] = \sum_{i=1}^{p} Y^i_{jk} \, \mathcal{D}_i$$

$\Longrightarrow$ Non-commutative differential algebra

- Syzygies (functional relations) among

  the differentiated invariants:

$$\Phi(\ \ldots \ \mathcal{D}_J I_\kappa \ \ldots \ ) \equiv 0$$

$\Longrightarrow$ Codazzi relations

Curves

Theorem. Let G be an ordinary* Lie group acting on the m-dimensional manifold M. Then, locally, there exist $m-1$ generating differential invariants $\kappa_1, \ldots, \kappa_{m-1}$. Every other differential invariant can be written as a function of the generating differential invariants and their derivatives with respect to the G-invariant arc length element ds.

* ordinary = transitive + no pseudo-stabilization.

$\Longrightarrow$ $m = 3$ — curvature κ & torsion τ

Equi-affine Surfaces

Theorem.

The algebra of equi-affine differential invariants for non-degenerate surfaces is generated by the Pick invariant through invariant differentiation.

In particular:

$$Q_\nu = \Phi_\nu(P, \mathcal{D}_1 P, \mathcal{D}_2 P, \ \ldots \ )$$

Euclidean Surfaces

Theorem.

The algebra of Euclidean differential invariants for suitably non-degenerate surfaces is generated by only the mean curvature through invariant differentiation.

In particular:

$$K = \Phi(H, \mathcal{D}_1 H, \mathcal{D}_2 H, \ \ldots \ )$$

Further Results

For suitably non-degenerate surfaces $S \subset \mathbb{R}^3$:

Theorem. $G = \mathrm{SO}(4,2)$

The algebra of conformal differential invariants is generated by a single third order differential invariant.

Theorem. $G = \mathrm{PSL}(4)$

The algebra of projective differential invariants is generated by a single fourth order differential invariant.

$\Longrightarrow$ (*with E. Hubert*)

Theorem. $G = \mathrm{GL}(3)$

The algebra of differential invariants for ternary forms is generated by a single third order differential invariant.

$\Longrightarrow$ (*with G. Gün Polat*)

Example. $G\colon\ (x,y,u) \longmapsto (\,x+a,\, y+b,\, u+P(x,y)\,)$

$a, b \in \mathbb{R}$, $\qquad P$ is an arbitrary polynomial of degree $\leq n$

Differential invariants:

$$u_{i,j} = \frac{\partial^{i+j} u}{\partial x^i \partial y^j} \qquad\qquad i+j \geq n+1$$

Invariant differential operators:

$$\mathcal{D}_1 = D_x, \qquad \mathcal{D}_2 = D_y.$$

Minimal generating set:

$$u_{i,j}, \qquad\qquad i+j = n+1$$

♠ For submanifolds of dimension $p \ \geq\ 2$, the number of generating differential invariants can be arbitrarily large.

Applications of Differential Invariants

- Equivalence and signatures of submanifolds

  $\Longrightarrow$ image processing

- Characterization of moduli spaces
- Invariant differential equations:

$$H(\ \ldots\ \mathcal{D}_J I_\kappa\ \ldots\ ) = 0$$

- Integration of ordinary differential equations
- Group splitting/foliation of PDEs

  — explicit solutions & Bäcklund transformations

- Invariant variational problems:

$$\int L(\ \ldots\ \mathcal{D}_J I_\kappa\ \ldots\ )\,\boldsymbol{\omega}$$

- Conservation laws and characteristic classes

Equivalence & Invariants

Cartan's solution to the equivalence problem for submanifolds under a transformation group relies on the functional relationships or syzygies among their differential invariants.

Theorem. Two regular submanifolds $S, \widetilde{S} \subset M$ are locally equivalent

$$\widetilde{S} = g \cdot S \qquad \text{for some} \qquad g \in G$$

if and only if they have identical syzygies among *all* their differential invariants.

Finiteness of Generators and Syzygies

♠ There are, in general, an infinite number of differential invariants and hence an infinite number of syzygies must be compared to establish equivalence.

♡ But the higher order differential invariants are always generated by invariant differentiation from a finite collection of basic differential invariants, and the higher order syzygies are all consequences of a finite number of low order syzygies!

Example — Plane Curves

If non-constant, both κ and κ_s depend on a single parameter, and so, locally, are subject to a syzygy:

$$\kappa_s = H(\kappa) \tag{$*$}$$

But then

$$\kappa_{ss} = \frac{d}{ds} H(\kappa) = H'(\kappa)\,\kappa_s = H'(\kappa)\,H(\kappa)$$

and similarly for κ_{sss}, etc.

Consequently, all the higher order syzygies are generated by the fundamental first order syzygy $(*)$.

Thus, for Euclidean (or equi-affine or projective or ...) plane curves we need only know a single syzygy between κ and κ_s in order to establish equivalence!

Signature Curves

Definition. The signature curve $\Sigma \subset \mathbb{R}^2$ of a plane curve $C \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants

$$\chi \; : \; C \longrightarrow \Sigma = \left\{ \left(\kappa \, , \, \frac{d\kappa}{ds} \right) \right\} \subset \mathbb{R}^2$$

$\Longrightarrow$ *Calabi, PJO, Shakiban, Tannenbaum, Haker*

Theorem. Two regular curves C and $\overline{C}$ are locally equivalent:

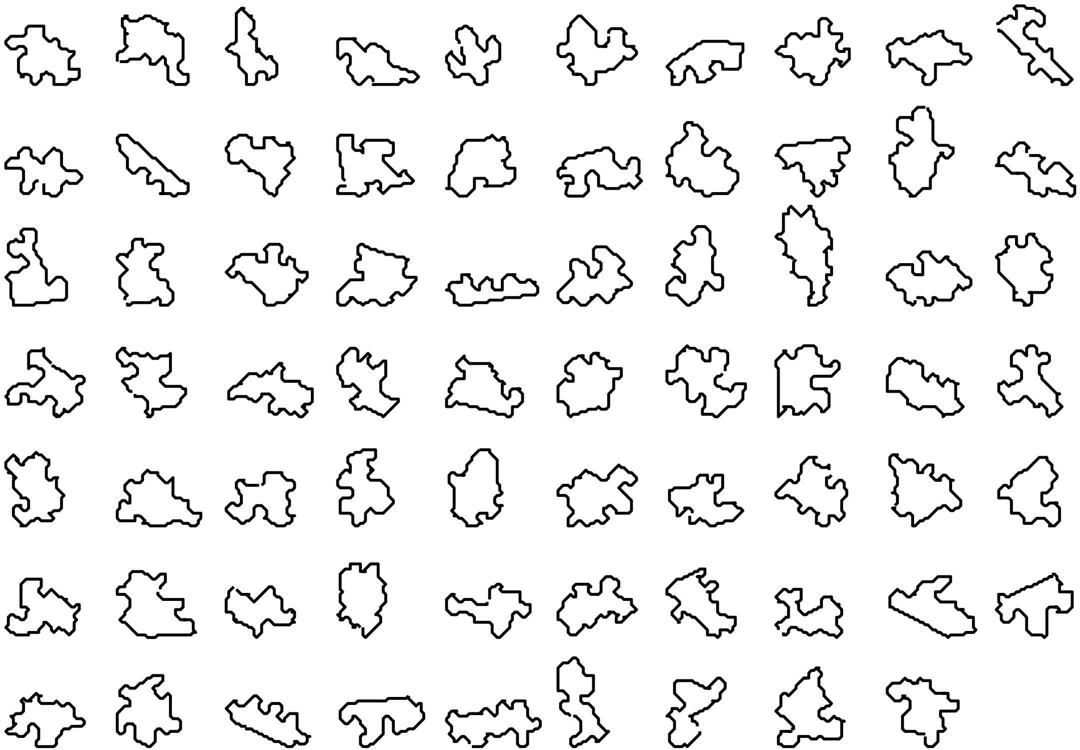
$$\overline{C} = g \cdot C$$

if and only if their signature curves are identical:

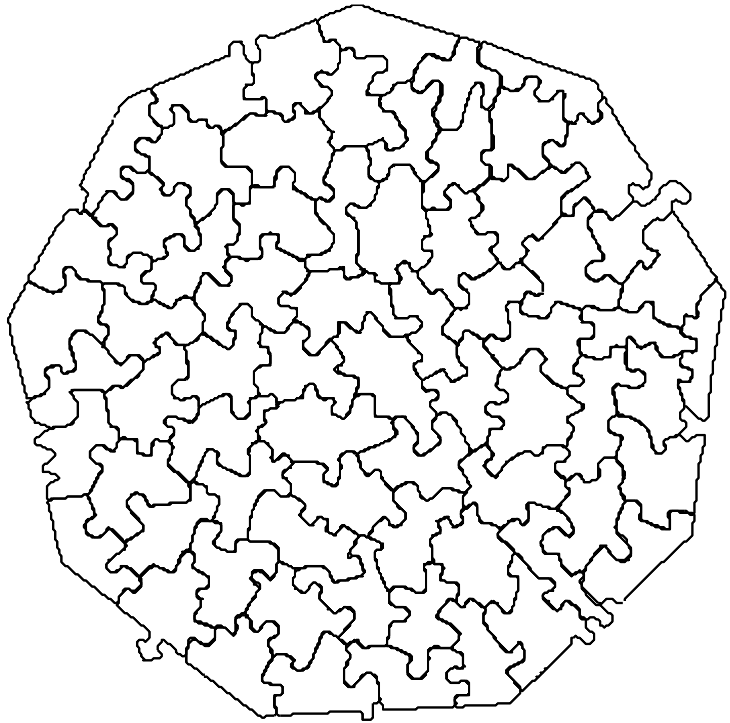
$$\overline{\Sigma} = \Sigma$$

$\Longrightarrow$ regular: $(\kappa_s, \kappa_{ss}) \neq 0.$

The Baffler Jigsaw Puzzle

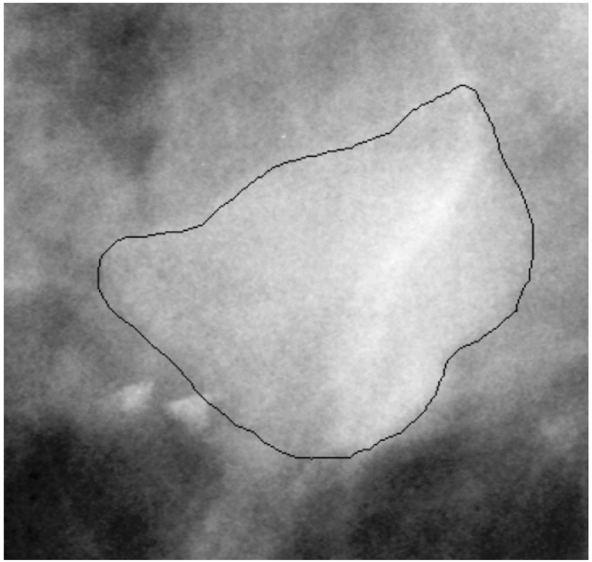


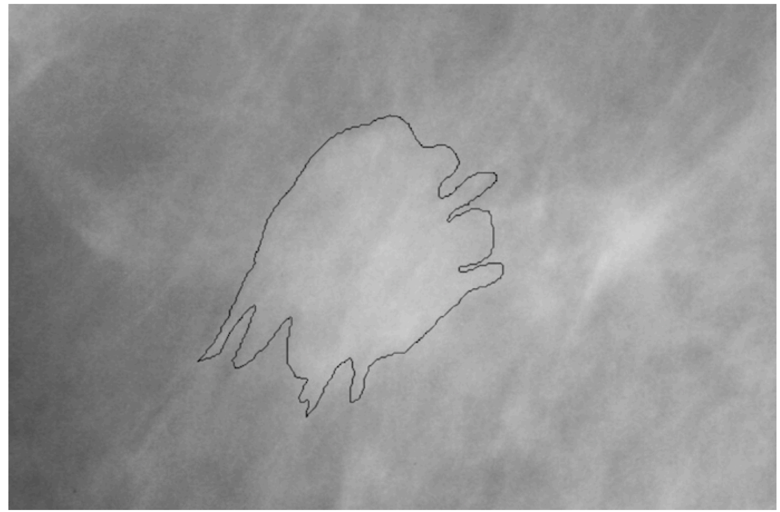
The Baffler Solved



⟹ Dan Hoff

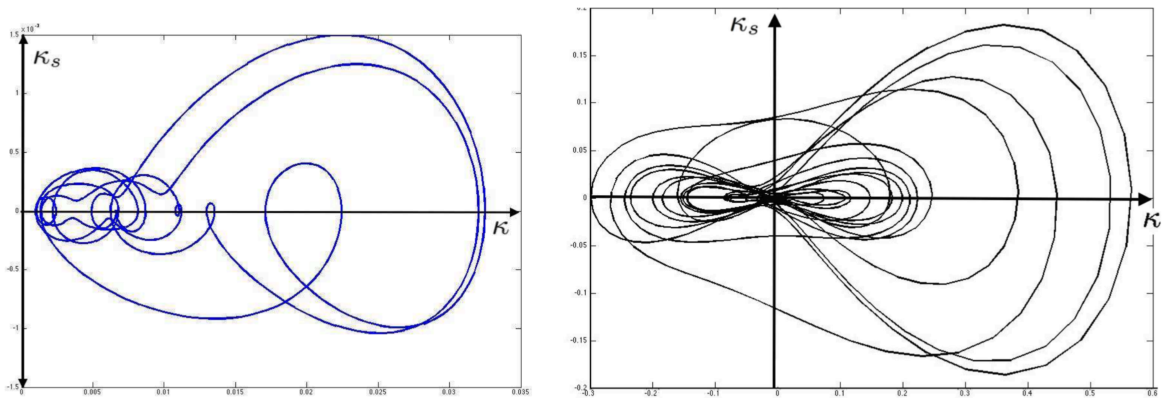
Benign vs. Malignant Tumors



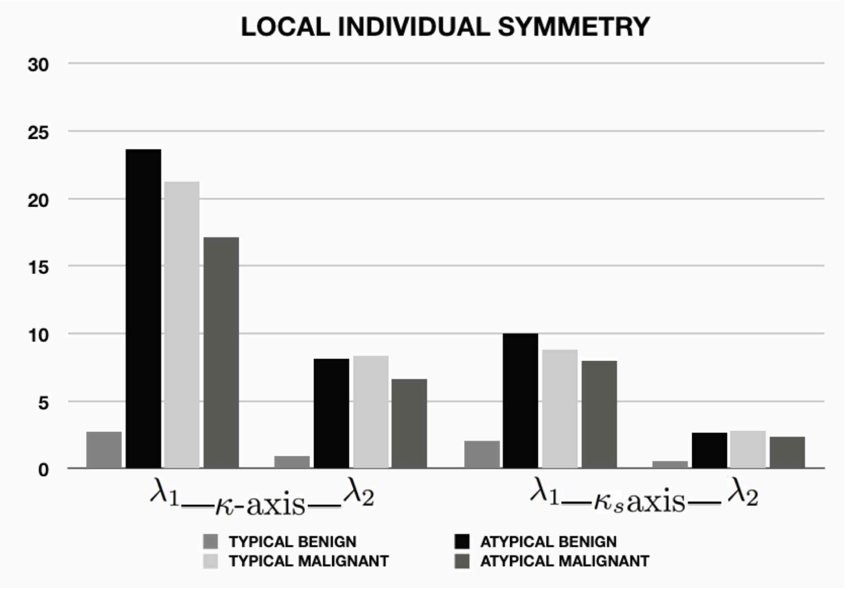


$\Longrightarrow$ A. Grim, C. Shakiban

Benign vs. Malignant Tumors



Benign vs. Malignant Tumors



3D Differential Invariant Signatures

Euclidean space curves: $C \subset \mathbb{R}^3$

$$\Sigma = \{\, (\, \kappa \,,\, \kappa_s \,,\, \tau \,) \,\} \subset \mathbb{R}^3$$

- κ — curvature, τ — torsion

Euclidean surfaces: $S \subset \mathbb{R}^3$ (generic)

$$\Sigma = \left\{ \left(H \,,\, K \,,\, H_{,1} \,,\, H_{,2} \,,\, K_{,1} \,,\, K_{,2} \right) \right\} \subset \mathbb{R}^6$$

or $$\widehat{\Sigma} = \left\{ \left(H \,,\, H_{,1} \,,\, H_{,2} \,,\, H_{,11} \right) \right\} \subset \mathbb{R}^4$$

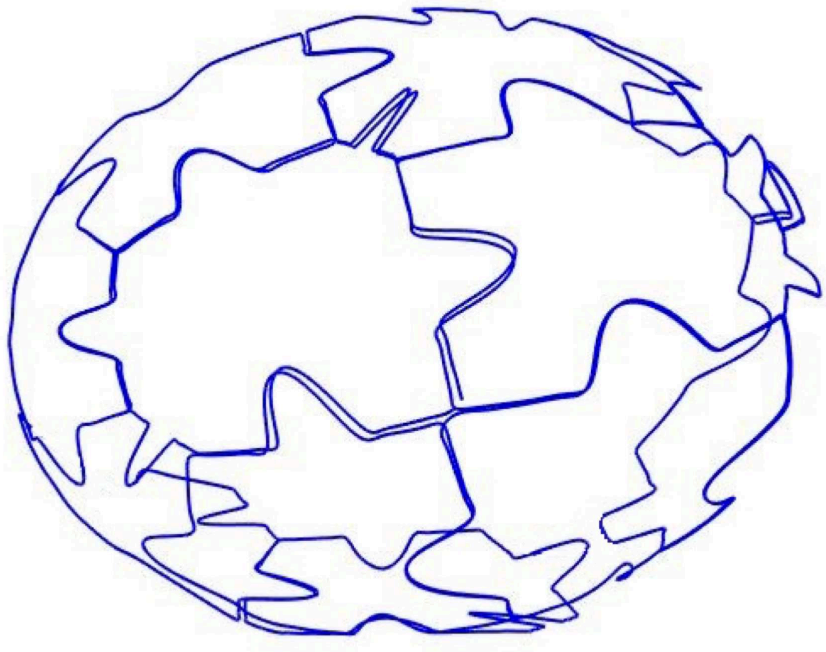
- H — mean curvature, K — Gauss curvature

