The Theory and Applications of Moving Frames

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Moving Frames

Classical contributions:

Bartels ($\sim$1800), Serret, Frénet, Darboux, Cotton,

Élie Cartan

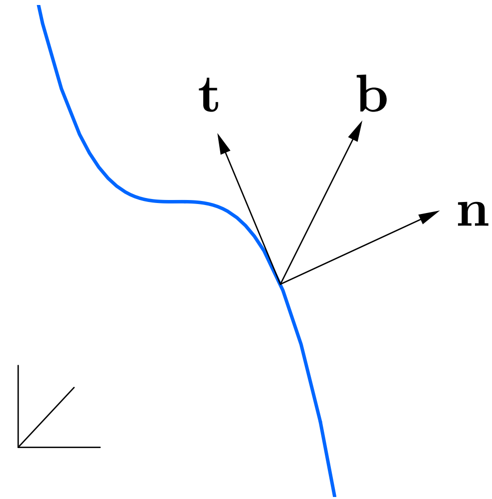
Modern developments: (1970's)

Chern, Green, Griffiths, Jensen, . . .

The equivariant approach: (1997 –)

PJO, Fels, Mansfield, Marí–Beffa, Kogan, Pohjanpelto, Kim, Boutin, Lewis, Hubert, Morozov, McLenaghan, Smirnov, Valiquette, Thompson, Benson, Arnaldsson, Popovych, Bihlo, Ruddy, Merker, Sabzevari, Z. Chen, . . .

Moving Frame — Space Curves



tangent	normal	binormal
$\mathbf{t} = \dfrac{dz}{ds}$	$\mathbf{n} = \dfrac{z_{ss}}{\| z_{ss} \|}$	$\mathbf{b} = \mathbf{t} \times \mathbf{n}$

s — arc length

Frénet–Serret equations

$$\frac{d\mathbf{t}}{ds} = \kappa\, \mathbf{n} \qquad \frac{d\mathbf{n}}{ds} = -\,\kappa\, \mathbf{t} + \tau\, \mathbf{b} \qquad \frac{d\mathbf{b}}{ds} = -\,\tau\, \mathbf{n}$$

κ — curvature $\qquad$ τ — torsion

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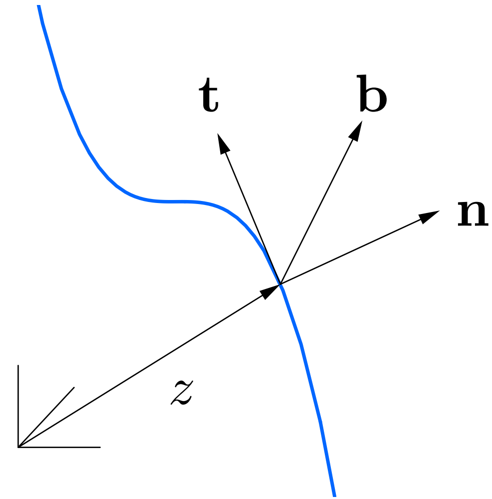

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$$\frac{d\mathbf{t}}{ds} = \kappa\, \mathbf{n} \qquad \frac{d\mathbf{n}}{ds} = -\kappa\, \mathbf{t} + \tau\, \mathbf{b} \qquad \frac{d\mathbf{b}}{ds} = -\tau\, \mathbf{n}$$

κ — curvature $\qquad$ τ — torsion

“I did not quite understand how he [Cartan] does this in general, though in the examples he gives the procedure is clear.”

“Nevertheless, I must admit I found the book, like most of Cartan’s papers, hard reading.”

— Hermann Weyl

“Cartan on groups and differential geometry”
Bull. Amer. Math. Soc. **44** (1938) 598–601

The Basic Equivalence Problem

M — smooth m-dimensional manifold.

G — transformation group acting on M

- finite-dimensional Lie group
- infinite-dimensional Lie pseudo-group
- finite or discrete group

Equivalence:

Determine when two p-dimensional submanifolds

$$N \quad \text{and} \quad \overline{N} \;\subset\; M$$

are *congruent*:

$$\overline{N} = g \cdot N \qquad \text{for} \qquad g \in G$$

Symmetry:

Find all symmetries,

i.e., self-equivalences or *self-congruences*:

$$N = g \cdot N$$

Classical Geometry — *F. Klein*

- **Euclidean group:** $$G = \begin{cases} \mathrm{SE}(m) = \mathrm{SO}(m) \ltimes \mathbb{R}^m \\ \mathrm{E}(m) = \mathrm{O}(m) \ltimes \mathbb{R}^m \end{cases}$$

  $$\boxed{z \longmapsto A \cdot z + b} \qquad A \in \mathrm{SO}(m) \text{ or } \mathrm{O}(m), \quad b \in \mathbb{R}^m, \quad z \in \mathbb{R}^m$$

  $\Rightarrow$ isometries: rotations, translations , (reflections)

- **Equi-affine group:** $G = \mathrm{SA}(m) = \mathrm{SL}(m) \ltimes \mathbb{R}^m$

  $A \in \mathrm{SL}(m)$ — volume-preserving

- **Affine group:** $G = \mathrm{A}(m) = \mathrm{GL}(m) \ltimes \mathbb{R}^m$

  $A \in \mathrm{GL}(m)$

- **Projective group:** $G = \mathrm{PSL}(m+1)$

  acting on $\mathbb{R}^m \subset \mathbb{RP}^m$

  $\Longrightarrow$ Applications in computer vision

Tennis, Anyone?

⋆ Projective (equi-affine) equivalence and symmetries

Classical Invariant Theory

Binary form:

$$Q(x) = \sum_{k=0}^{n} \binom{n}{k} a_k \, x^k$$

Equivalence of polynomials (binary forms):

$$Q(x) = (\gamma x + \delta)^n \, \overline{Q} \left(\frac{\alpha x + \beta}{\gamma x + \delta} \right) \qquad g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2)$$

- multiplier representation of GL(2)
- modular forms

$$Q(x) = (\gamma x + \delta)^n \, \overline{Q}\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right)$$

Transformation group:

$$g : (x, u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta}\,,\ \frac{u}{(\gamma x + \delta)^n}\right)$$

Equivalence of functions $\iff$ equivalence of graphs

$$\Gamma_Q = \{\, (x, u) = (x, Q(x)) \,\} \subset \mathbb{C}^2$$

Invariants

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⊙ Invariants describe the moduli space of objects under group transformations.

⋆ If G acts transitively, there are no (non-constant) invariants — in which case we need to "prolong" the action to a higher dimensional space.

Moving Frames

Definition.

A moving frame is a G-equivariant map

$$\rho : \; M \longrightarrow G$$

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Equivariance:

$$\rho(g \cdot z) = \begin{cases} g \cdot \rho(z) & \text{left moving frame} \\ \rho(z) \cdot g^{-1} & \text{right moving frame} \end{cases}$$

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$$\rho_{left}(z) = \rho_{right}(z)^{-1}$$

The Main Result

Theorem. A moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.

Isotropy & Freeness

Isotropy subgroup of a point $z \in M$**:**

$$G_z = \{\, g \mid g \cdot z = z \,\}$$

- free — the only group element $g \in G$ which fixes *one* point $z \in M$ is the identity

  $$\Longrightarrow \quad G_z = \{e\} \text{ for all } z \in M$$

- locally free — the orbits all have the same dimension as G

  $$\Longrightarrow \quad G_z \subset G \text{ is discrete for all } z \in M$$

- regular — the orbits form a regular foliation

  $$\not\approx \quad \text{irrational flow on the torus}$$

Proof of the Main Theorem

Necessity: Let $\rho : M \to G$ be a left moving frame.

Freeness: If $g \in G_z$, so $g \cdot z = z$, then by left equivariance:

$$\rho(z) = \rho(g \cdot z) = g \cdot \rho(z).$$

Therefore $g = e$, and hence $G_z = \{e\}$ for all $z \in M$.

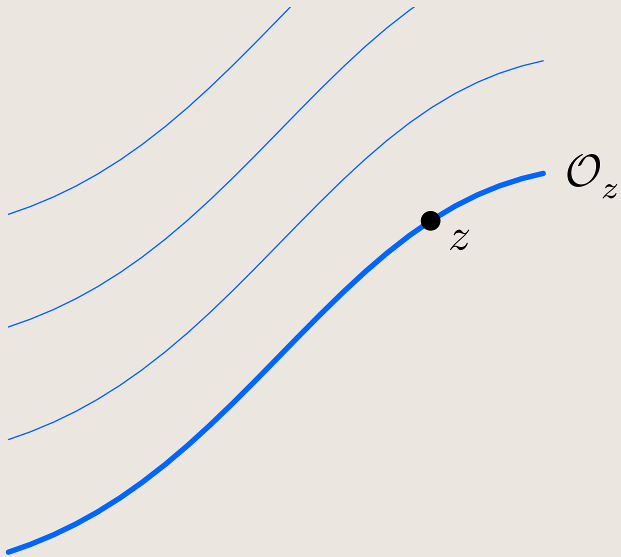
Regularity: Suppose $z_n = g_n \cdot z \longrightarrow z$ as $n \to \infty$.

By continuity, $\rho(z_n) = \rho(g_n \cdot z) = g_n \cdot \rho(z) \longrightarrow \rho(z)$.
Hence $g_n \longrightarrow e$ in G.

Sufficiency: By direct construction — "normalization".

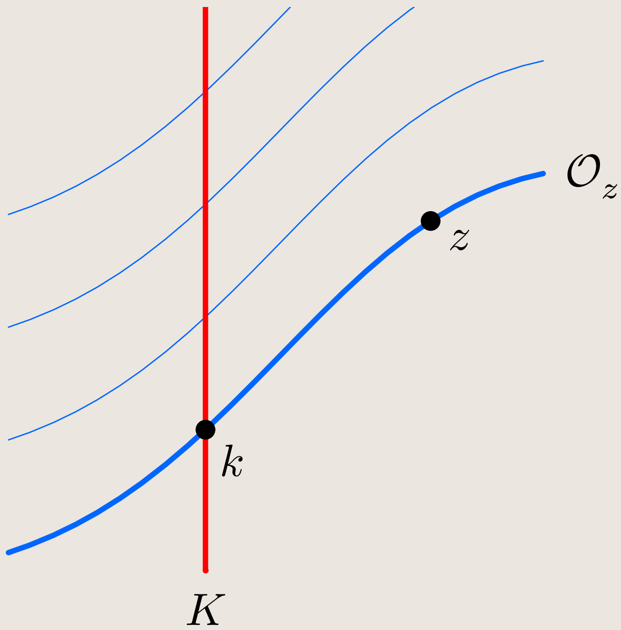
Q.E.D.

Geometric Construction



Normalization = choice of cross-section to the group orbits

Geometric Construction



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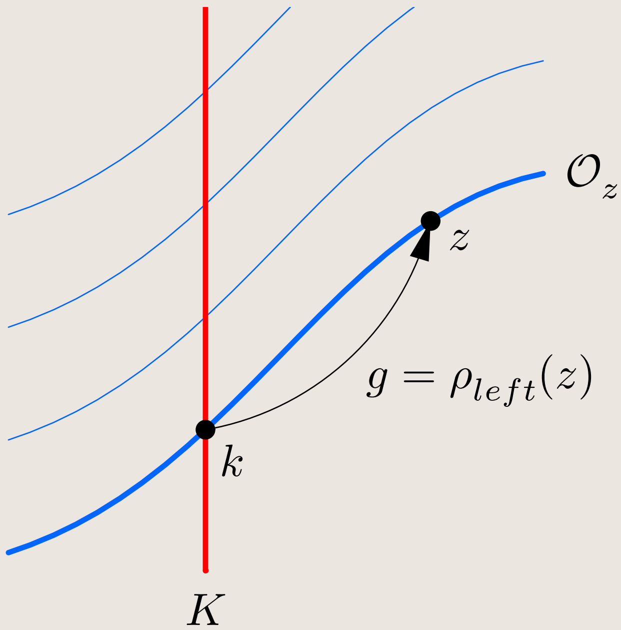

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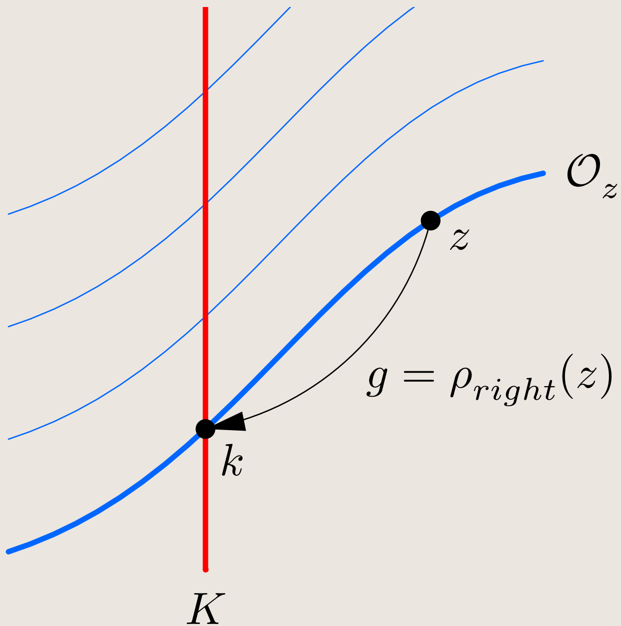

Normalization = choice of cross-section to the group orbits

K — cross-section to the group orbits

$\mathcal{O}_z$ — orbit through $z \in M$

$k \in K \cap \mathcal{O}_z$ — unique point in the intersection

- k is the *canonical form* of z
- the (nonconstant) coordinates of k are the fundamental invariants

$g \in G$ — *unique* group element mapping k to z

$\Longrightarrow$ freeness

$\rho(z) = g$ left moving frame $\rho(h \cdot z) = h \cdot \rho(z)$

$$k = \rho^{-1}(z) \cdot z = \rho_{right}(z) \cdot z$$

Algebraic Construction

$$r = \dim G \;\leq\; m = \dim M$$

Coordinate cross-section

$$K = \{\, z_1 = c_1, \;\ldots\; , z_r = c_r \,\}$$

left	right
$w(g, z) = g^{-1} \cdot z$	$w(g, z) = g \cdot z$

$g = (g_1, \ldots, g_r)$ — group parameters

$z = (z_1, \ldots, z_m)$ — coordinates on M

Choose $r = \dim G$ components to *normalize*:

$$w_1(g, z) = c_1 \qquad \ldots \qquad w_r(g, z) = c_r$$

Solve for the group parameters $g = (g_1, \ldots, g_r)$

$\Longrightarrow$ Implicit Function Theorem

The solution

$$g = \rho(z)$$

is a (local) moving frame.

The Fundamental Invariants

Substituting the moving frame formulae

$$g = \rho(z)$$

into the unnormalized components of $w(g, z)$ produces the fundamental invariants

$$I_1(z) = w_{r+1}(\rho(z), z) \quad \ldots \quad I_{m-r}(z) = w_m(\rho(z), z)$$

$\Longrightarrow$ These are the coordinates of the canonical form $k \in K$.

Completeness of Invariants

Theorem. Every invariant $I(z)$ can be (locally) uniquely written as a function of the fundamental invariants:

$$I(z) = H(I_1(z), \ \ldots \ , I_{m-r}(z))$$

Invariantization

Definition. The *invariantization* of a function $F : M \to \mathbb{R}$ with respect to a right moving frame $g = \rho(z)$ is the the invariant function $I = \iota(F)$ defined by

$$I(z) = F(\rho(z) \cdot z).$$

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$$\iota(z_1) = c_1, \ \ldots \ \iota(z_r) = c_r, \quad \iota(z_{r+1}) = I_1(z), \ \ldots \ \iota(z_m) = I_{m-r}(z).$$

cross-section variables — fundamental invariants

"phantom invariants"

$$\boxed{\iota\,[\,F(z_1, \ldots, z_m)\,] \ = \ F(c_1, \ldots, c_r, I_1(z), \ldots, I_{m-r}(z))}$$

Invariantization amounts to restricting F to the cross-section: $I \mid K = F \mid K$, and then requiring that $I = \iota(F)$ be constant along the orbits.