Equi–affine surfaces: $S \subset \mathbb{R}^3$ (generic)

$$\Sigma = \left\{ \left(P \,,\, P_{,1} \,,\, P_{,2} ,\, P_{,11} \right) \right\} \subset \mathbb{R}^4$$

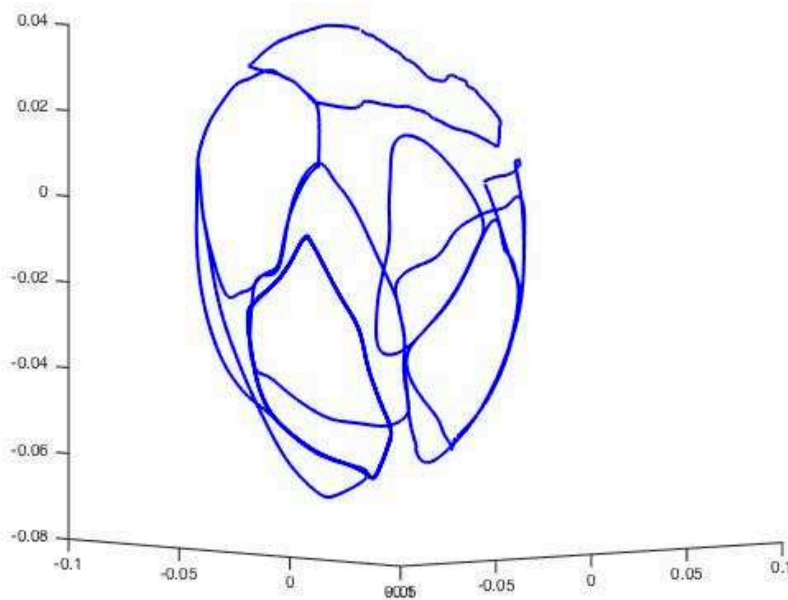
- P — Pick invariant

3D Jigsaw Puzzles



⟹ Anna Grim, Tim O'Connor, Ryan Schlecta
Cheri Shakiban, Rob Thompson, PJO

Reassembling Humpty Dumpty



⟹ Broken ostrich egg shell — Marshall Bern

Archaeology

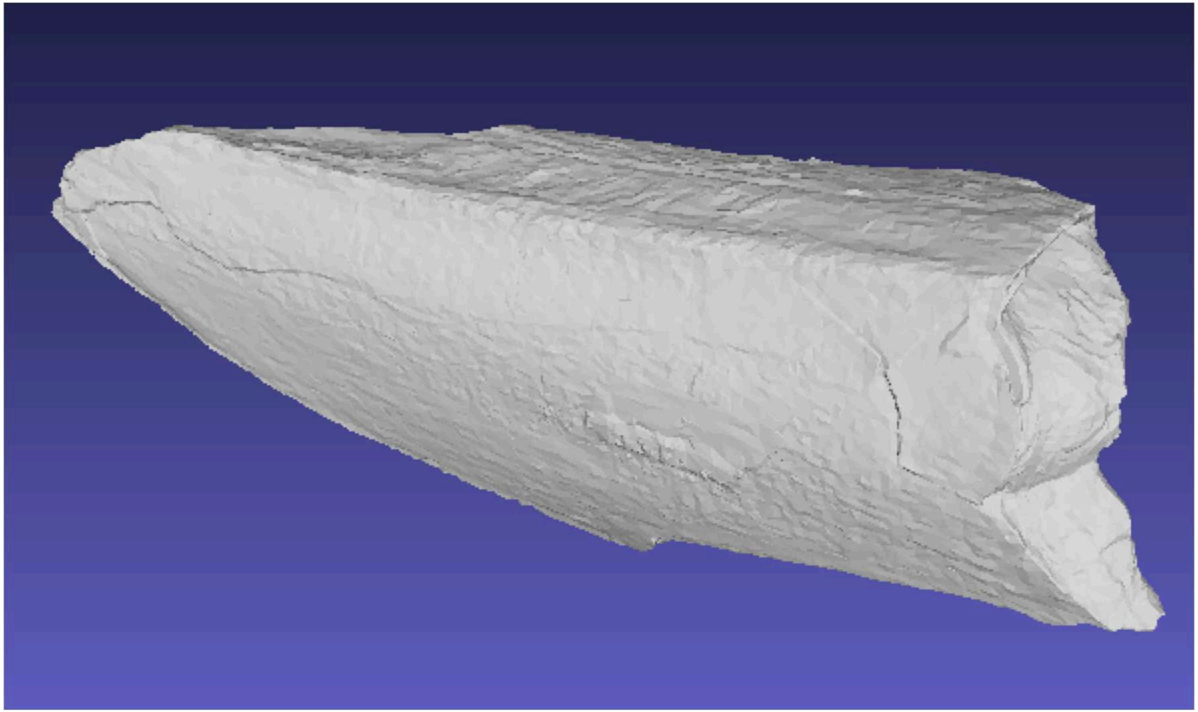
$\Longrightarrow$ **Virtual Archaeology**

Surgery

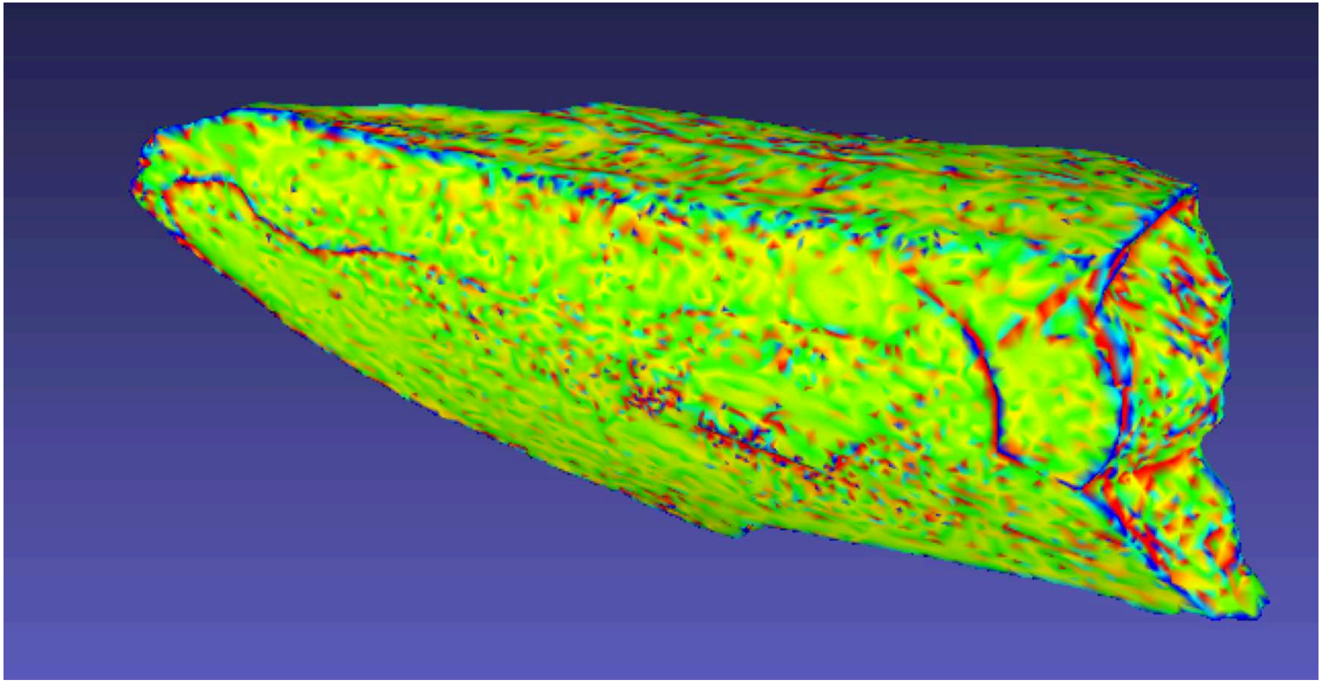
Anthropology

$\Longrightarrow$ Katrina Yezzi-Woodley, Jeff Calder, Pedro Angulo–Umana

Bone fragment



Bone fragment



⟹ Mean curvature

Invariant Differential Equations

Any (non-degenerate) differential equation that admits G as a symmetry group can be expressed in terms of the differential invariants:

$$\boxed{F(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ ) = 0}$$

★ Lie's integration method for ordinary differential equations.

★ Vessiot's Method of group foliation (group splitting) for partial differential equations to construct invariant and non-invariant solutions, as well as Bäcklund transformations, etc, for partial differential equations.

$\Longrightarrow$ Thompson–Valiquette

Invariant Variational Problems

Any G–invariant variational problem can be written in terms of the differential invariants:

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots \ \mathcal{D}_K I^\alpha \ \ \ldots \ ) \ \boldsymbol{\omega}$$

Moreover, its Euler–Lagrange equations admit G as a symmetry group, and hence can also be expressed in terms of the differential invariants:

$$E(L) \simeq F(\ \ldots \ \mathcal{D}_K I^\alpha \ \ \ldots \ ) = 0$$

Main Problem: Construct F directly from P.

⋆⋆ The general formula is a now known and a consequence of the structure of the differential invariant algebra and the corresponding invariant variational bicomplex.

The shape of a Möbius strip

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PJO

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The Möbius strip, obtained by taking a rectangular strip of plastic or paper, twisting one end through 180°, and then joining the ends, is the canonical example of a one-sided surface. Finding its characteristic developable shape has been an open problem ever since its first formulation in refs 1,2. Here we use the invariant variational bicomplex formalism to derive the first equilibrium equations for a wide developable strip undergoing large deformations, thereby giving the first non-trivial demonstration of the potential of this approach. We then formulate the boundary-value problem for the Möbius strip and solve it numerically. Solutions for increasing width show the formation of creases bounding nearly flat triangular regions, a feature also familiar from fabric draping[3] and paper crumpling[4,5]. This could give new insight into energy localization phenomena in unstretchable sheets[6], which might help to predict points of onset of tearing. It could also aid our understanding of the relationship between geometry and physical properties of nano- and microscopic Möbius strip structures[7–9].

It is fair to say that the Möbius strip is one of the few icons of mathematics that have been absorbed into wider culture. It has mathematical beauty and inspired artists such as Escher[10]. In engineering, pulley belts are often used in the form of Möbius strips to wear 'both' sides equally. At a much smaller scale, Möbius strips have recently been formed in ribbon-shaped $NbSe_3$ crystals under certain growth conditions involving a large temperature gradient[7,8].

Figure 1 Photo of a paper Möbius strip of aspect ratio 2π. The strip adopts a characteristic shape. Inextensibility of the material causes the surface to be developable. Its straight generators are drawn and the colouring varies according to the bending energy density.

The Algebra of Differential Invariants

$\star\star$ Moving frames furnish constructive algorithms for determining the full structure of the differential invariant algebra $\mathcal{I}(G)$!

Equivariant Moving Frames

Definition. An n^{th} order *moving frame* is a G-equivariant map

$$\rho^{(n)} : V^n \subset \mathrm{J}^n \longrightarrow G$$

- *Élie Cartan*
- *Guggenheimer, Griffiths, Green, Jensen*
- *Fels, Kogan, Pohjanpelto, PJO*

Equivariance:

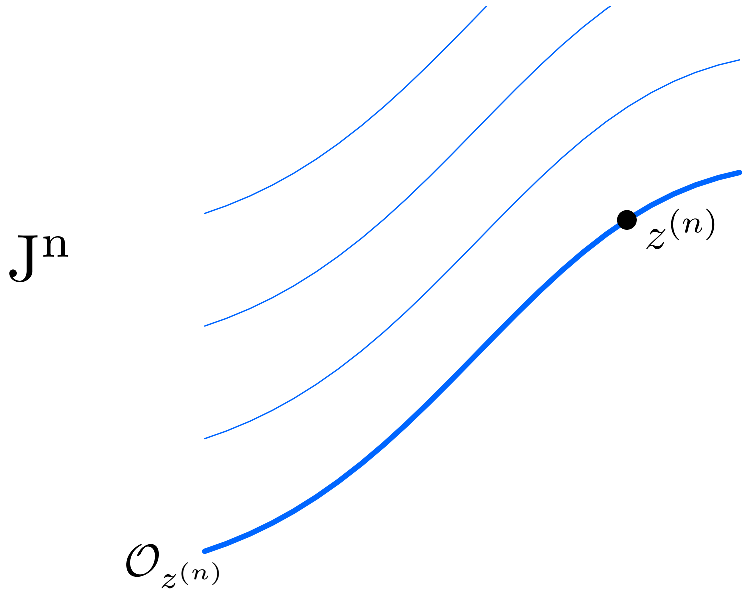
$$\rho(g^{(n)} \cdot z^{(n)}) = \begin{cases} g \cdot \rho(z^{(n)}) & \text{left moving frame} \\ \rho(z^{(n)}) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

Note: $\rho_{left}(z^{(n)}) = \rho_{right}(z^{(n)})^{-1}$

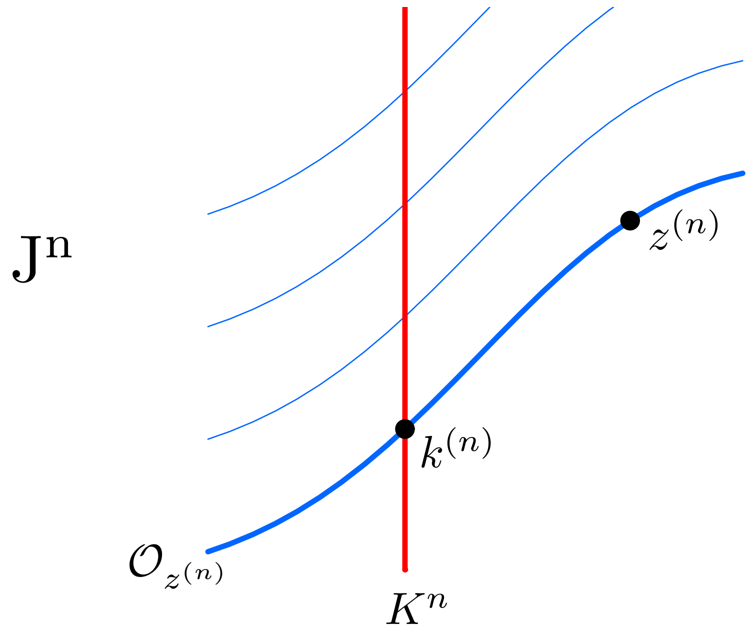
Theorem. A moving frame exists in a neighborhood of a jet $z^{(n)} \in \mathrm{J}^n$ if and only if G acts freely and regularly near $z^{(n)}$.

- free — the only group element $g \in G$ which fixes *one* point $z^{(n)} \in \mathrm{J}^n$ is the identity: $g^{(n)} \cdot z^{(n)} = z^{(n)} \iff g = e$.
- locally free — the orbits have the same dimension as G.
- regular — all orbits have the same dimension and intersect sufficiently small coordinate charts only once ($\not\approx$ irrational flow on the torus)

Geometric Construction

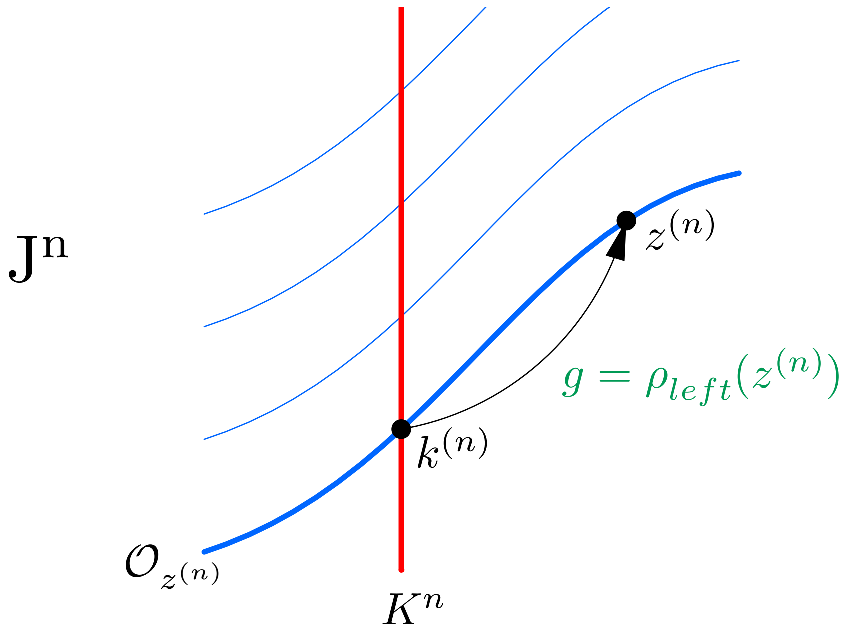


Geometric Construction



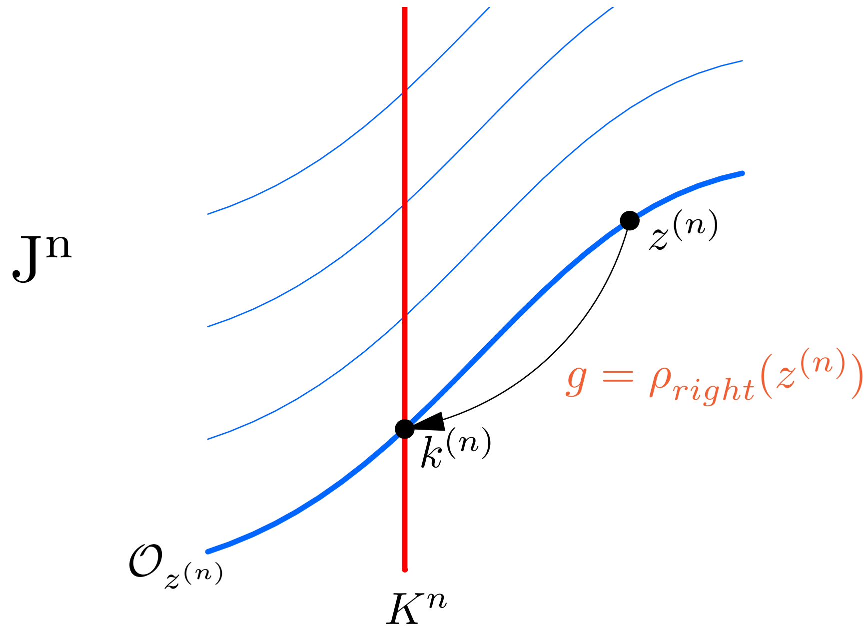
Normalization = choice of cross-section to the group orbits