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Replacement Rule:

$$I(z_1, \ldots, z_m) \;=\; I(c_1, \ldots, c_r, I_1(z), \ldots, I_{m-r}(z))$$

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Invariantization defines a canonical projection

$$\iota \;:\; \text{functions} \quad \longmapsto \quad \text{invariants}$$

The Rotation Group

$$G = \mathrm{SO}(2) \qquad \text{acting on} \qquad \mathbb{R}^2$$

$$z = (x, u) \longmapsto g \cdot z = (\, x \cos \phi - u \sin \phi \;,\; x \sin \phi + u \cos \phi \,)$$

$$\Longrightarrow \quad \text{Free on } M = \mathbb{R}^2 \setminus \{0\}$$

Left moving frame:

$$w(g, z) = g^{-1} \cdot z = (y, v)$$

$$y = x \cos \phi + u \sin \phi \qquad v = -\, x \sin \phi + u \cos \phi$$

Cross-section:

$$K = \{\, u = 0, \; x > 0 \,\}$$

Normalization equation:

$$v = -x \sin \phi + u \cos \phi = 0$$

Left moving frame:

$$\phi = \tan^{-1} \frac{u}{x} \quad \Longrightarrow \quad \phi = \rho(x, u) \in \mathrm{SO}(2)$$

Fundamental invariant:

$$r = \iota(x) = \sqrt{x^2 + u^2}$$

Invariantization:

$$\iota[\, F(x, u) \,] = F(r, 0)$$

Replacement theorem: if I is any invariant,

$$I(x, u) = I(r, 0)$$

Prolongation

Most interesting group actions (Euclidean, affine, projective, etc.) are *not* free!

Freeness typically fails because the dimension of the underlying manifold is not large enough, i.e., $m < r = \dim G$.

Thus, to make the action free, we must increase the dimension of the space via some natural prolongation process.

- Prolonging to derivatives (jet space)

$$G^{(n)} : \mathrm{J}^n(M,p) \longrightarrow \mathrm{J}^n(M,p)$$

$\Longrightarrow$ differential invariants

- Prolonging to Cartesian product actions

$$G^{\times n} : M \times \cdots \times M \longrightarrow M \times \cdots \times M$$

$\Longrightarrow$ joint invariants

- Prolonging to "multi-space"

$$G^{(n)} : M^{(n)} \longrightarrow M^{(n)}$$

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Euclidean Plane Curves

Special Euclidean group: $\quad G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$

acts on $M = \mathbb{R}^2$ via rigid motions: $\; w = R\, z + c$

To obtain the classical (left) moving frame we invert the group transformations:

$$\left.\begin{aligned} y &= \quad \cos\phi\,(x-a) + \sin\phi\,(u-b) \\ v &= -\sin\phi\,(x-a) + \cos\phi\,(u-b) \end{aligned}\right\} \qquad w = R^{-1}(z-c)$$

Assume for simplicity the curve is (locally) a graph:

$$\mathcal{C} = \{u = f(x)\}$$

$\Longrightarrow$ extensions to parametrized curves are straightforward

Prolong the action to J^n via implicit differentiation:

$$y = \cos\phi\,(x-a) + \sin\phi\,(u-b)$$

$$v = -\sin\phi\,(x-a) + \cos\phi\,(u-b)$$

$$v_y = \frac{-\sin\phi + u_x\cos\phi}{\cos\phi + u_x\sin\phi}$$

$$v_{yy} = \frac{u_{xx}}{(\cos\phi + u_x\sin\phi)^3}$$

$$v_{yyy} = \frac{(\cos\phi + u_x\sin\phi)u_{xxx} - 3u_{xx}^2\sin\phi}{(\cos\phi + u_x\sin\phi)^5}$$

$$\vdots$$

Normalization: $\quad r = \dim G = 3$

$$y = \cos\phi\,(x-a) + \sin\phi\,(u-b) = 0$$

$$v = -\sin\phi\,(x-a) + \cos\phi\,(u-b) = 0$$

$$v_y = \frac{-\sin\phi + u_x\cos\phi}{\cos\phi + u_x\sin\phi} = 0$$

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$$v_{yyy} = \frac{(\cos\phi + u_x\sin\phi)u_{xxx} - 3u_{xx}^2\sin\phi}{(\cos\phi + u_x\sin\phi)^5}$$

$$\vdots$$

Solve for the group parameters:

$$y = \cos\phi\,(x-a) + \sin\phi\,(u-b) = 0$$

$$v = -\sin\phi\,(x-a) + \cos\phi\,(u-b) = 0$$

$$v_y = \frac{-\sin\phi + u_x\cos\phi}{\cos\phi + u_x\sin\phi} = 0$$

$\Longrightarrow$ Left moving frame $\quad \rho : \mathrm{J}^1 \longrightarrow \mathrm{SE}(2)$

$$a = x \qquad b = u \qquad \phi = \tan^{-1} u_x$$

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Differential invariants

$$v_{yy} = \frac{u_{xx}}{(\cos \phi + u_x \sin \phi)^3} \longmapsto \kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}}$$

$$v_{yyy} = \cdots \qquad \longmapsto \frac{d\kappa}{ds} = \frac{(1 + u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1 + u_x^2)^3}$$

$$v_{yyyy} = \cdots \qquad \longmapsto \frac{d^2\kappa}{ds^2} - 3\kappa^3 = \cdots$$

$\Longrightarrow$ recurrence formulae

Contact invariant one-form — arc length

$$dy = (\cos \phi + u_x \sin \phi)\, dx \qquad \longmapsto \qquad ds = \sqrt{1 + u_x^2}\; dx$$

Dual invariant differential operator

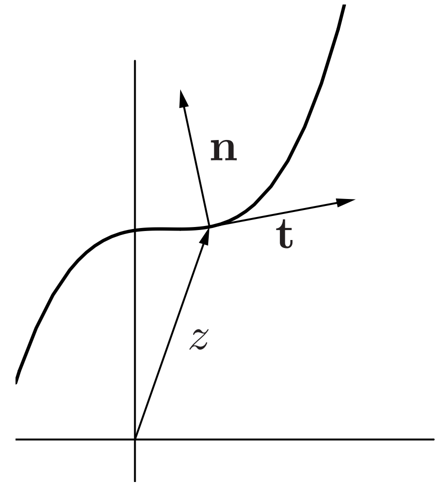
— arc length derivative

$$\frac{d}{dy} = \frac{1}{\cos\phi + u_x \sin\phi}\,\frac{d}{dx} \qquad \longmapsto \qquad \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx}$$

Theorem. All differential invariants are functions of the derivatives of curvature with respect to arc length:

$$\kappa, \qquad \frac{d\kappa}{ds}, \qquad \frac{d^2\kappa}{ds^2}, \qquad \cdots$$

The Classical Picture:



Moving frame $\qquad \rho : \; (x, u, u_x) \longmapsto (R, c) \in \mathrm{SE}(2)$

$$R = \frac{1}{\sqrt{1 + u_x^2}} \begin{pmatrix} 1 & -u_x \\ u_x & 1 \end{pmatrix} = (\, \mathbf{t}, \; \mathbf{n} \,) \qquad c = \begin{pmatrix} x \\ u \end{pmatrix} = z$$

Frenet frame

$$\mathbf{t} = \frac{d\mathbf{x}}{ds} = \begin{pmatrix} x_s \\ y_s \end{pmatrix}, \qquad \mathbf{n} = \mathbf{t}^{\perp} = \begin{pmatrix} -\,y_s \\ x_s \end{pmatrix}.$$

Frenet equations = Pulled-back Maurer–Cartan forms:

$$\frac{d\mathbf{x}}{ds} = \mathbf{t}, \qquad \frac{d\mathbf{t}}{ds} = \kappa\,\mathbf{n}, \qquad \frac{d\mathbf{n}}{ds} = -\,\kappa\,\mathbf{t}.$$

Equi-affine Curves $\qquad G = \mathrm{SA}(2)$

$$z \longmapsto A\, z + c \qquad A \in \mathrm{SL}(2), \qquad c \in \mathbb{R}^2$$

Invert for left moving frame:

$$\left. \begin{aligned} y &= \delta\,(x-a) - \beta\,(u-b) \\ v &= -\gamma\,(x-a) + \alpha\,(u-b) \end{aligned} \right\} \qquad w = A^{-1}(z-c)$$

$$\alpha\,\delta - \beta\,\gamma = 1$$

Prolong to J^3 via implicit differentiation

$$dy = (\delta - \beta\,u_x)\;dx \qquad D_y = \frac{1}{\delta - \beta\,u_x}\,D_x$$

Prolongation:

$$y = \delta\,(x - a) - \beta\,(u - b)$$

$$v = -\,\gamma\,(x - a) + \alpha\,(u - b)$$

$$v_y = -\,\frac{\gamma - \alpha\,u_x}{\delta - \beta\,u_x}$$

$$v_{yy} = \frac{u_{xx}}{(\delta - \beta\,u_x)^3}$$

$$v_{yyy} = \frac{(\delta - \beta\,u_x)\,u_{xxx} + 3\,\beta\,u_{xx}^2}{(\delta - \beta\,u_x)^5}$$

$$v_{yyyy} = \frac{u_{xxxx}(\delta - \beta\,u_x)^2 + 10\,\beta\,(\delta - \beta\,u_x)\,u_{xx}\,u_{xxx} + 15\,\beta^2\,u_{xx}^3}{(\delta - \beta\,u_x)^7}$$

$$v_{yyyyy} = \;\ldots$$

Normalization: $\qquad r = \dim G = 5$

$$y = \delta\,(x-a) - \beta\,(u-b) = 0$$

$$v = -\gamma\,(x-a) + \alpha\,(u-b) = 0$$

$$v_y = -\frac{\gamma - \alpha\, u_x}{\delta - \beta\, u_x} = 0$$

$$v_{yy} = \frac{u_{xx}}{(\delta - \beta\, u_x)^3} = 1$$

$$v_{yyy} = \frac{(\delta - \beta\, u_x)\, u_{xxx} + 3\,\beta\, u_{xx}^2}{(\delta - \beta\, u_x)^5} = 0$$

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$$v_{yyyyy} = \ \ldots$$

Equi-affine Moving Frame

$$\rho: \ (x, u, u_x, u_{xx}, u_{xxx}) \longmapsto (A, c) \in \mathrm{SA}(2)$$

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} u_{xx}^{-1/3} & -\frac{1}{3}\, u_{xx}^{-5/3} u_{xxx} \\ u_x u_{xx}^{-1/3} & u_{xx}^{1/3} - \frac{1}{3}\, u_x u_{xx}^{-5/3} u_{xxx} \end{pmatrix}$$

$$c = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x \\ u \end{pmatrix}$$

Nondegeneracy condition (freeness): $\quad u_{xx} \neq 0.$

Equi-affine arc length

$$dy = (\delta - \beta\, u_x)\ dx \quad \longmapsto \quad ds = \sqrt[3]{u_{xx}}\ dx$$

Equi-affine curvature

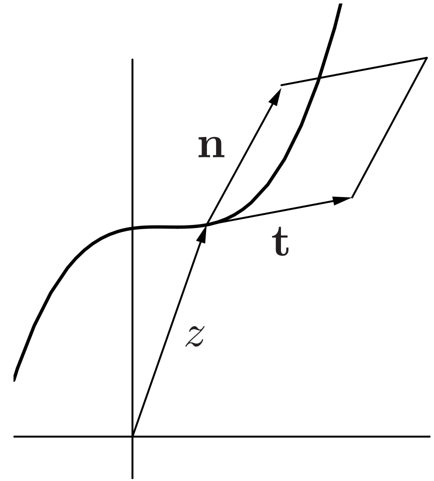
$$v_{yyyy} \quad \longmapsto \quad \kappa \;=\; \frac{5\, u_{xx} u_{xxxx} - 3\, u_{xxx}^2}{9\, u_{xx}^{8/3}}$$

$$v_{yyyyy} \quad \longmapsto \quad \frac{d\kappa}{ds}$$

$$v_{yyyyyy} \quad \longmapsto \quad \frac{d^2\kappa}{ds^2} - 5\kappa^2$$

⋆ ⋆ recurrence formulae

The Classical Picture:



$$A = \begin{pmatrix} u_{xx}^{-1/3} & -\frac{1}{3}\,u_{xx}^{-5/3}u_{xxx} \\ u_x u_{xx}^{-1/3} & u_{xx}^{1/3} - \frac{1}{3}\,u_x u_{xx}^{-5/3}u_{xxx} \end{pmatrix} = (\,\mathbf{t},\ \mathbf{n}\,)$$

$$c = \begin{pmatrix} x \\ u \end{pmatrix} = z$$

Frenet frame

$$\mathbf{t} = \frac{dz}{ds}\,, \qquad \mathbf{n} = \frac{d^2 z}{ds^2}\,.$$

Frenet equations = Pulled-back Maurer–Cartan forms:

$$\frac{dz}{ds} = \mathbf{t}, \qquad \frac{d\mathbf{t}}{ds} = \mathbf{n}, \qquad \frac{d\mathbf{n}}{ds} = \kappa\,\mathbf{t}.$$

Inductive and Recursive Methods

Given $H \subset G$ one can use a recursive method to construct the moving frame for G in terms of the moving frame and differential invariants of H. The calculations also provide expressions for the G differential invariants as functions of the H differential invariants and their invariant derivatives.

Kogan, I.A., Inductive construction of moving frames, *Contemp. Math.* **285** (2001), 157–170.

Olver, P.J., Recursive moving frames, *Results Math.* **60** (2011), 423–452.

Normal Forms

The moving frame normalizations based on a cross-section in the jet space can be reinterpreted as placing the submanifold in normal form, meaning that one uses group transformations to move it to a distinguished location and then successively normalizes the coefficients in the associated Taylor expansion. Once these are fixed, the remaining unnormalized coefficients are the differential invariants.

Normal Forms

For Euclidean plane curves $C \subset \mathbb{R}^2$, translations are used to make the curve go through the origin, and then a rotation makes its tangent horizontal there, producing the

Euclidean normal form

$$u_0(x) = \frac{1}{2}\kappa x^2 + \frac{1}{6}\kappa_s x^3 + \frac{1}{24}(\kappa_{ss} + 3\kappa^3)x^4 + \cdots$$

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Similarly, by employing a sequence of equi-affine transformations one deduces the equi-affine normal form for a plane curve:

$$u_0(x) = \frac{1}{2}x^2 + \frac{1}{4!}\kappa x^4 + \frac{1}{5!}\kappa_s x^5 + \frac{1}{6!}(\kappa_{ss} + 5\kappa^2)x^6 + \cdots ,$$

where κ is equi-affine curvature and ds equi-affine arc length

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where κ is equi-affine curvature and ds equi-affine arc length

$\Longrightarrow$ The formulas for the coefficients are differential invariants and found using the Recurrence Formulae.

The General Set-Up

$\dim M = p + q$ — for example $M = \mathbb{R}^p \times \mathbb{R}^q$

$p = \#$ independent variables $x = (x^1, \ldots, x^p)$;

$q = \#$ dependent variables $u = (u^1, \ldots, u^q)$.

$\mathrm{J}^n = \mathrm{J}^n(M, p)$ — jet space of order n

u^α_J — jet coordinates on J^n (representing partial derivatives of the u's with respect to the x's)

G — Lie (pseudo-)group of point transformations acting on M or of contact transformations on J^1 when $p = 1$

$G^{(n)}$ — prolonged action of G on J^n (implicit differentiation)

$g^{(n)}$ — prolonged infinitesimal generators

Differential Invariants

A differential invariant is a (locally defined) invariant function $I : \mathrm{J}^n \to \mathbb{R}$ for the prolonged (pseudo-)group action

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

$$\Longrightarrow \quad \text{curvature, torsion, } \ldots$$

Invariant differential operators:

$$\mathcal{D}_1, \ldots, \mathcal{D}_p \qquad \Longrightarrow \quad \text{arc length derivative}$$

- If I is a differential invariant, so is $\mathcal{D}_j I$.