Geometric Construction



Normalization = choice of cross-section to the group orbits

Geometric Construction



Normalization = choice of cross-section to the group orbits

Algebraic Construction

1. Write out the explicit formulas for the prolonged group action:

$$w^{(n)}(g, z^{(n)}) = g^{(n)} \cdot z^{(n)}$$

$\Longrightarrow$ *Implicit differentiation*

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2. From the components of $w^{(n)}$, choose $r = \dim G$ *normalization equations* to define the cross-section:

$$w_1(g, z^{(n)}) = c_1 \qquad \ldots \qquad w_r(g, z^{(n)}) = c_r$$

3. Solve the normalization equations for the group parameters $g = (g_1, \ldots, g_r)$:

$$g = \rho(z^{(n)}) = \rho(x, u^{(n)})$$

The solution is the right moving frame.

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4. Substitute the moving frame formulas

$$g = \rho(z^{(n)}) = \rho(x, u^{(n)})$$

for the group parameters into the un-normalized components of $w^{(n)}$ to produce a complete system of functionally independent differential invariants of order $\leq n$:

$$I_k(x, u^{(n)}) = w_k(\rho(z^{(n)}), z^{(n)})), \qquad k = r + 1, \ \ldots \ , \dim \mathrm{J}^n$$

Example: Euclidean Plane Curves

Rigid motions (translations and rotations):

$$G = \mathrm{SE}(2) \qquad \text{acting on} \qquad C \subset M = \mathbb{R}^2$$

Assume the curve is (locally) a graph:

$$\mathcal{C} = \{u = f(x)\}$$

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0. Write out the group transformations

$$\left.\begin{aligned} y &= x\cos\phi - u\sin\phi + a \\ v &= x\cos\phi + u\sin\phi + b \end{aligned}\right\} \qquad w = R\,z + c$$

1. Prolong to J^n via implicit differentiation

$$y = x\cos\phi - u\sin\phi + a \qquad v = x\sin\phi + u\cos\phi + b$$

$$v_y = \frac{\sin\phi + u_x\cos\phi}{\cos\phi - u_x\sin\phi} \qquad v_{yy} = \frac{u_{xx}}{(\cos\phi - u_x\sin\phi)^3}$$

$$v_{yyy} = \frac{(\cos\phi - u_x\sin\phi)\,u_{xxx} - 3\,u_{xx}^2\sin\phi}{(\cos\phi - u_x\sin\phi)^5} \qquad \vdots$$

2. Choose a cross-section, or, equivalently a set of $r = \dim G = 3$ normalization equations:

$$y = 0 \qquad v = 0 \qquad v_y = 0$$

3. Solve the normalization equations for the group parameters:

$$\phi = -\tan^{-1} u_x \qquad a = -\frac{x + u u_x}{\sqrt{1 + u_x^2}} \qquad b = \frac{x u_x - u}{\sqrt{1 + u_x^2}}$$

The result is the right moving frame $\rho : \mathrm{J}^1 \longrightarrow \mathrm{SE}(2)$

4. Substitute into the moving frame formulas for the group parameters into the remaining prolonged transformation formulae to produce the basic differential invariants:

$$
\begin{aligned}
v_{yy} &\longmapsto \kappa &&= \frac{u_{xx}}{(1+u_x^2)^{3/2}} \\
v_{yyy} &\longmapsto \frac{d\kappa}{ds} &&= \frac{(1+u_x^2)u_{xxx} - 3\,u_x u_{xx}^2}{(1+u_x^2)^3} \\
v_{yyyy} &\longmapsto \frac{d^2\kappa}{ds^2} + 3\,\kappa^3 &&= \cdots
\end{aligned}
$$

$\Longrightarrow$ recurrence formulae

Theorem. All differential invariants are functions of the derivatives of curvature with respect to arc length:

$$
\kappa \qquad \frac{d\kappa}{ds} \qquad \frac{d^2\kappa}{ds^2} \qquad \cdots
$$

5. The invariant differential operators and invariant differential forms are also obtained by substituting the moving frame formulas for the group parameters:

Invariant one-form — arc length

$$dy = (\cos\phi - u_x \sin\phi)\, dx \quad \longmapsto \quad ds = \sqrt{1 + u_x^2}\ dx$$

Invariant differential operator — arc length derivative

$$\frac{d}{dy} = \frac{1}{\cos\phi - u_x \sin\phi}\,\frac{d}{dx} \quad \longmapsto \quad \frac{d}{ds} = \frac{1}{\sqrt{1 + u_x^2}}\,\frac{d}{dx}$$

Invariantization

The process of replacing group parameters in transformed objects by their moving frame formulae is known as invariantization:

$$\iota : \left\{ \begin{array}{ccc} \text{Functions} & \longrightarrow & \text{Invariants} \\ \begin{array}{c}\text{Differential} \\ \text{Forms}\end{array} & \longrightarrow & \begin{array}{c}\text{Invariant Differential} \\ \text{Forms}\end{array} \\ \begin{array}{c}\text{Differential} \\ \text{Operators}\end{array} & \longrightarrow & \begin{array}{c}\text{Invariant Differential} \\ \text{Operators}\end{array} \\ \begin{array}{c}\text{Variational} \\ \text{Problems}\end{array} & \longrightarrow & \begin{array}{c}\text{Invariant Variational} \\ \text{Problems}\end{array} \\ \vdots & & \vdots \end{array} \right.$$

- Invariantization defines an (exterior) algebra morphism.
- Invariantization does not affect invariants: $\iota(I) = I$

The Fundamental Differential Invariants

Invariantized jet coordinate functions:

$$\boxed{H^i(x, u^{(n)}) = \iota(x^i) \qquad I^\alpha_K(x, u^{(l)}) = \iota(u^\alpha_K)}$$

- The constant differential invariants, as defined by the cross-section coordiantes, are known as the phantom invariants.
- The remaining non-constant differential invariants are the basic invariants and form a complete system of functionally independent differential invariants for the prolonged group action.

Invariantization of general differential functions:

$$\iota\,[\,F(\,\ldots\;x^i\;\ldots\;u^\alpha_J\;\ldots\,)\,] = F(\,\ldots\;H^i\;\ldots\;I^\alpha_J\;\ldots\,)$$

The Replacement Theorem: (Rewrite Rule)

If J is a differential invariant, then $\iota(J) = J$.

$$J(\,\ldots\;x^i\;\ldots\;u^\alpha_J\;\ldots\,) = J(\,\ldots\;H^i\;\ldots\;I^\alpha_J\;\ldots\,)$$

Key fact: Invariantization and differentiation do not commute:

$$\iota(D_i F) \;\neq\; \mathcal{D}_i \iota(F)$$

Infinitesimal Generators

Infinitesimal generators of action of G on M:

$$\mathbf{v}_\kappa = \sum_{i=1}^{p} \xi^i_\kappa(x,u)\,\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha_\kappa(x,u)\,\frac{\partial}{\partial u^\alpha} \qquad \kappa = 1,\ldots,r$$

Prolonged infinitesimal generators on J^n:

$$\mathbf{v}_\kappa^{(n)} = \mathbf{v}_\kappa + \sum_{\alpha=1}^{q} \sum_{j=\#J=1}^{n} \varphi^\alpha_{J,\kappa}(x,u^{(j)})\ \frac{\partial}{\partial u^\alpha_J}$$

Prolongation formula:

$$\varphi^\alpha_{J,\kappa} = D_K\left(\varphi^\alpha_\kappa - \sum_{i=1}^{p} u^\alpha_i\,\xi^i_\kappa\right) + \sum_{i=1}^{p} u^\alpha_{J,i}\,\xi^i_\kappa$$

$D_1,\ldots,D_p$ — total derivatives

Recurrence Formulae

$$\mathcal{D}_j\, \iota(F) = \iota(D_j F) \;+\; \sum_{\kappa=1}^{r} R_j^\kappa\, \iota(\mathbf{v}_\kappa^{(n)}(F))$$

$\omega^i = \iota(dx^i)$ — invariant coframe

$\mathcal{D}_i = \iota(D_{x^i})$ — dual invariant differential operators

$\mathbf{v}_1^{(n)}, \;\ldots\; \mathbf{v}_r^{(n)} \in \mathfrak{g}$ — prolonged infinitesimal generators

R_j^κ — Maurer–Cartan invariants

Recurrence Formulae

$$\mathcal{D}_j\, \iota(F) = \iota(D_j F) \;+\; \sum_{\kappa=1}^{r} R_j^\kappa\, \iota(\mathbf{v}_\kappa^{(n)}(F))$$

♠ If $\iota(F) = c$ is a phantom differential invariant, then the left hand side of the recurrence formula is zero. The collection of all such phantom recurrence formulae form a linear algebraic system of equations that can be uniquely solved for the Maurer–Cartan invariants R_j^κ!

♡ Once the Maurer–Cartan invariants are replaced by their explicit formulae, the induced recurrence relations completely determine the structure of the differential invariant algebra $\mathcal{I}(G)$!

The Universal Recurrence Formula

If Ω is any differential form on J^n:

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \gamma^{\kappa} \wedge\ \iota[\mathbf{v}_{\kappa}(\Omega)]$$

Invariantized Maurer–Cartan forms:

$$\gamma^{\kappa} = \rho^{*}(\mu^{\kappa}) \equiv \sum_{j=1}^{p} R_j^{\kappa}\,\omega^j$$

$\Longrightarrow$ The Invariant Variational Bicomplex

The Commutator Invariants

By the Universal Recurrence Formula:

$$d\omega^i = d[\iota(dx^i)] = \iota(d^2x^i) + \sum_{\kappa=1}^{r} \gamma^\kappa \wedge \iota[\mathbf{v}_\kappa(dx^i)]$$

$$= - \sum_{j<k} Y^i_{jk}\, \omega^j \wedge \omega^k + \cdots$$

$$\boxed{[\,\mathcal{D}_j, \mathcal{D}_k\,] = \sum_{i=1}^{p} Y^i_{jk}\, \mathcal{D}_i}$$

The Differential Invariant Algebra

Thus, remarkably, the structure of $\mathcal{I}(G)$ can be determined without knowing the explicit formulae for either the moving frame, or the differential invariants, or the invariant differential operators!

The only required ingredients are the specification of the cross-section, and the standard formulae for the prolonged infinitesimal generators.

Theorem. If G acts transitively on M, or if the infinitesimal generator coefficients depend rationally in the coordinates, then all recurrence formulae are rational in the basic differential invariants and so $\mathcal{I}(G)$ is a rational, non-commutative differential algebra.

Euclidean Surfaces

Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$ acts on surfaces $S \subset \mathbb{R}^3$.

For simplicity, we assume the surface is (locally) the graph of a function

$$z = u(x, y)$$

Infinitesimal generators:

$$\mathbf{v}_1 = -y\,\partial_x + x\,\partial_y, \qquad \mathbf{v}_2 = -u\,\partial_x + x\,\partial_u, \qquad \mathbf{v}_3 = -u\,\partial_y + y\,\partial_u,$$

$$\mathbf{w}_1 = \partial_x, \qquad \mathbf{w}_2 = \partial_y, \qquad \mathbf{w}_3 = \partial_u.$$

- The translations $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ will be ignored, as they play no role in the higher order recurrence formulae.

Cross-section (Darboux frame):

$$x = y = u = u_x = u_y = u_{xy} = 0.$$

Phantom differential invariants:

$$\iota(x) = \iota(y) = \iota(u) = \iota(u_x) = \iota(u_y) = \iota(u_{xy}) = 0$$

Principal curvatures

$$\kappa_1 = \iota(u_{xx}), \qquad \kappa_2 = \iota(u_{yy})$$

Mean curvature and Gauss curvature:

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2), \qquad K = \kappa_1 \kappa_2$$

Higher order differential invariants — invariantized jet coordinates:

$$I_{jk} = \iota(u_{jk}) \qquad \text{where} \qquad u_{jk} = \frac{\partial^{j+k} u}{\partial x^j \partial y^k}$$

★ ★ Nondegeneracy condition: non-umbilic point $\kappa_1 \neq \kappa_2$.

The Algebra of Euclidean Differential Invariants

Principal curvatures:

$$\kappa_1 = \iota(u_{xx}), \qquad \kappa_2 = \iota(u_{yy})$$

Mean curvature and Gauss curvature:

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2), \qquad K = \kappa_1 \kappa_2$$

Invariant differentiation operators:

$$\mathcal{D}_1 = \iota(D_x), \qquad \mathcal{D}_2 = \iota(D_y)$$

$\Longrightarrow$ Differentiation with respect to the diagonalizing Darboux frame.

The recurrence formulae enable one to express the higher order differential invariants in terms of the principal curvatures, or, equivalently, the mean and Gauss curvatures, and their invariant derivatives:

$$\begin{aligned} I_{jk} = \iota(u_{jk}) &= \widetilde{\Phi}_{jk}(\kappa_1, \kappa_2, \mathcal{D}_1\kappa_1, \mathcal{D}_2\kappa_1, \mathcal{D}_1\kappa_2, \mathcal{D}_2\kappa_2, \mathcal{D}_1^2\kappa_1, \ \ldots\ ) \\ &= \Phi_{jk}(H, K, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ \ldots\ ) \end{aligned}$$

Recurrence Formulae

$$\iota(D_i u_{jk}) = \mathcal{D}_i\, \iota(u_{jk}) - \sum_{\kappa=1}^{3} R_i^\kappa\, \iota[\,\varphi_\kappa^{jk}(x,y,u^{(j+k)})\,], \qquad j+k \geq 1$$

$I_{jk} = \iota(u_{jk})$ — normalized differential invariants

R_i^κ — Maurer–Cartan invariants

$\varphi_\kappa^{jk}(0,0,I^{(j+k)}) = \iota[\,\varphi_\kappa^{jk}(x,y,u^{(j+k)})\,]$

— invariantized prolonged infinitesimal generator coefficients.

$$I_{j+1,k} = \mathcal{D}_1 I_{jk} - \sum_{\kappa=1}^{3} \varphi_\kappa^{jk}(0,0,I^{(j+k)})\, R_1^\kappa$$

$$I_{j,k+1} = \mathcal{D}_1 I_{jk} - \sum_{\kappa=1}^{3} \varphi_\kappa^{jk}(0,0,I^{(j+k)})\, R_2^\kappa$$

Prolonged infinitesimal generators:

$$\begin{aligned}
\mathrm{pr}\,\mathbf{v}_1 &= -y\,\partial_x + x\,\partial_y - u_y\partial_{u_x} + u_x\partial_{u_y} \\
&\quad - 2\,u_{xy}\partial_{u_{xx}} + (u_{xx} - u_{yy})\,\partial_{u_{xy}} - 2\,u_{xy}\partial_{u_{yy}} + \;\cdots\;, \\
\mathrm{pr}\,\mathbf{v}_2 &= -u\,\partial_x + x\,\partial_u + (1 + u_x^2)\,\partial_{u_x} + u_x u_y\partial_{u_y} \\
&\quad + 3\,u_x u_{xx}\partial_{u_{xx}} + (u_y u_{xx} + 2\,u_x u_{xy})\,\partial_{u_{xy}} + (2\,u_y u_{xy} + u_x u_{yy})\,\partial_{u_{yy}} + \;\cdots\;, \\
\mathrm{pr}\,\mathbf{v}_3 &= -u\,\partial_y + y\,\partial_u + u_x u_y\partial_{u_x} + (1 + u_y^2)\,\partial_{u_y} \\
&\quad + (u_y u_{xx} + 2\,u_x u_{xy})\,\partial_{u_{xx}} + (2\,u_y u_{xy} + u_x u_{yy})\,\partial_{u_{xy}} + 3\,u_y u_{yy}\partial_{u_{yy}} + \;\cdots\;.
\end{aligned}$$

Normalized differential invariants: $I_{jk} = \iota(u_{jk})$

Phantom differential invariants: $I_{00} = I_{10} = I_{01} = I_{11} = 0$

Principal curvatures: $I_{20} = \kappa_1 \qquad I_{02} = \kappa_2$

Phantom recurrence formulae:

$$\kappa_1 = I_{20} = \mathcal{D}_1 I_{10} - R_1^2 = -R_1^2,$$

$$0 = I_{11} = \mathcal{D}_1 I_{01} - R_1^3 = -R_1^3,$$

$$I_{21} = \mathcal{D}_1 I_{11} - (\kappa_1 - \kappa_2)\, R_1^1 = -(\kappa_1 - \kappa_2)\, R_1^1,$$

$$0 = I_{11} = \mathcal{D}_2 I_{10} - R_2^2 = -R_2^2,$$

$$\kappa_2 = I_{02} = \mathcal{D}_2 I_{01} - R_2^3 = -R_2^3,$$

$$I_{12} = \mathcal{D}_2 I_{11} - (\kappa_1 - \kappa_2)\, R_2^1 = -(\kappa_1 - \kappa_2)\, R_2^1.$$