$\mathcal{I}(G)$ — the algebra of differential invariants

The Basis Theorem

Theorem. The differential invariant algebra $\mathcal{I}(G)$ is locally generated by a finite number of differential invariants

$$I_1, \ \ldots \ , I_\ell$$

and $p = \dim S$ invariant differential operators

$$\mathcal{D}_1, \ \ldots \ , \mathcal{D}_p$$

meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa .$$

$\Longrightarrow$ Lie groups: *Lie, Ovsiannikov, Fels–O*

$\Longrightarrow$ Lie pseudo-groups: *Tresse, Kumpera, Kruglikov–Lychagin, Muñoz–Muriel–Rodríguez, Pohjanpelto–O*

Key Issues

- Minimal basis of generating invariants: $I_1, \ldots, I_\ell$
- Commutation formulae for

  the invariant differential operators:

$$[\,\mathcal{D}_j, \mathcal{D}_k\,] = \sum_{i=1}^{p} Y_{jk}^i\, \mathcal{D}_i$$

Y_{jk}^i — commutator invariants

$\Longrightarrow$ Non-commutative differential algebra

- Syzygies (functional relations) among

  the differentiated invariants:

$$\Phi(\ \ldots\ \mathcal{D}_J I_\kappa\ \ldots\ ) \equiv 0$$

$\Longrightarrow$ Codazzi relations

Recurrence Formulae

★★ Invariantization and differentiation *do not commute.*

$$\mathcal{D}_j\, \iota(F) = \iota(D_j F) \;+\; \sum_{\kappa=1}^{r} R_j^\kappa\, \iota(\mathbf{v}_\kappa^{(n)}(F))$$

$\omega^i = \iota(dx^i)$ — invariant horizontal coframe

$\mathcal{D}_i = \iota(D_{x^i})$ — dual invariant differential operators

$\mathbf{v}_\kappa^{(n)}$ — basis for $g^{(n)}$ (prolonged infinitesimal generators)

R_j^κ — Maurer–Cartan invariants

Recurrence Formulae

$$\mathcal{D}_j\, \iota(F) = \iota(D_j F) \; + \; \sum_{\kappa=1}^{r} R_j^\kappa \; \iota(\mathbf{v}_\kappa^{(n)}(F))$$

♠ If $\iota(F) = c$ is a phantom differential invariant, then the left hand side of the recurrence formula is zero. The collection of all such phantom recurrence formulae form a linear algebraic system of equations that can be uniquely solved for the Maurer–Cartan invariants R_j^κ!

♡ Once the Maurer–Cartan invariants are replaced by their explicit formulae, the induced recurrence relations completely determine the structure of the differential invariant algebra $\mathcal{I}(G)$!

The Maurer–Cartan Invariants

$\mathbf{v}_1, \ \ldots \ \mathbf{v}_r \in \mathfrak{g}$ — basis for infinitesimal generators

$\mu^1, \ \ldots \ \mu^r \in \mathfrak{g}^*$ — dual basis of Maurer–Cartan forms

Invariantized Maurer–Cartan forms:

$$\gamma^\kappa = \rho^*(\mu^\kappa) \equiv \sum_{j=1}^{p} R_j^\kappa \, \omega^j$$

$\omega^1, \ \ldots \ \omega^p$ — invariant horizontal coframe

R_j^κ — Maurer–Cartan invariants

The Universal Recurrence Formula

For *any* function or differential form Ω on J^n:

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \gamma^\kappa \wedge \iota[\,\mathbf{v}_\kappa^{(n)}(\Omega)\,]$$

$\mathbf{v}_1^{(n)}, \ldots, \mathbf{v}_r^{(n)}$ — basis for prolonged infinitesimal generators

$\gamma^1, \ldots, \gamma^r$ — dual invariantized Maurer–Cartan forms

⋆⋆ The γ^κ are uniquely determined by the recurrence formulae for the phantom differential invariants

$$d\,\iota(\Omega) = \iota(d\Omega) + \sum_{\kappa=1}^{r} \gamma^{\kappa} \wedge \iota[\,\mathbf{v}_{\kappa}(\Omega)\,]$$

★ ★ ★ All identities, commutation formulae, syzygies, etc., among differential invariants and, more generally, the invariant variational bicomplex follow from this universal recurrence formula by letting Ω range over the basic functions and differential forms!

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★ ★ ★ All identities, commutation formulae, syzygies, etc., among differential invariants and, more generally, the invariant variational bicomplex follow from this universal recurrence formula by letting Ω range over the basic functions and differential forms!

★ ★ ★ Therefore, the entire structure of the differential invariant algebra and invariant variational bicomplex can be completely determined using only linear differential algebra; this does not require explicit formulas for the moving frame, the differential invariants, the invariant differential forms, or the group transformations!

The Commutator Invariants

Explicit formulae:

$$\boxed{Y^i_{jk} = \sum_{\kappa=1}^{r} R^\kappa_k \, \iota(D_j \xi^i_\kappa) - R^\kappa_j \, \iota(D_k \xi^i_\kappa) \,.}$$

Follows from the recurrence formulae for

$$d\omega^i = d[\iota(dx^i)] = \iota(d^2 x^i) + \sum_{\kappa=1}^{r} \gamma^\kappa \wedge \iota[\mathbf{v}_\kappa(dx^i)]$$

$$= - \sum_{j<k} Y^i_{jk} \, \omega^j \wedge \omega^k + \cdots$$

Generating Differential Invariants

Theorem. (*Fels–O*) If the moving frame has order n, then the set of normalized differential invariants of order $\leq n+1$ forms a generating set.

Theorem. (*O–Hubert*) Given a *minimal order cross-section*, meaning that, for each $k = 0, 1, \ldots, n$,

$$Z_1(x, u^{(k)}) = c_1, \quad \ldots \quad Z_{r_k}(x, u^{(k)}) = c_{r_k},$$

defines a cross-section for the action of $G^{(k)}$ on J^k, then the differential invariants $\iota(D_i Z_j)$ for $i = 1, \ldots, p$, $j = 1, \ldots, r$ and, in the intransitive case, the order zero invariants, form a generating set.

Theorem. (*Hubert*) The Maurer–Cartan invariants and, in the intransitive case, the order zero invariants serve to generate the differential invariant algebra $\mathcal{I}(G)$.

The Differential Invariant Algebra

Thus, remarkably, the structure of $\mathcal{I}(G)$ can be determined without knowing the explicit formulae for either the moving frame, or the differential invariants, or the invariant differential operators!

The only required ingredients are the specification of the cross-section, and the standard formulae for the prolonged infinitesimal generators.

Theorem. If G acts transitively on M, or if the infinitesimal generator coefficients depend rationally in the coordinates, then all recurrence formulae are rational in the basic differential invariants and so $\mathcal{I}(G)$ is a rational, non-commutative differential algebra.

Curves

Theorem. Let G be an ordinary* Lie group acting on the m-dimensional manifold M. Then, locally, there exist $m-1$ generating differential invariants $\kappa_1, \ldots, \kappa_{m-1}$. Every other differential invariant can be written as a function of the generating differential invariants and their derivatives with respect to the G-invariant arc length element ds.

* ordinary $=$ transitive $+$ no pseudo-stabilization.

Curves

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* ordinary = transitive + no pseudo-stabilization.

$\Longrightarrow \quad m = 3$ — curvature κ & torsion τ

Euclidean Surfaces

Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$ acts on surfaces $S \subset \mathbb{R}^3$.

For simplicity, we assume the surface is (locally) the graph of a function

$$z = u(x, y)$$

Infinitesimal generators:

$$\mathbf{v}_1 = -y\,\partial_x + x\,\partial_y, \qquad \mathbf{v}_2 = -u\,\partial_x + x\,\partial_u, \qquad \mathbf{v}_3 = -u\,\partial_y + y\,\partial_u,$$

$$\mathbf{w}_1 = \partial_x, \qquad \mathbf{w}_2 = \partial_y, \qquad \mathbf{w}_3 = \partial_u.$$

- The translations $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ will be ignored, as they play no role in the higher order recurrence formulae.

Cross-section (Darboux frame):

$$x = y = u = u_x = u_y = u_{xy} = 0.$$

Phantom differential invariants:

$$\iota(x) = \iota(y) = \iota(u) = \iota(u_x) = \iota(u_y) = \iota(u_{xy}) = 0$$

Principal curvatures

$$\kappa_1 = \iota(u_{xx}), \qquad \kappa_2 = \iota(u_{yy})$$

Mean curvature and Gauss curvature:

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2), \qquad K = \kappa_1 \kappa_2$$

Higher order differential invariants — invariantized jet coordinates:

$$I_{jk} = \iota(u_{jk}) \qquad \text{where} \qquad u_{jk} = \frac{\partial^{j+k} u}{\partial x^j \partial y^k}$$

★ ★ Nondegeneracy condition: non-umbilic point $\kappa_1 \neq \kappa_2$.

Algebra of Euclidean Differential Invariants

Principal curvatures:

$$\kappa_1 = \iota(u_{xx}), \qquad \kappa_2 = \iota(u_{yy})$$

Mean curvature and Gauss curvature:

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2), \qquad K = \kappa_1 \kappa_2$$

Invariant differentiation operators:

$$\mathcal{D}_1 = \iota(D_x), \qquad \mathcal{D}_2 = \iota(D_y)$$

$\Longrightarrow$ Differentiation with respect to the diagonalizing Darboux frame.

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Invariant differentiation operators:

$$\mathcal{D}_1 = \iota(D_x), \qquad \mathcal{D}_2 = \iota(D_y)$$

$\Longrightarrow$ Differentiation with respect to the diagonalizing Darboux frame.

The recurrence formulae enable one to express the higher order differential invariants in terms of the principal curvatures, or, equivalently, the mean and Gauss curvatures, and their invariant derivatives:

$$\begin{aligned} I_{jk} = \iota(u_{jk}) &= \widetilde{\Phi}_{jk}(\kappa_1, \kappa_2, \mathcal{D}_1\kappa_1, \mathcal{D}_2\kappa_1, \mathcal{D}_1\kappa_2, \mathcal{D}_2\kappa_2, \mathcal{D}_1^2\kappa_1, \ \ldots\ ) \\ &= \Phi_{jk}(H, K, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ \ldots\ ) \end{aligned}$$

Recurrence Formulae

$$\iota(D_i u_{jk}) = \mathcal{D}_i \, \iota(u_{jk}) - \sum_{\kappa=1}^{3} R_i^\kappa \, \iota[\, \varphi_\kappa^{jk}(x, y, u^{(j+k)}) \,], \qquad j + k \geq 1$$

$I_{jk} = \iota(u_{jk})$ — normalized differential invariants

R_i^κ — Maurer–Cartan invariants

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$$\iota(D_i u_{jk}) = \mathcal{D}_i\, \iota(u_{jk}) - \sum_{\kappa=1}^{3} R_i^\kappa\, \iota[\, \varphi_\kappa^{jk}(x,y,u^{(j+k)})\,], \qquad j+k \geq 1$$

$I_{jk} = \iota(u_{jk})$ — normalized differential invariants

R_i^κ — Maurer–Cartan invariants

$\varphi_\kappa^{jk}(0,0,I^{(j+k)}) = \iota[\, \varphi_\kappa^{jk}(x,y,u^{(j+k)})\,]$

— invariantized prolonged infinitesimal generator coefficients.

$$I_{j+1,k} = \mathcal{D}_1 I_{jk} - \sum_{\kappa=1}^{3} \varphi_\kappa^{jk}(0,0,I^{(j+k)})\, R_1^\kappa$$

$$I_{j,k+1} = \mathcal{D}_1 I_{jk} - \sum_{\kappa=1}^{3} \varphi_\kappa^{jk}(0,0,I^{(j+k)})\, R_2^\kappa$$

Prolonged infinitesimal generators:

$$\begin{aligned}
\mathrm{pr}\,\mathbf{v}_1 &= -y\,\partial_x + x\,\partial_y - u_y\partial_{u_x} + u_x\partial_{u_y} \\
&\quad - 2\,u_{xy}\partial_{u_{xx}} + (u_{xx} - u_{yy})\,\partial_{u_{xy}} - 2\,u_{xy}\partial_{u_{yy}} + \ \cdots\ , \\
\mathrm{pr}\,\mathbf{v}_2 &= -u\,\partial_x + x\,\partial_u + (1 + u_x^2)\,\partial_{u_x} + u_x u_y\partial_{u_y} \\
&\quad + 3\,u_x u_{xx}\partial_{u_{xx}} + (u_y u_{xx} + 2\,u_x u_{xy})\,\partial_{u_{xy}} + (2\,u_y u_{xy} + u_x u_{yy})\,\partial_{u_{yy}} + \ \cdots\ , \\
\mathrm{pr}\,\mathbf{v}_3 &= -u\,\partial_y + y\,\partial_u + u_x u_y\partial_{u_x} + (1 + u_y^2)\,\partial_{u_y} \\
&\quad + (u_y u_{xx} + 2\,u_x u_{xy})\,\partial_{u_{xx}} + (2\,u_y u_{xy} + u_x u_{yy})\,\partial_{u_{xy}} + 3\,u_y u_{yy}\partial_{u_{yy}} + \ \cdots\ .
\end{aligned}$$

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\mathrm{pr}\,\mathbf{v}_2 &= -u\,\partial_x + x\,\partial_u + (1 + u_x^2)\,\partial_{u_x} + u_x\,u_y\partial_{u_y} \\
&\qquad + 3\,u_x\,u_{xx}\partial_{u_{xx}} + (u_y\,u_{xx} + 2\,u_x\,u_{xy})\,\partial_{u_{xy}} + (2\,u_y\,u_{xy} + u_x\,u_{yy})\,\partial_{u_{yy}} + \ \cdots\ ,\\
\mathrm{pr}\,\mathbf{v}_3 &= -u\,\partial_y + y\,\partial_u + u_x\,u_y\partial_{u_x} + (1 + u_y^2)\,\partial_{u_y} \\
&\qquad + (u_y\,u_{xx} + 2\,u_x\,u_{xy})\,\partial_{u_{xx}} + (2\,u_y\,u_{xy} + u_x\,u_{yy})\,\partial_{u_{xy}} + 3\,u_y\,u_{yy}\partial_{u_{yy}} + \ \cdots\ .
\end{aligned}$$

$$I_{jk} = \iota(u_{jk})$$

Phantom differential invariants:

$$I_{00} = I_{10} = I_{01} = I_{11} = 0$$

Principal curvatures:

$$I_{20} = \kappa_1 \qquad I_{02} = \kappa_2$$

Phantom recurrence formulae:

$$\begin{aligned}
\kappa_1 = I_{20} &= \mathcal{D}_1 I_{10} - R_1^2 = -R_1^2, \\
0 = I_{11} &= \mathcal{D}_1 I_{01} - R_1^3 = -R_1^3, \\
I_{21} &= \mathcal{D}_1 I_{11} - (\kappa_1 - \kappa_2)\, R_1^1 = -(\kappa_1 - \kappa_2)\, R_1^1, \\
0 = I_{11} &= \mathcal{D}_2 I_{10} - R_2^2 = -R_2^2, \\
\kappa_2 = I_{02} &= \mathcal{D}_2 I_{01} - R_2^3 = -R_2^3, \\
I_{12} &= \mathcal{D}_2 I_{11} - (\kappa_1 - \kappa_2)\, R_2^1 = -(\kappa_1 - \kappa_2)\, R_2^1.
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\kappa_2 = I_{02} &= \mathcal{D}_2 I_{01} - R_2^3 = -R_2^3, \\
I_{12} &= \mathcal{D}_2 I_{11} - (\kappa_1 - \kappa_2)\, R_2^1 = -(\kappa_1 - \kappa_2)\, R_2^1.
\end{aligned}$$

Maurer–Cartan invariants:

$$\begin{aligned}
R_1^1 &= -Y_1, \qquad R_1^2 = -\kappa_1, \qquad R_1^3 = 0, \\
R_2^1 &= -Y_2, \qquad R_2^2 = 0, \qquad R_2^3 = -\kappa_2.
\end{aligned}$$

Commutator invariants:

$$Y_1 = \frac{I_{21}}{\kappa_1 - \kappa_2} = \frac{\mathcal{D}_1 \kappa_2}{\kappa_1 - \kappa_2} \qquad\qquad Y_2 = \frac{I_{12}}{\kappa_1 - \kappa_2} = \frac{\mathcal{D}_2 \kappa_1}{\kappa_2 - \kappa_1}$$

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I_{12} &= \mathcal{D}_2 I_{11} - (\kappa_1 - \kappa_2)\, R_2^1 = -(\kappa_1 - \kappa_2)\, R_2^1.
\end{aligned}$$

Maurer–Cartan invariants:

$$R_1^1 = -Y_1, \qquad R_1^2 = -\kappa_1, \qquad R_1^3 = 0,$$

$$R_2^1 = -Y_2, \qquad R_2^2 = 0, \qquad R_2^3 = -\kappa_2.$$

Commutator invariants:

$$Y_1 = \frac{I_{21}}{\kappa_1 - \kappa_2} = \frac{\mathcal{D}_1 \kappa_2}{\kappa_1 - \kappa_2} \qquad\qquad Y_2 = \frac{I_{12}}{\kappa_1 - \kappa_2} = \frac{\mathcal{D}_2 \kappa_1}{\kappa_2 - \kappa_1}$$

$$\boxed{[\,\mathcal{D}_1, \mathcal{D}_2\,] = \mathcal{D}_1\, \mathcal{D}_2 - \mathcal{D}_2\, \mathcal{D}_1 = Y_2\, \mathcal{D}_1 - Y_1\, \mathcal{D}_2,}$$

Third order recurrence relations:

$$I_{30} = \mathcal{D}_1\kappa_1 = \kappa_{1,1}, \quad I_{21} = \mathcal{D}_2\kappa_1 = \kappa_{1,2}, \quad I_{12} = \mathcal{D}_1\kappa_2 = \kappa_{2,1}, \quad I_{03} = \mathcal{D}_2\kappa_2 = \kappa_{2,2},$$

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Fourth order recurrence relations:

$$\begin{aligned}
I_{40} &= \kappa_{1,11} - \frac{3\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} + 3\,\kappa_1^3, \\
I_{31} &= \kappa_{1,12} - \frac{3\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2} && = \kappa_{1,21} + \frac{\kappa_{1,1}\kappa_{1,2} - 2\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2}, \\
I_{22} &= \kappa_{1,22} + \frac{\kappa_{1,1}\kappa_{2,1} - 2\,\kappa_{2,1}^2}{\kappa_1 - \kappa_2} + \kappa_1\kappa_2^2 && = \kappa_{2,11} - \frac{\kappa_{1,2}\kappa_{2,2} - 2\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} + \kappa_1^2\kappa_2, \\
I_{13} &= \kappa_{2,21} + \frac{3\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2} && = \kappa_{2,12} - \frac{\kappa_{2,1}\kappa_{2,2} - 2\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2}, \\
I_{04} &= \kappa_{2,22} + \frac{3\,\kappa_{2,1}^2}{\kappa_1 - \kappa_2} + 3\,\kappa_2^3.
\end{aligned}$$

★ The two expressions for I_{31} and I_{13} follow from the commutator formula.