Maurer–Cartan invariants:

$$R_1^1 = -Y_1, \qquad R_1^2 = -\kappa_1, \qquad R_1^3 = 0,$$

$$R_1^2 = -Y_2, \qquad R_2^2 = 0, \qquad R_3^2 = -\kappa_2.$$

Commutator invariants:

$$Y_1 = \frac{I_{21}}{\kappa_1 - \kappa_2} = \frac{\mathcal{D}_1 \kappa_2}{\kappa_1 - \kappa_2} \qquad Y_2 = \frac{I_{12}}{\kappa_1 - \kappa_2} = \frac{\mathcal{D}_2 \kappa_1}{\kappa_2 - \kappa_1}$$

$$\boxed{[\,\mathcal{D}_1, \mathcal{D}_2\,] = \mathcal{D}_1\, \mathcal{D}_2 - \mathcal{D}_2\, \mathcal{D}_1 = Y_2\, \mathcal{D}_1 - Y_1\, \mathcal{D}_2,}$$

Third order recurrence relations:

$$I_{30} = \mathcal{D}_1\kappa_1 = \kappa_{1,1}, \quad I_{21} = \mathcal{D}_2\kappa_1 = \kappa_{1,2}, \quad I_{12} = \mathcal{D}_1\kappa_2 = \kappa_{2,1}, \quad I_{03} = \mathcal{D}_2\kappa_2 = \kappa_{2,2},$$

Fourth order recurrence relations:

$$\begin{aligned}
I_{40} &= \kappa_{1,11} - \frac{3\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} + 3\,\kappa_1^3 \\
I_{31} &= \kappa_{1,12} - \frac{3\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2} &&= \kappa_{1,21} + \frac{\kappa_{1,1}\kappa_{1,2} - 2\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2} \\
I_{22} &= \kappa_{1,22} + \frac{\kappa_{1,1}\kappa_{2,1} - 2\,\kappa_{2,1}^2}{\kappa_1 - \kappa_2} + \kappa_1\kappa_2^2 &&= \kappa_{2,11} - \frac{\kappa_{1,2}\kappa_{2,2} - 2\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} + \kappa_1^2\kappa_2 \\
I_{13} &= \kappa_{2,21} + \frac{3\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2} &&= \kappa_{2,12} - \frac{\kappa_{2,1}\kappa_{2,2} - 2\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2} \\
I_{04} &= \kappa_{2,22} + \frac{3\,\kappa_{2,1}^2}{\kappa_1 - \kappa_2} + 3\,\kappa_2^3
\end{aligned}$$

★ The two expressions for I_{31} and I_{13} follow from the commutator formula.

Fourth order recurrence relations

$$I_{40} = \kappa_{1,11} - \frac{3\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} + 3\,\kappa_1^3$$

$$I_{31} = \kappa_{1,12} - \frac{3\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2} \qquad = \kappa_{1,21} + \frac{\kappa_{1,1}\kappa_{1,2} - 2\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2}$$

$$I_{22} = \kappa_{1,22} + \frac{\kappa_{1,1}\kappa_{2,1} - 2\,\kappa_{2,1}^2}{\kappa_1 - \kappa_2} + \kappa_1\kappa_2^2 = \kappa_{2,11} - \frac{\kappa_{1,2}\kappa_{2,2} - 2\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} + \kappa_1^2\kappa_2$$

$$I_{13} = \kappa_{2,21} + \frac{3\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2} \qquad = \kappa_{2,12} - \frac{\kappa_{2,1}\kappa_{2,2} - 2\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2}$$

$$I_{04} = \kappa_{2,22} + \frac{3\,\kappa_{2,1}^2}{\kappa_1 - \kappa_2} + 3\,\kappa_2^3$$

★ ★ The two expressions for I_{22} imply the Codazzi syzygy

$$\kappa_{1,22} - \kappa_{2,11} + \frac{\kappa_{1,1}\,\kappa_{2,1} + \kappa_{1,2}\,\kappa_{2,2} - 2\,\kappa_{2,1}^2 - 2\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} - \kappa_1\,\kappa_2\,(\kappa_1 - \kappa_2) = 0$$

which can be written compactly as

$$K = \kappa_1\kappa_2 = -\,(\mathcal{D}_1 + Y_1)\,Y_1 - (\mathcal{D}_2 + Y_2)\,Y_2$$

$\Longrightarrow$ Gauss' Theorema Egregium

The Commutator Trick

$$\boxed{K = \kappa_1 \kappa_2 = -(\mathcal{D}_1 + Y_1)Y_1 - (\mathcal{D}_2 + Y_2)Y_2}$$

To determine the commutator invariants:

$$\begin{aligned} \mathcal{D}_1\mathcal{D}_2 H - \mathcal{D}_2\mathcal{D}_1 H &= Y_2\, \mathcal{D}_1 H - Y_1\, \mathcal{D}_2 H \\ \mathcal{D}_1\mathcal{D}_2\mathcal{D}_J H - \mathcal{D}_2\mathcal{D}_1\mathcal{D}_J H &= Y_2\, \mathcal{D}_1\mathcal{D}_J H - Y_1\, \mathcal{D}_2\mathcal{D}_J H \end{aligned} \qquad (*)$$

Non-degeneracy condition:

$$\det \begin{pmatrix} \mathcal{D}_1 H & \mathcal{D}_2 H \\ \mathcal{D}_1\mathcal{D}_J H & \mathcal{D}_2\mathcal{D}_J H \end{pmatrix} \neq 0,$$

Solve $(*)$ for Y_1, Y_2 in terms of derivatives of H, producing a universal formula

$$\boxed{K = \Psi(H, \mathcal{D}_1 H, \mathcal{D}_2 H, \ \ldots\ )}$$

for the Gauss curvature as a rational function of the mean curvature and its invariant derivatives!

Definition. A surface $S \subset \mathbb{R}^3$ is mean curvature degenerate if, near any non-umbilic point $p_0 \in S$, there exist scalar functions $F_1(t), F_2(t)$ such that

$$\mathcal{D}_1 H = F_1(H), \qquad \mathcal{D}_2 H = F_2(H).$$

- surfaces with symmetry: rotation, helical;
- minimal surfaces;
- constant mean curvature surfaces;
- ???

Theorem. If a surface is mean curvature non-degenerate then the algebra of Euclidean differential invariants is generated entirely by the mean curvature and its successive invariant derivatives.

Equi-affine Surfaces

$$M = \mathbb{R}^3 \qquad G = \mathrm{SA}(3) = \mathrm{SL}(3) \ltimes \mathbb{R}^3 \qquad \dim G = 11.$$

$$g \cdot z = A\,z + b, \qquad \det A = 1, \qquad z = \begin{pmatrix} x \\ y \\ u \end{pmatrix} \in \mathbb{R}^3.$$

Surfaces $S \subset M = \mathbb{R}^3$:

$$u = f(x, y)$$

Hyperbolic case

$$u_{xx}u_{yy} - u_{xy}^2 < 0$$

Cross-section:

$$x = y = u = u_x = u_y = u_{xy} = 0, \qquad u_{xx} = 1, \qquad u_{yy} = -1,$$

$$u_{xyy} = u_{xxx}, \qquad u_{xxy} = u_{yyy} = 0.$$

Power series normal form:

$$u(x,y) = \tfrac{1}{2}(x^2 - y^2) + \tfrac{1}{6}\,c\,(x^3 + 3\,xy^2) + \cdots$$

$$\Longrightarrow \quad \textit{Nonsingular}: \ c \neq 0.$$

Invariantization — differential invariants: $\qquad I_{jk} = \iota(u_{jk})$

Phantom differential invariants:

$$\iota(x) = \iota(y) = \iota(u) = \iota(u_x) = \iota(u_y) = \iota(u_{xy}) = \iota(u_{xxy}) = \iota(u_{yyy}) = 0,$$

$$\iota(u_{xx}) = 1, \qquad \iota(u_{yy}) = -1, \qquad \iota(u_{xxx}) - \iota(u_{xyy}) = 0.$$

Pick invariant:

$$P = \iota(u_{xxx}) = \iota(u_{xyy}).$$

Basic differential invariants of order 4:

$$Q_0 = \iota(u_{xxxx}), \quad Q_1 = \iota(u_{xxxy}), \quad Q_2 = \iota(u_{xxyy}),$$

$$Q_3 = \iota(u_{xyyy}), \quad Q_4 = \iota(u_{yyyy}),$$

Invariant differential operators:

$$\mathcal{D}_1 = \iota(D_x), \qquad \mathcal{D}_2 = \iota(D_y).$$

- Since the moving frame has order 3, one can generate all higher order differential invariants from the basic differential invariants of order ≤ 4.

- This is a consequence of a general theorem, that follows directly from the recurrence formulae.

- Thus, to prove that the Pick invariant generates $\mathcal{I}(G)$, it suffices to generate $Q_0, \ldots, Q_4$ from P by invariant differentiation.

Infinitesimal generators:

$$\mathbf{v}_1 = x\,\partial_x - u\,\partial_u, \qquad \mathbf{v}_2 = y\,\partial_y - u\,\partial_u,$$

$$\mathbf{v}_3 = y\,\partial_x, \qquad \mathbf{v}_4 = u\,\partial_x, \qquad \mathbf{v}_5 = x\,\partial_y,$$

$$\mathbf{v}_6 = u\,\partial_y, \qquad \mathbf{v}_7 = x\,\partial_u, \qquad \mathbf{v}_8 = y\,\partial_u,$$

$$\mathbf{w}_1 = \partial_x, \qquad \mathbf{w}_2 = \partial_y, \qquad \mathbf{w}_3 = \partial_u,$$

- The translations will be ignored, as they play no role in the higher order recurrence formulae.

Recurrence formulae

$$\mathcal{D}_i \iota(u_{jk}) = \iota(D_i u_{jk}) + \sum_{\kappa=1}^{8} \varphi_\kappa^{jk}(x,y,u^{(j+k)}) R_i^\kappa, \qquad j+k \geq 1$$

$$\mathcal{D}_1 I_{jk} = I_{j+1,k} + \sum_{\kappa=1}^{8} \varphi_\kappa^{jk}(0,0,I^{(j+k)}) R_1^\kappa$$

$$\mathcal{D}_2 I_{jk} = I_{j,k+1} + \sum_{\kappa=1}^{8} \varphi_\kappa^{jk}(0,0,I^{(j+k)}) R_2^\kappa$$

$\varphi_\kappa^{jk}(0,0,I^{(j+k)}) = \iota[\,\varphi_\kappa^{jk}(x,y,u^{(j+k)})\,]$ — invariantized prolonged infinitesimal generator coefficients

R_i^κ — Maurer–Cartan invariants

Phantom recurrence formulae:

$$0 = \mathcal{D}_1 I_{10} = 1 + R_1^7, \qquad 0 = \mathcal{D}_2 I_{10} = R_2^7,$$

$$0 = \mathcal{D}_1 I_{01} = R_1^8, \qquad 0 = \mathcal{D}_2 I_{01} = -1 + R_2^8,$$

$$0 = \mathcal{D}_1 I_{20} = I_{30} - 3R_1^1 - R_1^2, \qquad 0 = \mathcal{D}_2 I_{20} = -3R_2^1 - R_2^2,$$

$$0 = \mathcal{D}_1 I_{11} = -R_1^3 + R_1^5, \qquad 0 = \mathcal{D}_2 I_{11} = I_{30} - R_2^3 + R_2^5,$$

$$0 = \mathcal{D}_1 I_{02} = I_{12} + R_1^1 + 3R_1^2, \qquad 0 = \mathcal{D}_2 I_{02} = R_2^1 + 3R_2^2,$$

$$0 = \mathcal{D}_1 I_{21} = I_{31} - I_{30} R_1^3 - 2I_{30} R_1^5 + R_1^6,$$

$$0 = \mathcal{D}_2 I_{21} = I_{22} - I_{30} R_2^3 - 2I_{30} R_2^5 + R_2^6,$$

$$0 = \mathcal{D}_1 I_{03} = I_{13} - 3I_{30} R_2^3 - 3R_2^6, \qquad 0 = \mathcal{D}_2 I_{03} = I_{04} - 3I_{30} R_2^3 - 3R_2^6.$$

Maurer–Cartan invariants:

$$R_1 = \left(\tfrac{1}{2}\,P, -\tfrac{1}{2}\,P, \frac{3\,Q_1 + Q_3}{12\,P}, \tfrac{1}{4}\,Q_0 - \tfrac{1}{4}\,Q_2 - \tfrac{1}{2}\,P^2, \frac{3\,Q_1 + Q_3}{12\,P}, -\tfrac{1}{4}\,Q_1 + \tfrac{1}{4}\,Q_3, -1, 0 \right)$$

$$R_2 = \left(0, 0, \frac{3\,Q_2 + Q_4}{12\,P} + \tfrac{1}{2}\,P, \tfrac{1}{4}\,Q_1 - \tfrac{1}{4}\,Q_3, \frac{3\,Q_2 + Q_4}{12\,P} - \tfrac{1}{2}\,P, -\tfrac{1}{4}\,Q_2 + \tfrac{1}{4}\,Q_4 - \tfrac{1}{2}\,P^2, 0, 1 \right)$$

Fourth order invariants:

$$P_1 = \mathcal{D}_1 P = \tfrac{1}{4}\,Q_0 + \tfrac{3}{4}\,Q_2, \qquad P_2 = \mathcal{D}_2 P = \tfrac{1}{4}\,Q_1 + \tfrac{3}{4}\,Q_3.$$

Commutator:

$$\mathcal{D}_3 = [\,\mathcal{D}_1, \mathcal{D}_2\,] = \mathcal{D}_1\,\mathcal{D}_2 - \mathcal{D}_2\,\mathcal{D}_1 = Y_1\,\mathcal{D}_1 + Y_2\,\mathcal{D}_2,$$

Commutator invariants:

$$Y_1 = R_2^1 - R_1^3 = -\,\frac{3\,Q_1 + Q_3}{12\,P}, \qquad Y_2 = R_2^5 - R_1^2 = \frac{3\,Q_2 + Q_4}{12\,P}.$$

Another fourth order invariant:

$$P_3 = \mathcal{D}_3 P = \mathcal{D}_1 \mathcal{D}_2 P - \mathcal{D}_2 \mathcal{D}_1 P = Y_1 P_1 + Y_2 P_2. \qquad (*)$$

Nondegeneracy condition: If

$$\det \begin{pmatrix} P_1 & P_2 \\ \mathcal{D}_1 P_j & \mathcal{D}_2 P_j \end{pmatrix} \neq 0 \qquad \text{for} \qquad j = 1, 2, \text{ or } 3,$$

we can solve $(*)$ and

$$\mathcal{D}_3 P_j = Y_1 \mathcal{D}_1 P_j + Y_2 \mathcal{D}_2 P_j$$

for the fourth order commutator invariants:

$$Y_1 = -\frac{3\,Q_1 + Q_3}{12\,P}, \qquad Y_2 = \frac{3\,Q_2 + Q_4}{12\,P}.$$

So far, we have constructed four combinations of the fourth order differential invariants

$$S_1 = Q_0 + 3\,Q_2, \qquad S_2 = Q_1 + 3\,Q_3,$$

$$S_3 = 3\,Q_1 + Q_3, \qquad S_4 = 3\,Q_2 + Q_4.$$

as rational functions of the invariant derivatives of the Pick invariant. To obtain the final fourth order differential invariant:

$$12\,P(\mathcal{D}_1 S_4 - \mathcal{D}_2 S_3) = 48\,P^2 Q_0 - 30\,P^2 S_1 + 18\,P^2 S_4 \\ - 3\,S_2 S_3 - S_3^2 + 3\,S_1 S_4 + S_4^2.$$

⋆ ⋆ ⋆ This completes the proof ⋆ ⋆ ⋆

Minimal Generating Invariants

A set of differential invariants is a generating system if all other differential invariants can be written in terms of them and their invariant derivatives.

Euclidean curves $C \subset \mathbb{R}^3$:	curvature κ and torsion τ
Equi–affine curves $C \subset \mathbb{R}^3$:	affine curvature κ and torsion τ
Euclidean surfaces $S \subset \mathbb{R}^3$:	mean curvature H
Equi–affine surfaces $S \subset \mathbb{R}^3$:	Pick invariant P.
Conformal surfaces $S \subset \mathbb{R}^3$:	third order invariant J_3.
Projective surfaces $S \subset \mathbb{R}^3$:	fourth order invariant K_4.

$\Longrightarrow$ (*with E. Hubert*)

Example. $G\colon\ (x,y,u) \longmapsto (\,x+a, y+b, u+P(x,y)\,)$

$$a,b \in \mathbb{R}, \qquad P \text{ is an arbitrary polynomial of degree} \leq n$$

Differential invariants:

$$u_{i,j} = \frac{\partial^{i+j} u}{\partial x^i \partial y^j} \qquad\qquad i+j \geq n+1$$

Invariant differential operators:

$$\mathcal{D}_1 = D_x, \qquad \mathcal{D}_2 = D_y.$$

Minimal generating set:

$$u_{i,j}, \qquad\qquad i+j = n+1$$

♣ For submanifolds of dimension $p \geq 2$, the number of generating differential invariants can be arbitrarily large.

♠ In general, finding a minimal generating set appears to be very difficult. (No known bound on order of syzygies.)

General Issues

The equivariant moving frame calculus is
completely constructive, and can be applied to

all finite-dimensional Lie transformation groups

most infinite-dimensional Lie pseudo-groups
arising in applications (eventually locally freely acting)

Fully determines the recurrence relations and hence, in principle, all identities for the algebra of differential invariants $\mathcal{I}(G)$.

★★ Structure theory for differential invariant algebras?

In particular, minimal generating sets require a syzygy bound:

$$\boxed{K = \Psi(H, \; \ldots \; , \mathcal{D}^{(n)} H) \qquad n \leq N \;???}$$

THANK YOU!