Fourth order recurrence relations

$$I_{40} = \kappa_{1,11} - \frac{3\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} + 3\,\kappa_1^3,$$

$$I_{31} = \kappa_{1,12} - \frac{3\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2} = \kappa_{1,21} + \frac{\kappa_{1,1}\kappa_{1,2} - 2\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2},$$

$$I_{22} = \kappa_{1,22} + \frac{\kappa_{1,1}\kappa_{2,1} - 2\,\kappa_{2,1}^2}{\kappa_1 - \kappa_2} + \kappa_1\kappa_2^2 = \kappa_{2,11} - \frac{\kappa_{1,2}\kappa_{2,2} - 2\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} + \kappa_1^2\kappa_2,$$

$$I_{13} = \kappa_{2,21} + \frac{3\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2} = \kappa_{2,12} - \frac{\kappa_{2,1}\kappa_{2,2} - 2\,\kappa_{1,2}\kappa_{2,1}}{\kappa_1 - \kappa_2},$$

$$I_{04} = \kappa_{2,22} + \frac{3\,\kappa_{2,1}^2}{\kappa_1 - \kappa_2} + 3\,\kappa_2^3.$$

★ ★ The two expressions for I_{22} imply the Codazzi syzygy

$$\kappa_{1,22} - \kappa_{2,11} + \frac{\kappa_{1,1}\,\kappa_{2,1} + \kappa_{1,2}\,\kappa_{2,2} - 2\,\kappa_{2,1}^2 - 2\,\kappa_{1,2}^2}{\kappa_1 - \kappa_2} - \kappa_1\,\kappa_2\,(\kappa_1 - \kappa_2) = 0,$$

which can be written compactly as

$$K = \kappa_1\kappa_2 = -\,(\mathcal{D}_1 + Y_1)\,Y_1 - (\mathcal{D}_2 + Y_2)\,Y_2.$$

$\Longrightarrow$ Gauss' Theorema Egregium

Generating Differential Invariants

♡ From the general structure of the recurrence relations, one proves that the Euclidean differential invariant algebra $\mathcal{I}_{\mathrm{SE}(3)}$ is generated by the principal curvatures κ_1, κ_2 or, equivalently, the mean and Gauss curvatures, H, K, through the process of invariant differentiation:

$$I = \Phi(H, K, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ \ldots\ )$$

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♢ Remarkably, for suitably generic surfaces, the Gauss curvature can be written as a universal rational function of the mean curvature and its invariant derivatives of order ≤ 4:

$$K = \Psi(H, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1^2 H, \ \ldots\ , \mathcal{D}_2^4 H)$$

and hence $\mathcal{I}_{\mathrm{SE}(3)}$ is generated by mean curvature alone!

Generating Differential Invariants

♡ From the general structure of the recurrence relations, one proves that the Euclidean differential invariant algebra $\mathcal{I}_{\mathrm{SE}(3)}$ is generated by the principal curvatures κ_1, κ_2 or, equivalently, the mean and Gauss curvatures, H, K, through the process of invariant differentiation:

$$I = \Phi(H, K, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ \ldots \ )$$

♢ Remarkably, for suitably generic surfaces, the Gauss curvature can be written as a universal rational function of the mean curvature and its invariant derivatives of order ≤ 4:

$$K = \Psi(H, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1^2 H, \ \ldots \ , \mathcal{D}_2^4 H)$$

and hence $\mathcal{I}_{\mathrm{SE}(3)}$ is generated by mean curvature alone!

♠ To prove this, in view of the Codazzi syzygy

$$K = \kappa_1 \kappa_2 = -\,(\mathcal{D}_1 + Y_1)\, Y_1 - (\mathcal{D}_2 + Y_2)\, Y_2,$$

it suffices to write the commutator invariants Y_1, Y_2 in terms of H.

The Commutator Trick

$$\boxed{K = \kappa_1 \kappa_2 = -(\mathcal{D}_1 + Y_1) Y_1 - (\mathcal{D}_2 + Y_2) Y_2}$$

To determine the commutator invariants:

$$\begin{aligned} \mathcal{D}_1 \mathcal{D}_2 H - \mathcal{D}_2 \mathcal{D}_1 H &= Y_2 \, \mathcal{D}_1 H - Y_1 \, \mathcal{D}_2 H \\ \mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_J H - \mathcal{D}_2 \mathcal{D}_1 \mathcal{D}_J H &= Y_2 \, \mathcal{D}_1 \mathcal{D}_J H - Y_1 \, \mathcal{D}_2 \mathcal{D}_J H \end{aligned} \qquad (*)$$

Non-degeneracy condition:

$$\det \begin{pmatrix} \mathcal{D}_1 H & \mathcal{D}_2 H \\ \mathcal{D}_1 \mathcal{D}_J H & \mathcal{D}_2 \mathcal{D}_J H \end{pmatrix} \neq 0,$$

Solve $(*)$ for Y_1, Y_2 in terms of derivatives of H, producing a universal formula

$$\boxed{K = \Psi(H, \mathcal{D}_1 H, \mathcal{D}_2 H, \ \ldots \ )}$$

for the Gauss curvature as a rational function of the mean curvature and its invariant derivatives!

Definition. A surface $S \subset \mathbb{R}^3$ is mean curvature degenerate if, near any non-umbilic point $p_0 \in S$, there exist scalar functions $F_1(t), F_2(t)$ such that

$$\mathcal{D}_1 H = F_1(H), \qquad \mathcal{D}_2 H = F_2(H).$$

- surfaces with symmetry: rotation, helical;
- minimal surfaces;
- constant mean curvature surfaces;
- ???

Theorem. If a surface is mean curvature non-degenerate then the algebra of Euclidean differential invariants is generated entirely by the mean curvature and its successive invariant derivatives.

Minimal Generating Invariants

Euclidean curves $C \subset \mathbb{R}^3$:	curvature κ and torsion τ
Equi–affine curves $C \subset \mathbb{R}^3$:	affine curvature κ and torsion τ

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Equi–affine surfaces $S \subset \mathbb{R}^3$:	Pick invariant P.
Conformal surfaces $S \subset \mathbb{R}^3$:	third order invariant J_3.
Projective surfaces $S \subset \mathbb{R}^3$:	fourth order invariant K_4.
Ternary forms $u = P(x, y)$:	third order invariant L_3.

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Ternary forms $u = P(x, y)$:	third order invariant L_3.

$\Longrightarrow$ For any $n \geq 1$, there exists a Lie group G_N acting on surfaces $S \subset \mathbb{R}^3$ such that its differential invariant algebra requires n generating invariants!

♠ Finding a minimal generating set appears to be a very difficult problem.
(No known bound on order of syzygies.)

Equivalence & Invariants

- Equivalent submanifolds $N \approx \overline{N}$
  must have the same invariants: $I = \bar{I}$.

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Non-constant invariants are not useful in isolation, because an equivalence map can drastically alter the dependence on the submanifold parameters:

$$\text{e.g.} \qquad \kappa = x^3 \qquad \text{versus} \qquad \overline{\kappa} = \sinh x$$

However, a functional dependency or syzygy among the invariants *is* intrinsic:

e.g. $\kappa_s = \kappa^3 - 1 \quad \Longleftrightarrow \quad \overline{\kappa}_{\bar{s}} = \overline{\kappa}^3 - 1$

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- Universal syzygies — Gauss–Codazzi
- Distinguishing syzygies.

Theorem. (Cartan) Two regular submanifolds are (locally) equivalent if and only if they have identical syzygies among *all* their differential invariants.

Finiteness of Generators and Syzygies

♠ There are, in general, an infinite number of differential invariants and hence an infinite number of syzygies must be compared to establish equivalence.

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♡ But the higher order differential invariants are always generated by invariant differentiation from a finite collection of basic differential invariants, and the higher order syzygies are all consequences of a finite number of low order syzygies!

♢ A suitable collection of low order fundamental differential invariants will parametrize a signature Σ of the original submanifold N. Two regular submanifolds are (locally) equivalent: $\overline{N} = g \cdot N$ if and only if they have identical signatures: $\overline{\Sigma} = \Sigma$.

Example — Plane Curves

If non-constant, both κ and κ_s depend on a single parameter, and so, locally, are subject to a syzygy:

$$\kappa_s = H(\kappa) \qquad (*)$$

But then

$$\kappa_{ss} = \frac{d}{ds} H(\kappa) = H'(\kappa)\,\kappa_s = H'(\kappa)\,H(\kappa)$$

and similarly for κ_{sss}, etc.

Consequently, all the higher order syzygies are generated by the fundamental first order syzygy $(*)$.

Thus, for Euclidean (or equi-affine or projective or . . .) plane curves we need only know a single syzygy between the fundamental differential invariants κ and κ_s in order to establish equivalence!

The Signature Map

The generating syzygies are encoded by the signature map

$$\chi : N \longrightarrow \Sigma$$

of the submanifold N, which is parametrized by the fundamental differential invariants:

$$\chi(x) = (I_1(x), \ldots, I_m(x))$$

The image

$$\Sigma = \operatorname{Im} \chi$$

is the signature subset (or submanifold) of N.

Equivalence & Signature

Theorem. Two regular submanifolds are equivalent:

$$\overline{N} = g \cdot N$$

if and only if their signatures are identical:

$$\overline{\Sigma} \;=\; \Sigma$$

Signature Curves

Definition. The signature curve $\Sigma \subset \mathbb{R}^2$ of a plane curve $C \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants

$$\chi \; : \; C \longrightarrow \Sigma = \left\{ \left(\kappa \, , \, \frac{d\kappa}{ds} \right) \right\} \subset \mathbb{R}^2$$

$\Longrightarrow$ *Calabi, PJO, Shakiban, Tannenbaum, Haker*

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Theorem. Two regular curves C and $\overline{C}$ are locally equivalent:

$$\overline{C} = g \cdot C$$

if and only if their signature curves are identical:

$$\overline{\Sigma} = \Sigma$$

$\Longrightarrow$ regular: $(\kappa_s, \kappa_{ss}) \neq 0.$

3D Differential Invariant Signatures

Euclidean space curves: $C \subset \mathbb{R}^3$

$$\Sigma = \{ \, (\, \kappa \, , \, \kappa_s \, , \, \tau \,) \, \} \; \subset \; \mathbb{R}^3$$

- κ — curvature, τ — torsion

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Euclidean surfaces: $S \subset \mathbb{R}^3$ (generic)

$$\Sigma = \left\{ \left(H \, , \, K \, , \, H_{,1} \, , \, H_{,2} \, , \, K_{,1} \, , \, K_{,2} \right) \right\} \, \subset \, \mathbb{R}^6$$

or $$\widehat{\Sigma} = \left\{ \left(H \, , \, H_{,1} \, , \, H_{,2} \, , \, H_{,11} \right) \right\} \, \subset \, \mathbb{R}^4$$

- H — mean curvature, K — Gauss curvature

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Equi–affine surfaces: $S \subset \mathbb{R}^3$ (generic)

$$\Sigma = \Big\{ \Big(P \, , \, P_{,1} \, , \, P_{,2} , \, P_{,11} \Big) \Big\} \; \subset \; \mathbb{R}^4$$

- P — Pick invariant

Symmetry and Signature

$$G_S = \text{(local) symmetry group(oid) of } S$$

$$= \{\, g \in G \mid \; g \cdot (S \cap U) \subset S \,\}$$

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★ Regular submanifolds:
the (local) dimension of the signature equals
the co-dimension of the (local) symmetry group:

$$\dim \Sigma = \dim S - \dim G_S$$

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⋆ Regular submanifolds:
the (local) dimension of the signature equals
the co-dimension of the (local) symmetry group:

$$\dim \Sigma = \dim S - \dim G_S$$

- Maximally symmetric: $\dim \Sigma = 0$

$\Longleftrightarrow$ all the differential invariants are constant

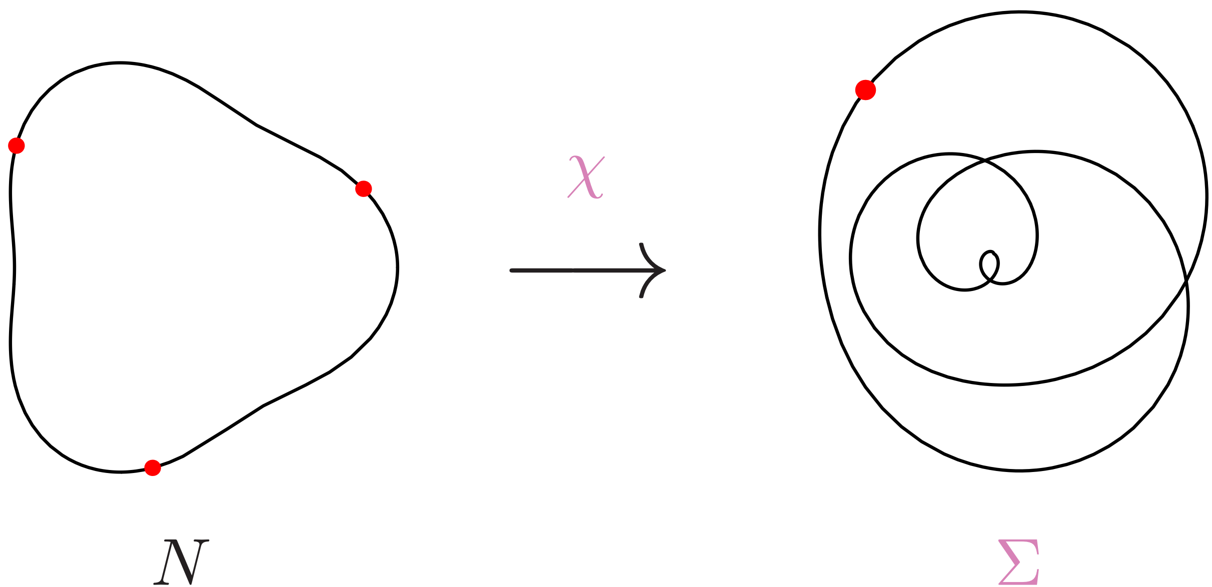
$\Longleftrightarrow$ $\dim G_S = \dim S = p$

$\Longleftrightarrow$ $S \subset H \cdot z_0$ is a piece of
an orbit of a p-dimensional subgroup $H \subset G$

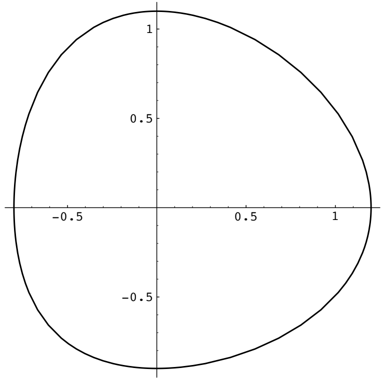
- Discrete symmetries: $\dim \Sigma = p = \dim S$

  The number of discrete (local) symmetries: $\# \, G_S$

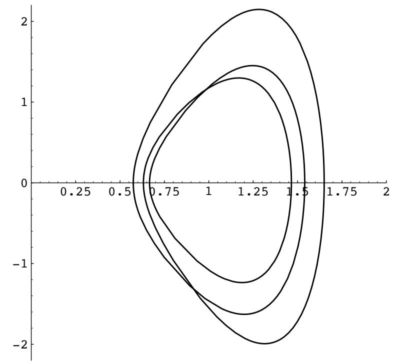
  equals the (local) index of the signature.



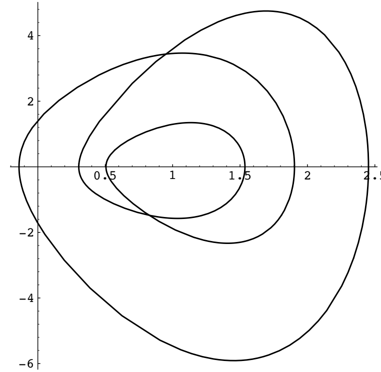
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \;\; y = \sin t + \frac{1}{10}\sin^2 t$



The Original Curve

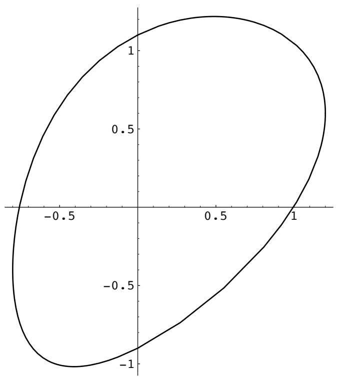


Euclidean Signature

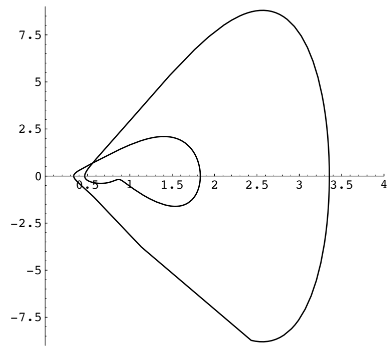


Equi-affine Signature

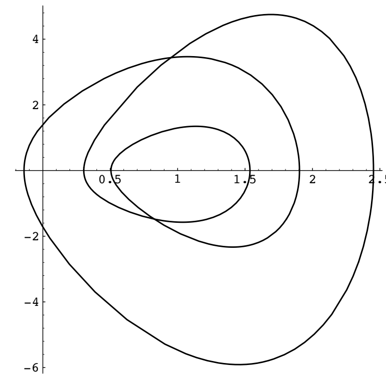
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \frac{1}{2}x + \sin t + \frac{1}{10}\sin^2 t$



The Original Curve

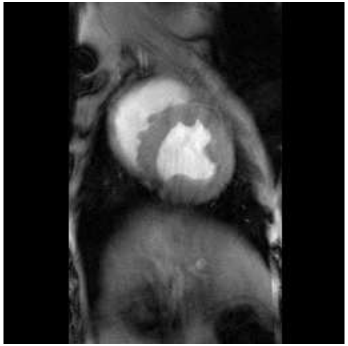


Euclidean Signature

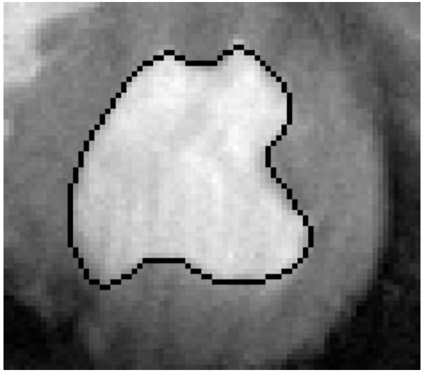


Equi-affine Signature

Canine Left Ventricle Signature

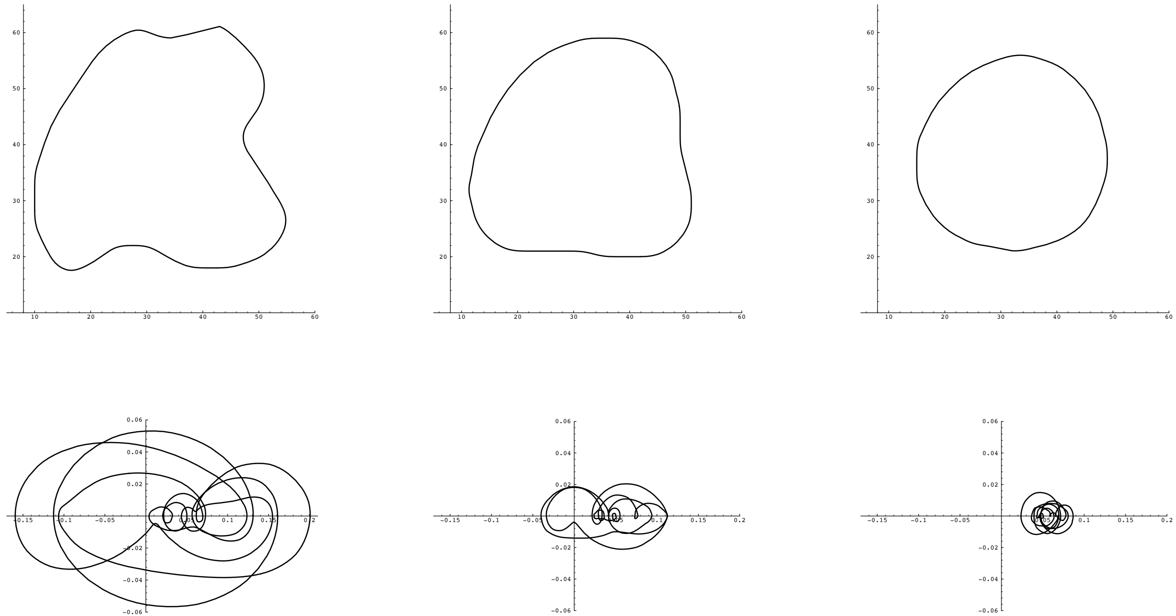


Original Canine Heart
MRI Image

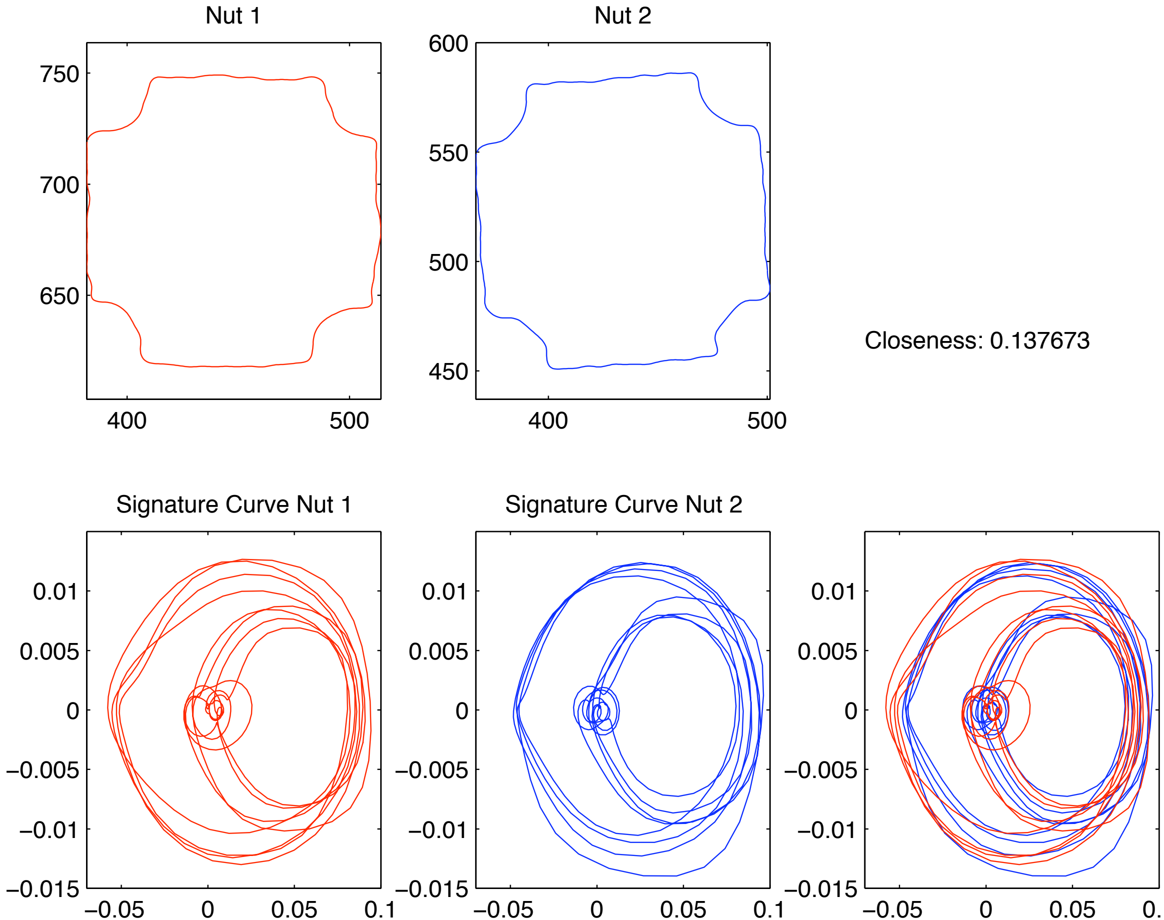


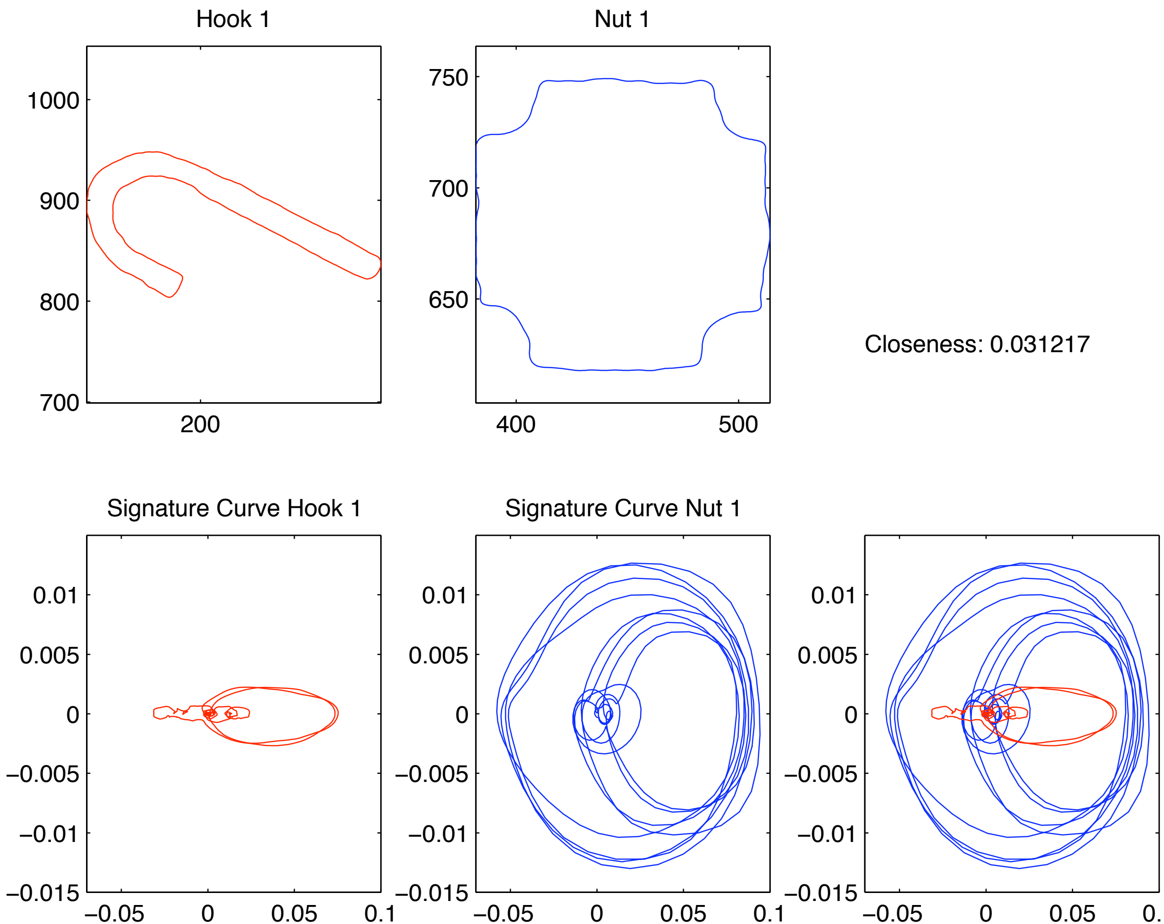
Boundary of Left Ventricle

Smoothed Ventricle Signature



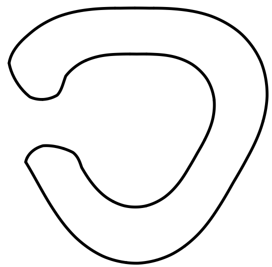
$\Longrightarrow$ Steve Haker


Nut 1
Nut 2
Closeness: 0.137673
Signature Curve Nut 1
Signature Curve Nut 2
750
700
650
400
500
600
550
500
450
400
500
0.01
0.005
0
−0.005
−0.01
−0.015
−0.05
0
0.05
0.1



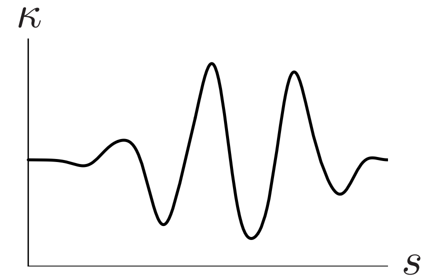
Hook 1
Nut 1
Closeness: 0.031217
Signature Curve Hook 1
Signature Curve Nut 1

Signatures

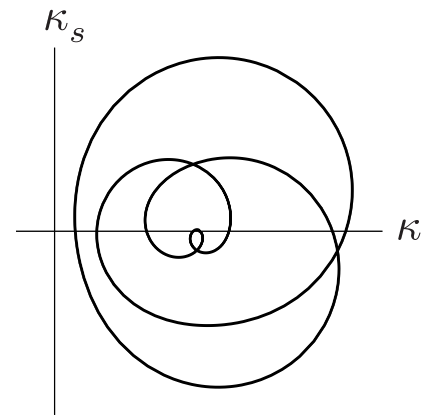


Original curve

$\longrightarrow$

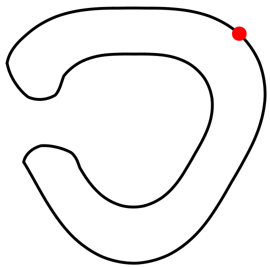


Classical signature

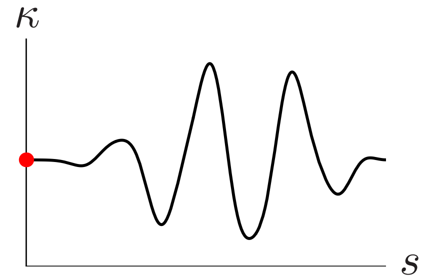


Differential invariant signature

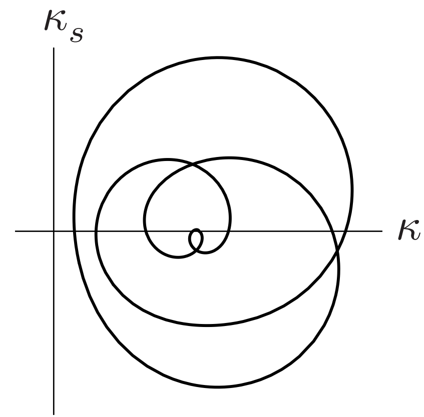
Signatures



Original curve

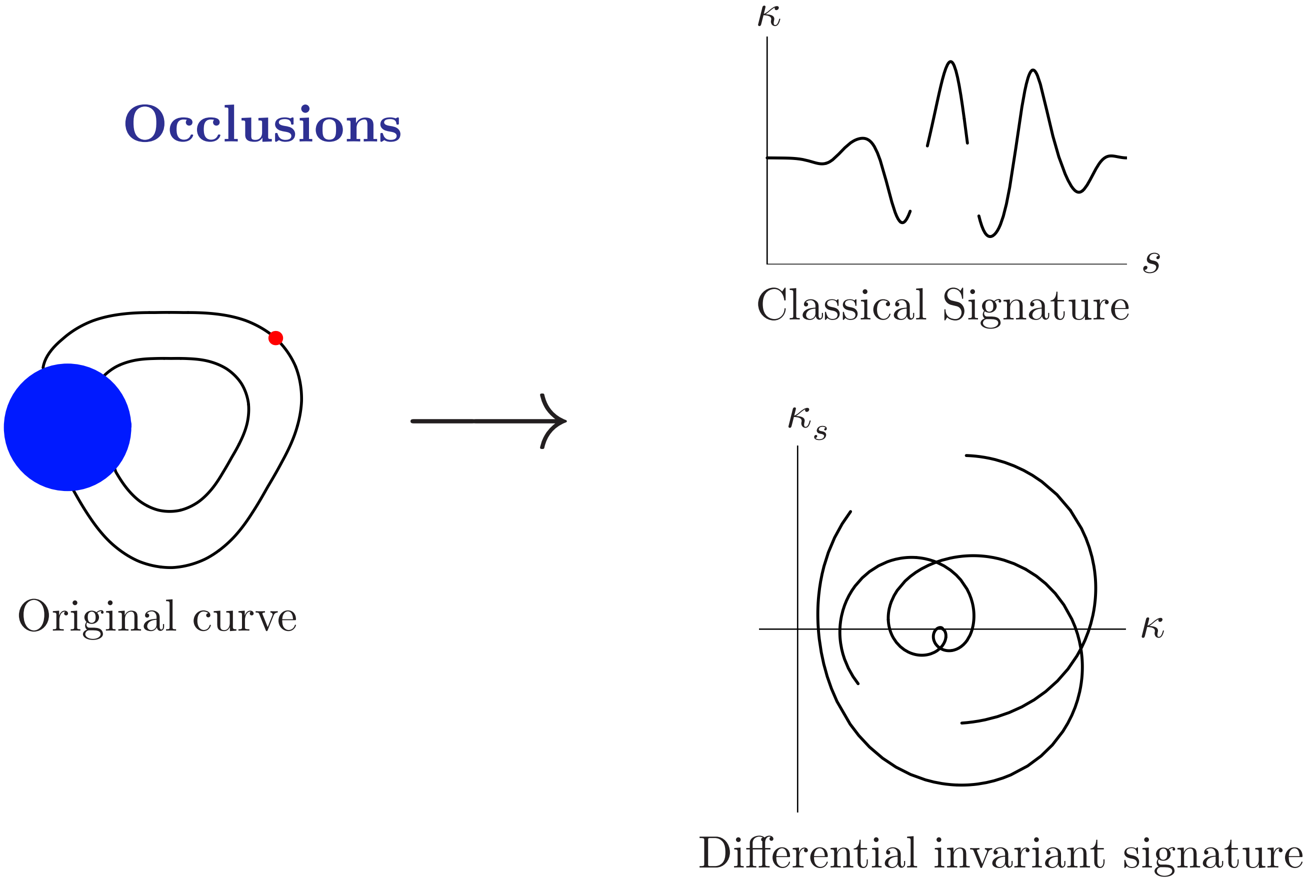


Classical signature



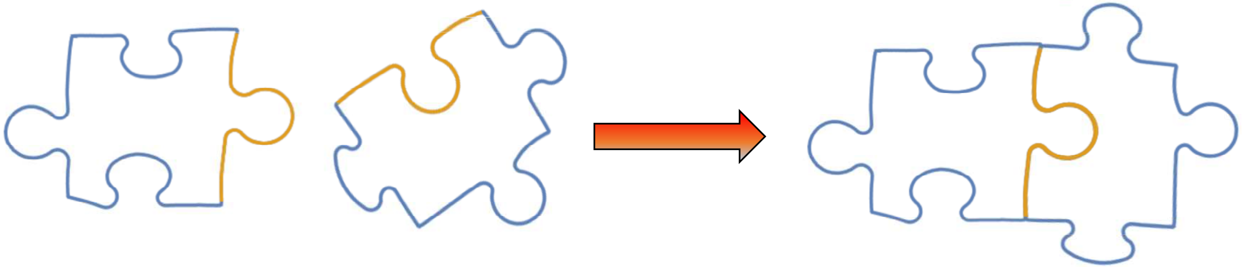
Differential invariant signature

Occlusions



κ
s
Classical Signature
κ_s
κ
Original curve
Differential invariant signature

Automatic puzzle reassembly



Step 0. Digitally photograph and smooth the puzzle pieces.

Step 1. Numerically compute invariant signatures of (parts of) pieces.

Step 2. Compare signatures to find potential fits.

Step 3. Put them together, if they fit, as closely as possible.

Repeat steps 1–3 until puzzle is assembled….

Vertices of Euclidean Curves

Ordinary vertex: local extremum of curvature

Generalized vertex: $\kappa_s \equiv 0$

- critical point
- circular arc
- straight line segment

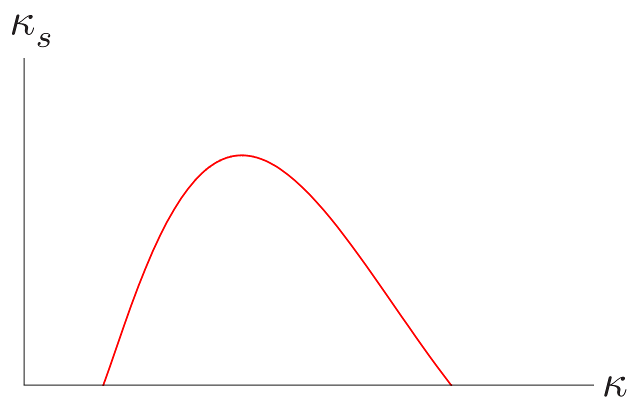
Mukhopadhya's Four Vertex Theorem:

A simple closed, non-circular plane curve has $n \geq 4$ generalized vertices.

Localization of Signatures

Bivertex arc: $\kappa_s \neq 0$ everywhere on the arc $B \subset C$
except $\kappa_s = 0$ at the two endpoints

The signature $\Sigma = \chi(B)$ of a bivertex arc is a single arc that starts and ends on the κ–axis.



Bivertex Decomposition

v-regular curve — finitely many generalized vertices

$$C = \bigcup_{j=1}^{m} B_j \;\cup\; \bigcup_{k=1}^{n} V_k$$

$B_1, \ldots, B_m$ — bivertex arcs

$V_1, \ldots, V_n$ — generalized vertices: $n \geq 4$

Main Idea: Compare individual bivertex arcs, and then decide whether the rigid equivalences are (approximately) the same.

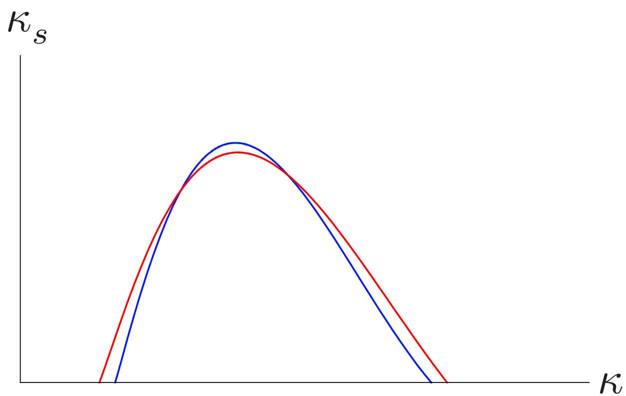
D. Hoff & PJO, Extensions of invariant signatures for object recognition, *J. Math. Imaging Vision* **45** (2013), 176–185.

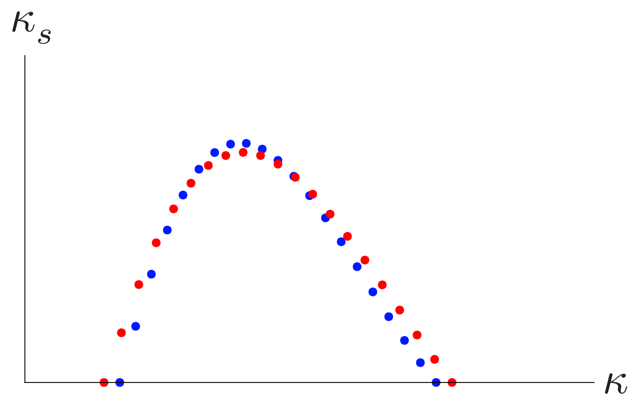
Measuring Closeness of Signatures

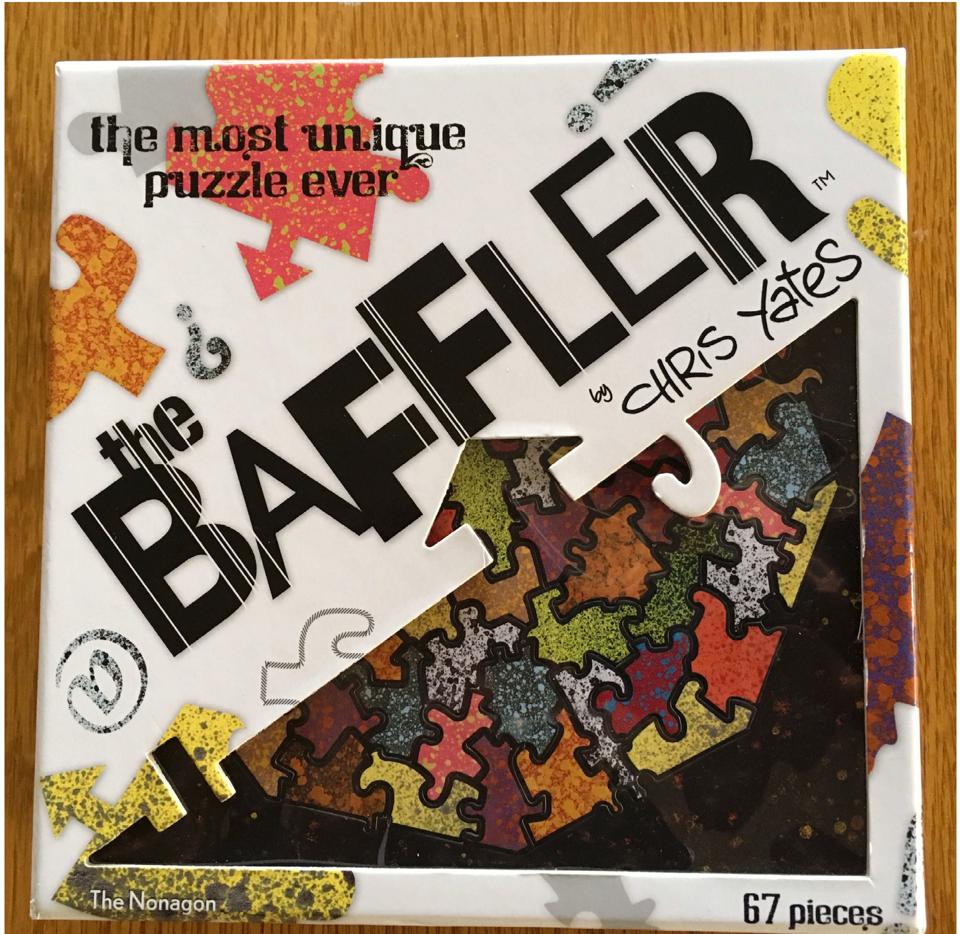
- Hausdorff distance
- Monge–Kantorovich optimal transport
- Electrostatic repulsion
- Latent semantic analysis
- Histograms
- Gromov–Hausdorff & Gromov–Wasserstein metric

Gravitational/Electrostatic Attraction

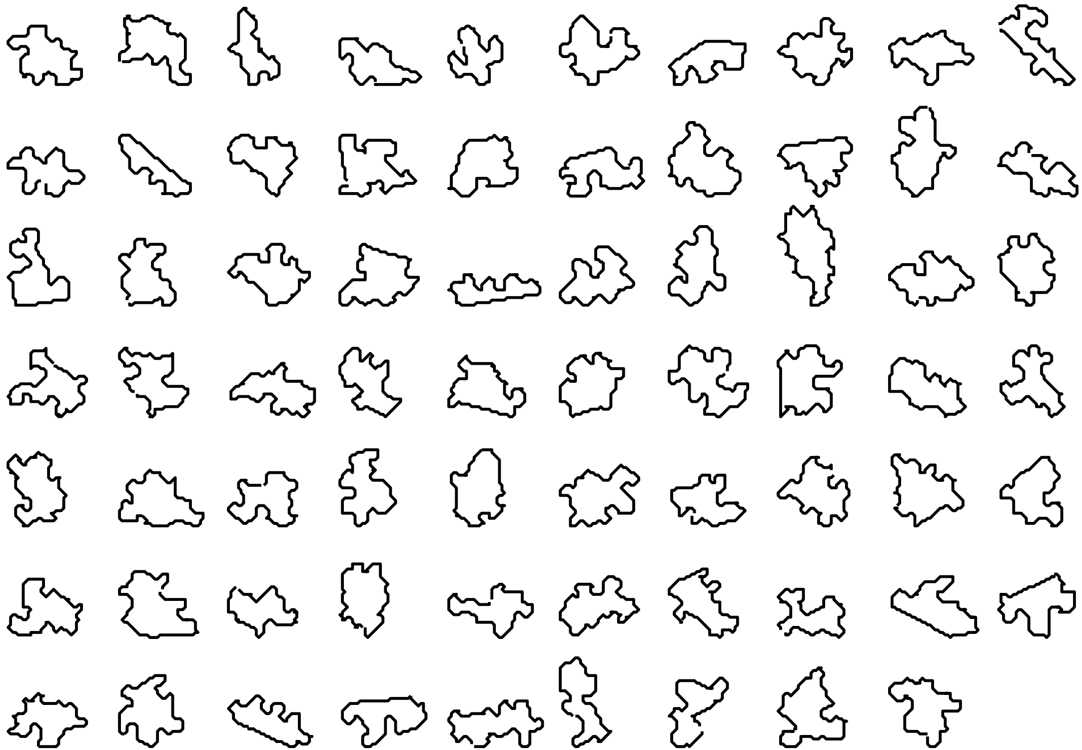
- ⋆ Treat the two bivertex arc sigantures as masses or as oppositely charged wires. The higher their mutual attraction, the closer they are together.
- ⋆ In practice, we are dealing with discrete data (pixels) and so treat the curves and signatures as point masses/charges.



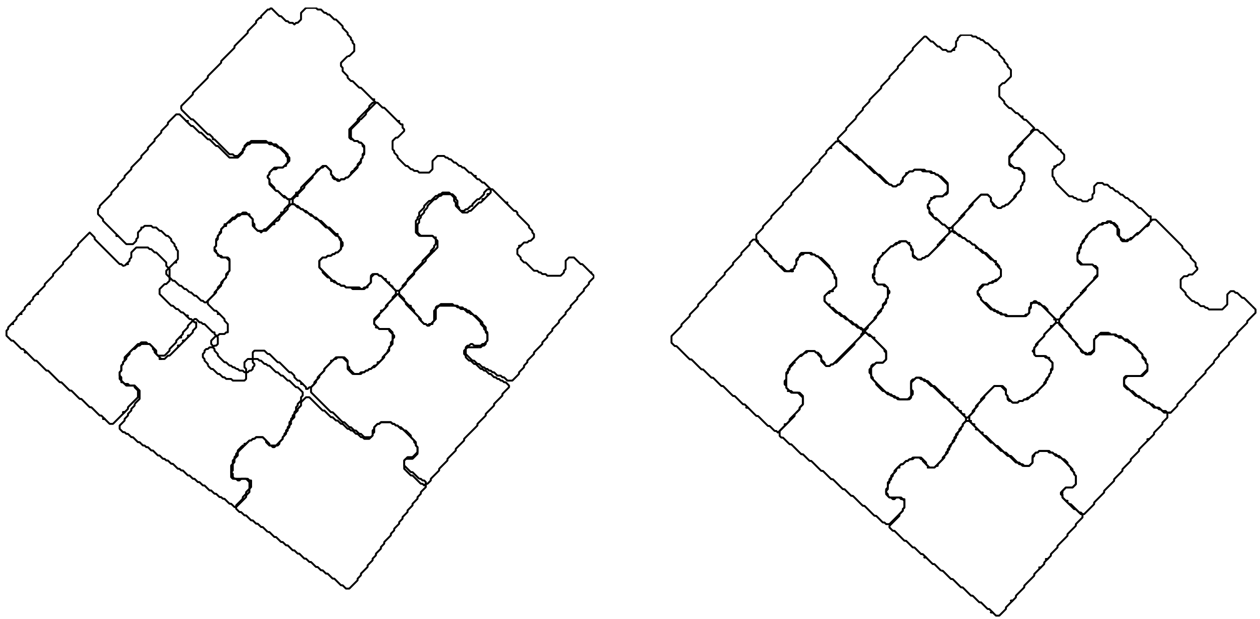



the most unique
puzzle ever
the BAFFLER™
by CHRIS YATES
The Nonagon
67 pieces

The Baffler Jigsaw Puzzle

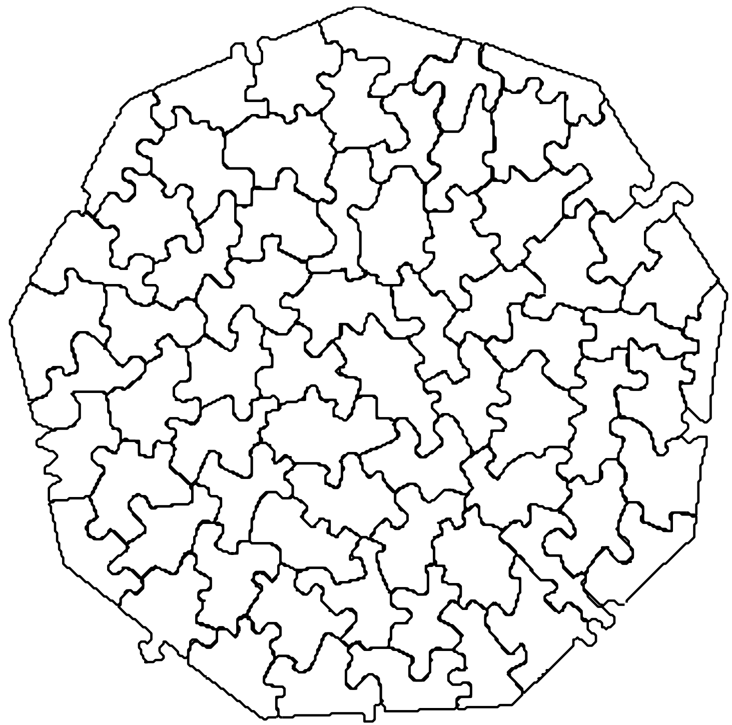


Piece Locking

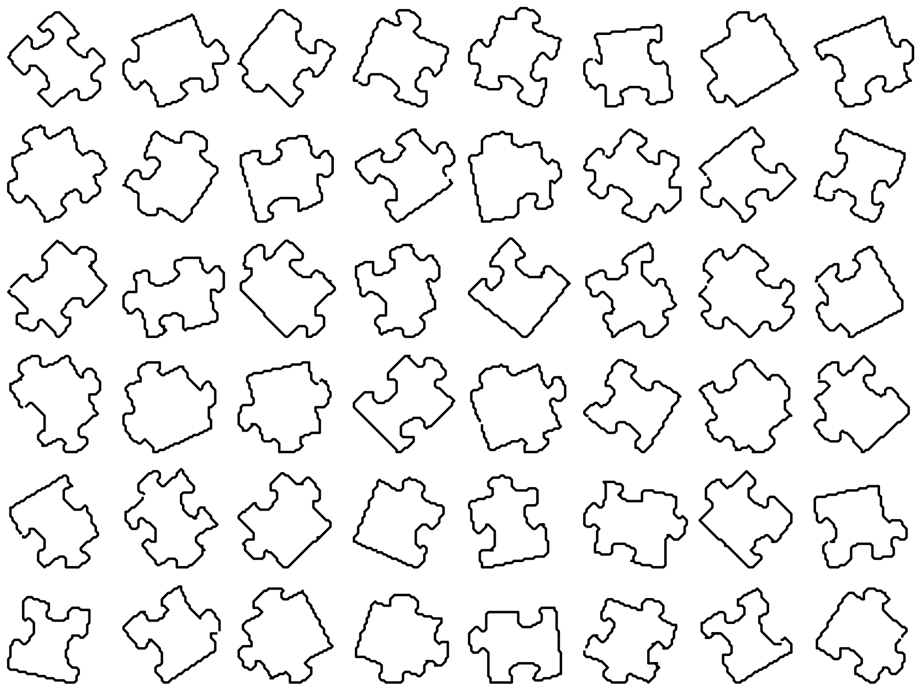


⋆⋆ Minimize force and torque based on gravitational attraction of the two matching edges.

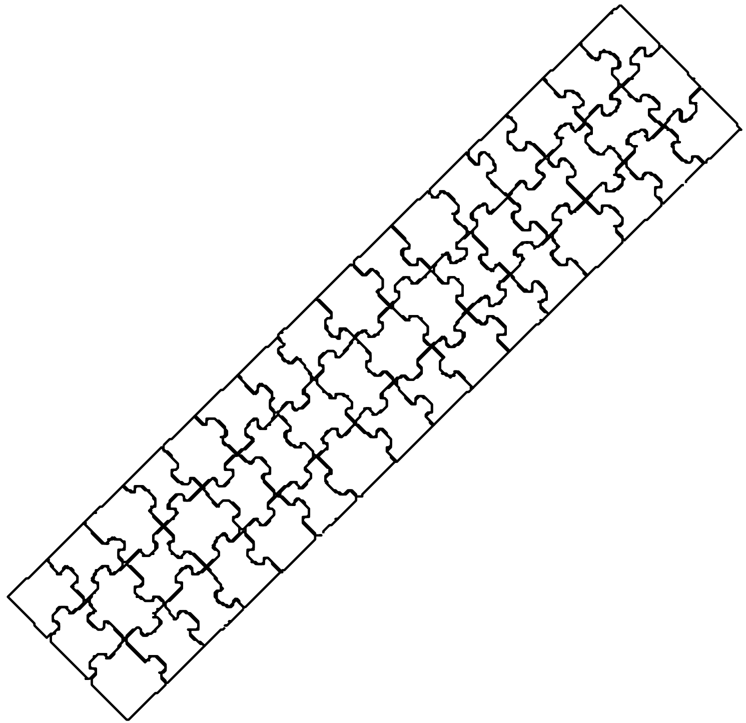
The Baffler Solved



The Rain Forest Giant Floor Puzzle

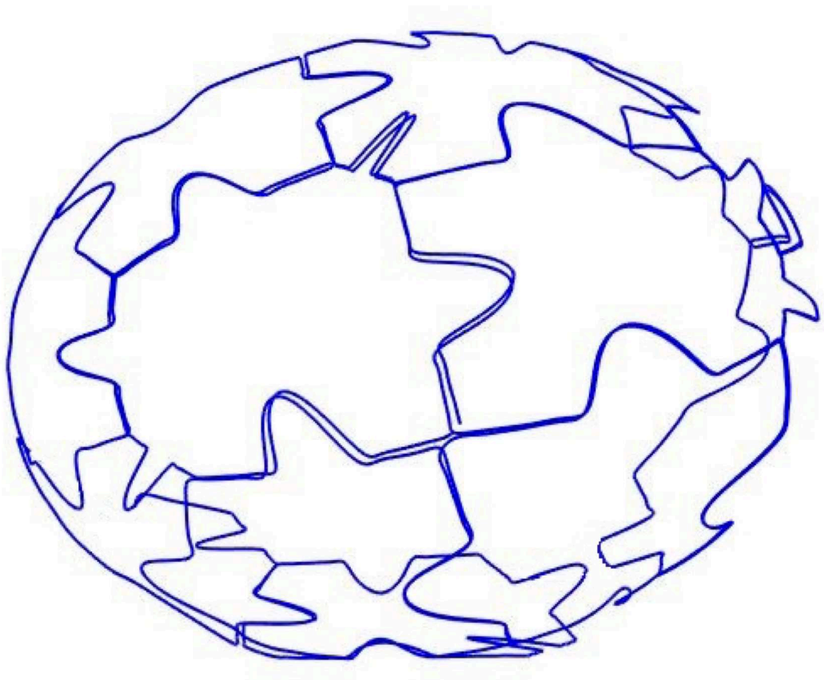


The Rain Forest Puzzle Solved



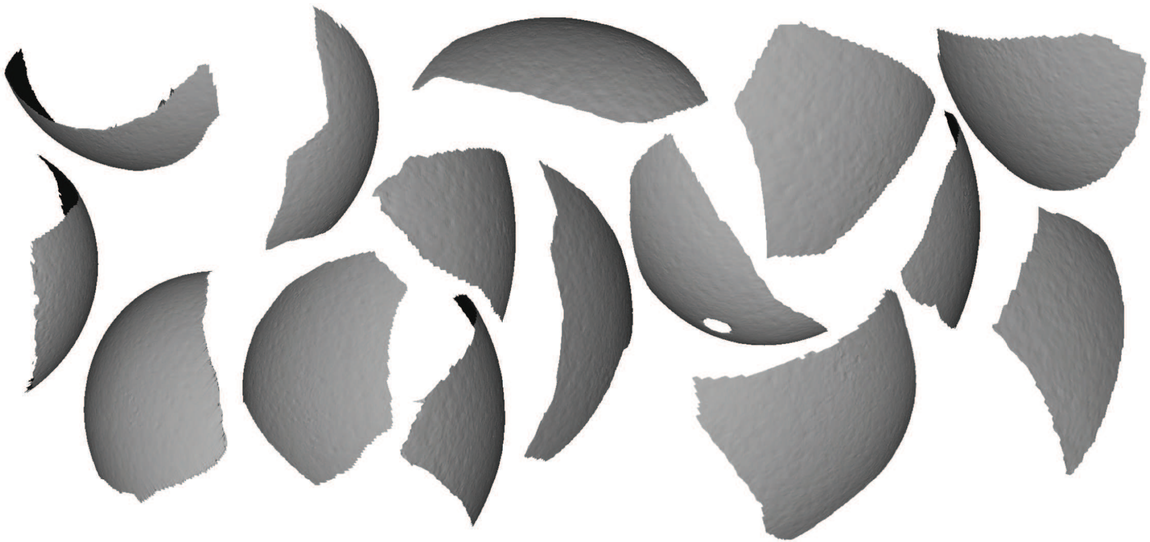
⟹ D. Hoff & PJO, Automatic solution of jigsaw puzzles, *J. Math. Imaging Vision* **49** (2014) 234–250.

3D Jigsaw Puzzles



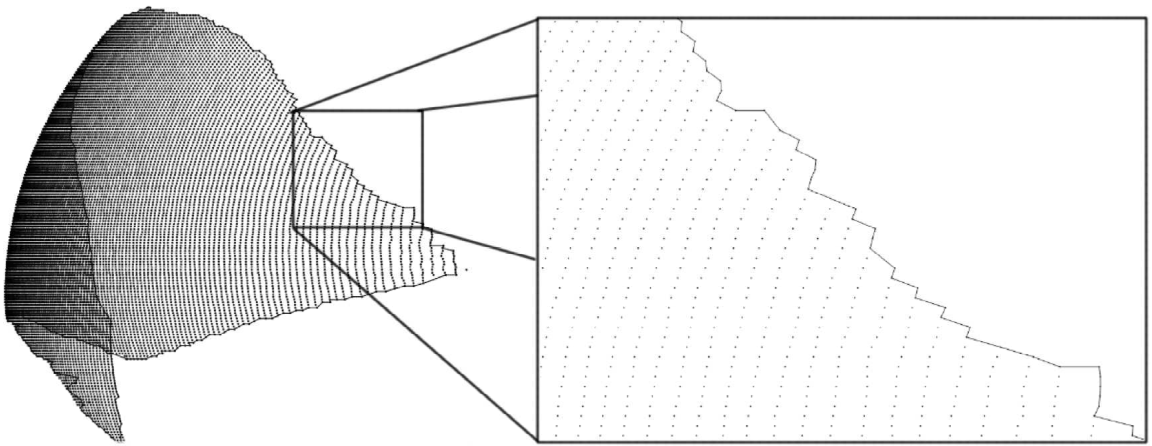
⟹ Anna Grim, Tim O'Connor, Ryan Schlecta
Cheri Shakiban, Rob Thompson, PJO

A broken ostrich egg

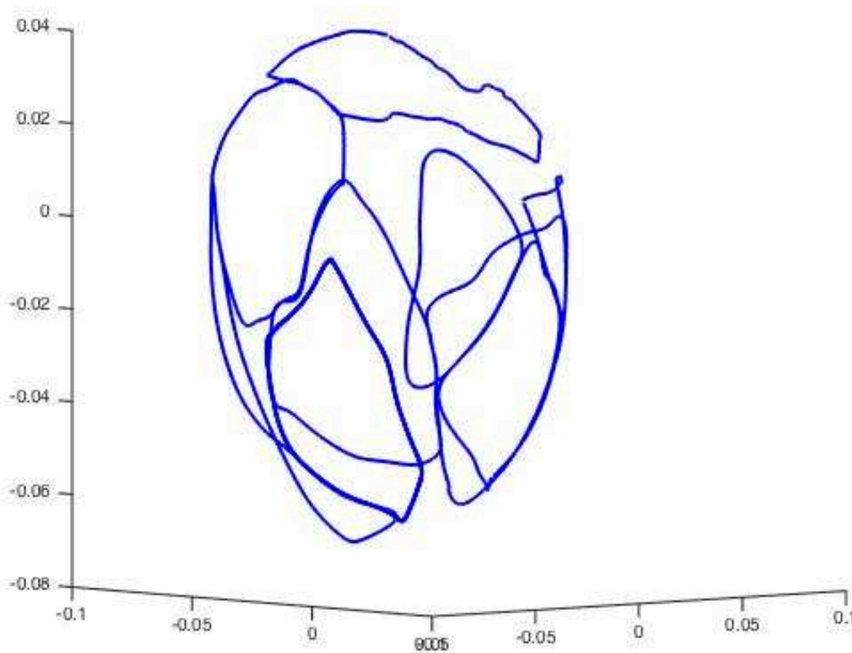


(Scanned by M. Bern, Xerox PARC)

An Eggshell Piece



Reassembling Humpty Dumpty



Archaeology

$\Longrightarrow$ **Virtual Archaeology**

Surgery

Anthropology

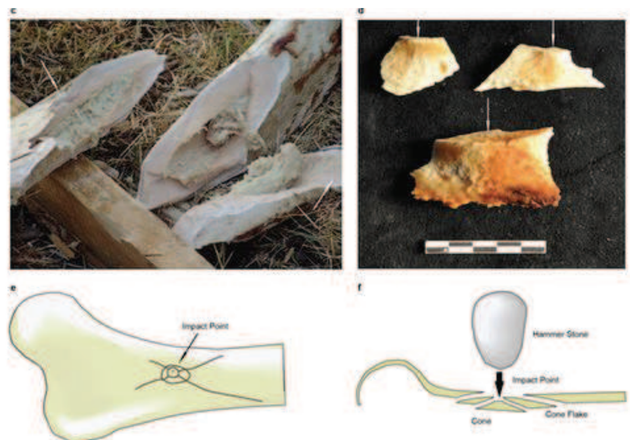
theguardian

Could history of humans in North America be rewritten by broken bones?

Smashed mastodon bones show humans arrived over 100,000 years earlier than previously thought say researchers, although other experts are sceptical

Ian Sample Science editor

Wednesday 26 April 2017 13.00 EDT



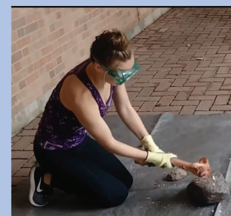
AMAAZE

Breaking Bones

Carnivore

Crocuta crocuta = *hyena*

Hominin



Batting

Hammerstone and anvil

Hammerstone only

Geological

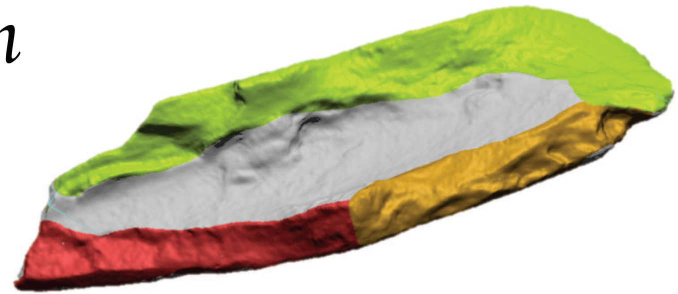
Rock fall

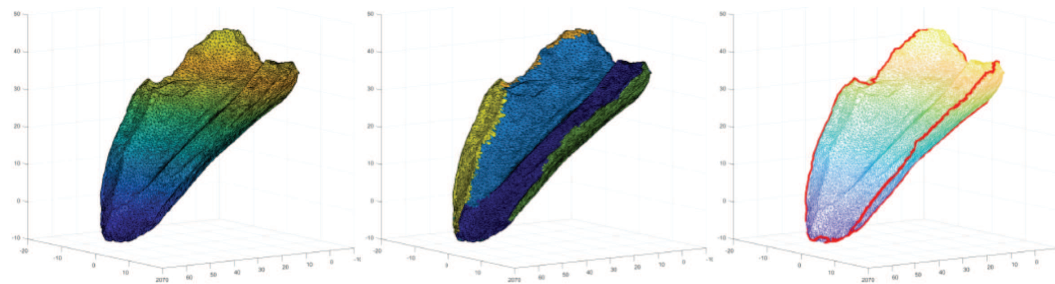
amaaze.umn.edu

Working Hypothesis

The geometry of the bone fragments,
their identity (taxon and element),
and how they are reassembled
will tell us the actor of breakage

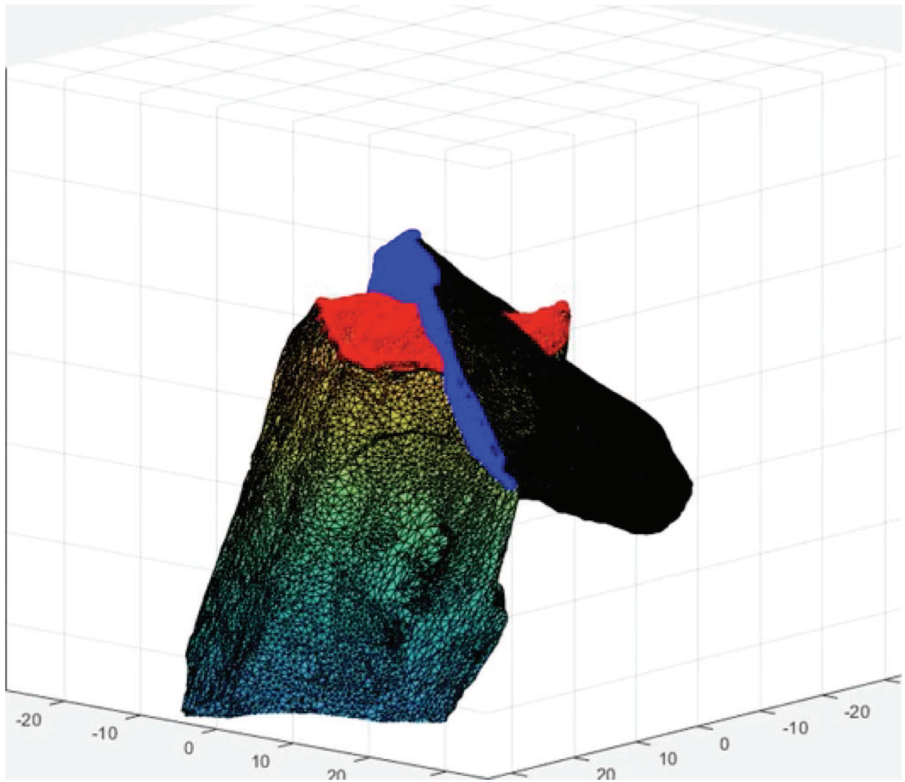
Segmentation





(a) Bone fragment (b) Face segmentation (c) Edge tracing

FIGURE 1: Results of preliminary experiments with face segmentation and edge tracing.


-20
-10
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10
20
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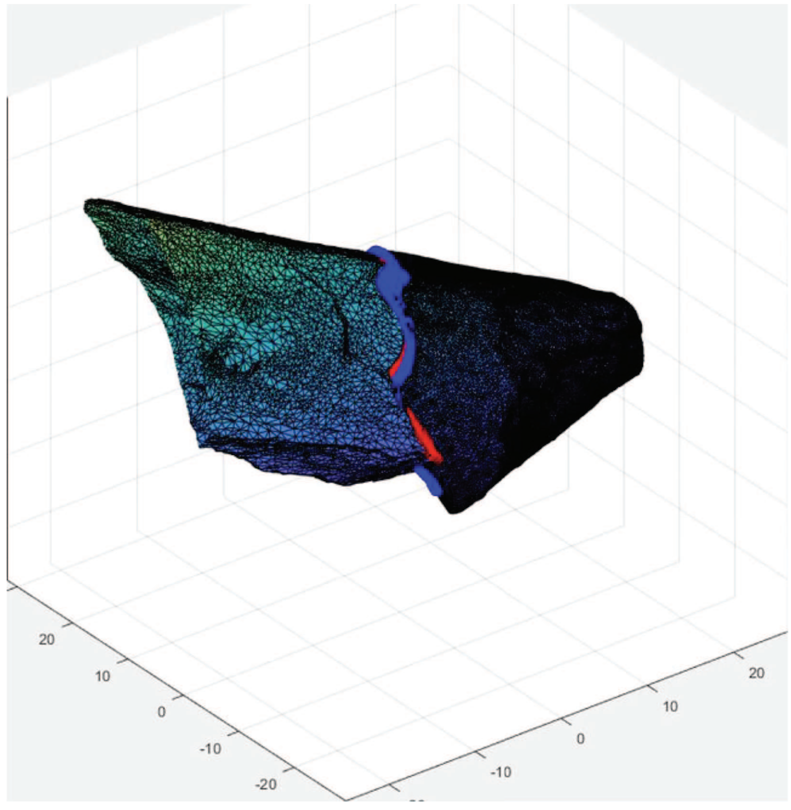

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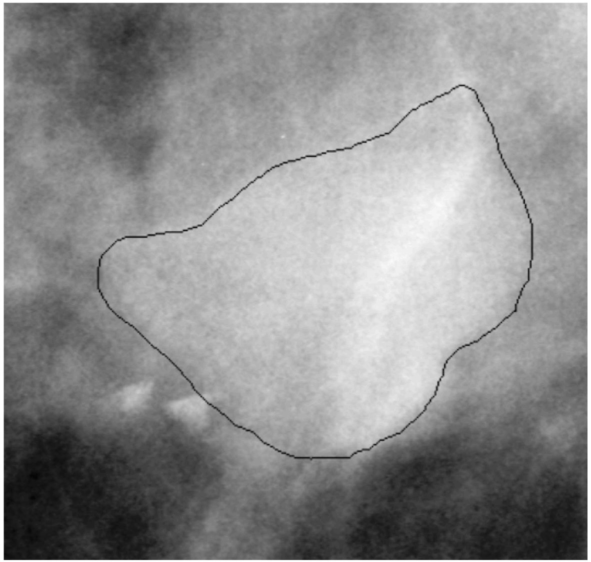
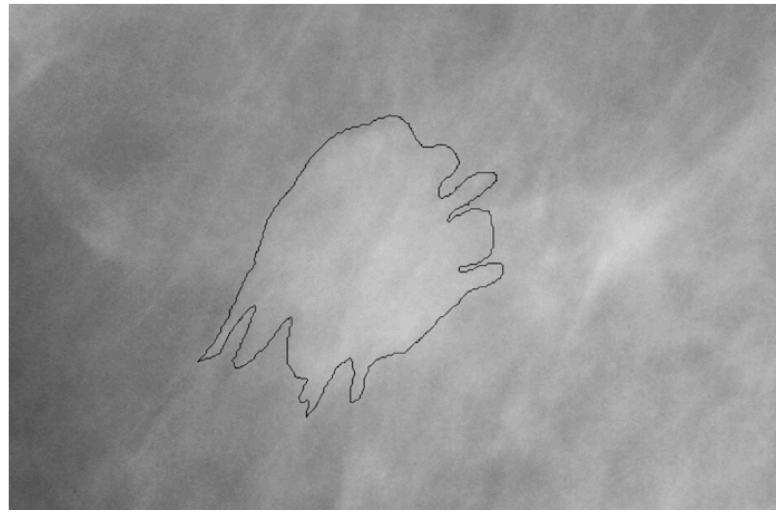
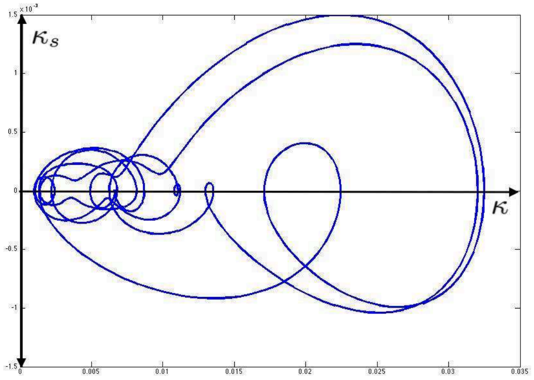
Benign vs. Malignant Tumors

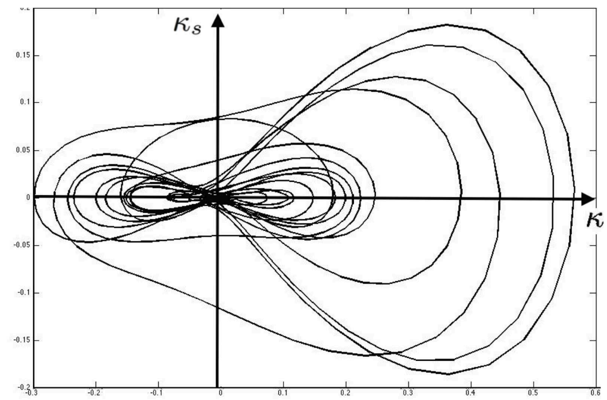




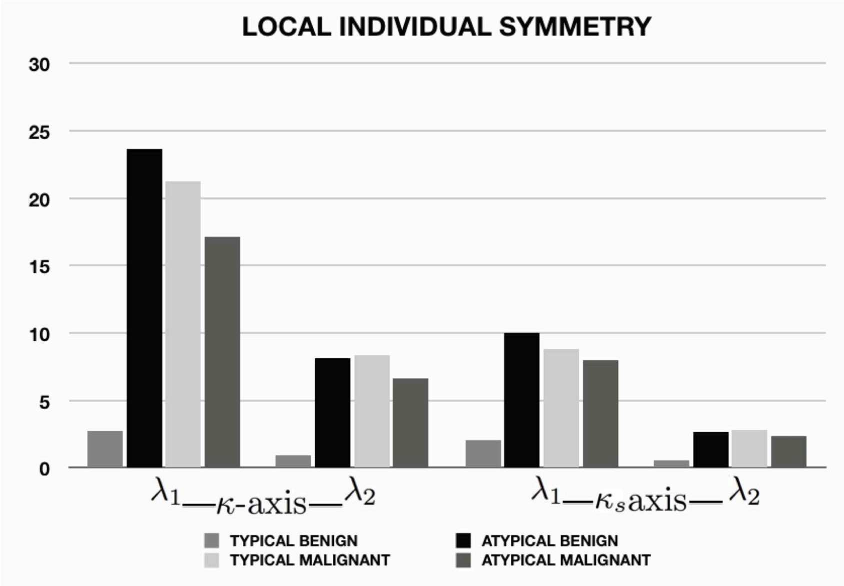
$\Longrightarrow$ A. Grim, C. Shakiban

Benign vs. Malignant Tumors





Benign vs. Malignant Tumors



Classical Invariant Theory

$$M = \mathbb{R}^2 \setminus \{ u = 0 \}$$

$$G = \mathrm{GL}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \;\middle|\; \Delta = \alpha\,\delta - \beta\,\gamma \neq 0 \right\}$$

$$(x, u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta}\, , \; \frac{u}{(\gamma x + \delta)^n} \right) \qquad n \neq 0, 1$$

Prolongation:

$$y = \frac{\alpha\, x + \beta}{\gamma\, x + \delta} \qquad \sigma = \gamma\, x + \delta$$

$$v = \sigma^{-n}\, u \qquad \Delta = \alpha\,\delta - \beta\,\gamma$$

$$v_y = \frac{\sigma\, u_x - n\,\gamma\, u}{\Delta\, \sigma^{n-1}}$$

$$v_{yy} = \frac{\sigma^2\, u_{xx} - 2(n-1)\gamma\,\sigma\, u_x + n(n-1)\gamma^2\, u}{\Delta^2\, \sigma^{n-2}}$$

$$v_{yyy} = \cdots$$

Normalization:

$$y = \frac{\alpha\, x + \beta}{\gamma\, x + \delta} = 0 \qquad\qquad \sigma = \gamma\, x + \delta$$

$$v = \sigma^{-n}\, u = 1 \qquad\qquad \Delta = \alpha\,\delta - \beta\,\gamma$$

$$v_y = \frac{\sigma\, u_x - n\,\gamma\, u}{\Delta\, \sigma^{n-1}} = 0$$

$$v_{yy} = \frac{\sigma^2\, u_{xx} - 2(n-1)\,\gamma\,\sigma\, u_x + n(n-1)\gamma^2\, u}{\Delta^2\, \sigma^{n-2}} = \frac{1}{n(n-1)}$$

$$v_{yyy} = \cdots$$

Moving frame:

$$\alpha = u^{(1-n)/n}\sqrt{H} \qquad \beta = -x\,u^{(1-n)/n}\sqrt{H}$$

$$\gamma = \tfrac{1}{n}\,u^{(1-n)/n} \qquad \delta = u^{1/n} - \tfrac{1}{n}\,x u^{(1-n)/n}$$

Hessian:

$$H = n(n-1)u\,u_{xx} - (n-1)^2 u_x^2 \neq 0$$

Note: $H \equiv 0$ if and only if $Q(x) = (a\,x + b)^n$

$\Longrightarrow$ Totally singular forms

Differential invariants:

$$v_{yyy} \longmapsto \frac{J}{n^2(n-1)} \approx \kappa \qquad v_{yyyy} \longmapsto \frac{K + 3(n-2)}{n^3(n-1)} \approx \frac{d\kappa}{ds}$$

Absolute rational covariants:

$$J^2 = \frac{T^2}{H^3} \qquad K = \frac{U}{H^2}$$

$$H = \tfrac{1}{2}(Q,\, Q)^{(2)} = n(n-1)QQ'' - (n-1)^2 Q'^2 \sim Q_{xx}Q_{yy} - Q_{xy}^2$$

$$T = (Q,\, H)^{(1)} = (2n-4)Q'H - nQH' \sim Q_x H_y - Q_y H_x$$

$$U = (Q,\, T)^{(1)} = (3n-6)Q'T - nQT' \sim Q_x T_y - Q_y T_x$$

$$\deg Q = n \quad \deg H = 2n-4 \quad \deg T = 3n-6 \quad \deg U = 4n-8$$

Signatures of Binary Forms

Signature curve of a nonsingular binary form $Q(x)$:

$$\Sigma_Q = \left\{ \left(J(x)^2, K(x) \right) = \left(\frac{T(x)^2}{H(x)^3}, \frac{U(x)}{H(x)^2} \right) \right\}$$

Nonsingular: $\quad H(x) \neq 0$ and $(J'(x), K'(x)) \neq 0.$

Theorem. Two nonsingular binary forms are equivalent if and only if their signature curves are identical.

Maximally Symmetric Binary Forms

Theorem. If $u = Q(x)$ is a polynomial, then the following are equivalent:

- $Q(x)$ admits a one-parameter symmetry group
- T^2 is a constant multiple of H^3
- $Q(x) \simeq x^k$ is complex-equivalent to a monomial
- the signature curve degenerates to a single point
- all the (absolute) differential invariants of Q are constant
- the graph of Q coincides with the orbit of a one-parameter subgroup

Symmetries of Binary Forms

Theorem. The symmetry group of a nonzero binary form $Q(x) \not\equiv 0$ of degree n is:

- A two-parameter group if and only if $H \equiv 0$ if and only if Q is equivalent to a constant. $\Longrightarrow$ totally singular
- A one-parameter group if and only if $H \not\equiv 0$ and $T^2 = c\,H^3$ if and only if Q is complex-equivalent to a monomial x^k, with $k \neq 0, n$. $\Longrightarrow$ maximally symmetric
- In all other cases, a finite group whose cardinality equals the index of the signature curve, and is bounded by

$$\iota_Q \quad \leq \quad \begin{cases} 6n - 12 & U = cH^2 \\ 4n - 8 & \text{otherwise} \end{cases}$$

Noise Reduction

Strategy #1:

Use lower order invariants to construct a signature:

- joint invariants
- joint differential invariants
- integral invariants
- topological invariants
- . . .

Joint Invariants

A joint invariant is an invariant of the k-fold Cartesian product action of G on $M \times \cdots \times M$:

$$\boxed{I(g \cdot z_1, \ldots, g \cdot z_k) \; = \; I(z_1, \ldots, z_k)}$$

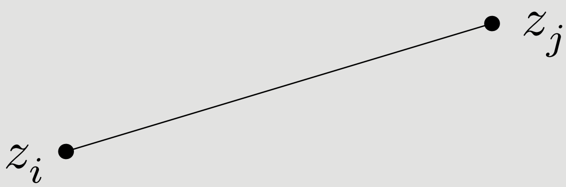
A joint differential invariant or semi-differential invariant is an invariant depending on the derivatives at several points $z_1, \ldots, z_k \in N$ on the submanifold:

$$\boxed{I(g \cdot z_1^{(n)}, \ldots, g \cdot z_k^{(n)}) \; = \; I(z_1^{(n)}, \ldots, z_k^{(n)})}$$

Joint Euclidean Invariants

Theorem. Every joint Euclidean invariant is a function of the interpoint distances

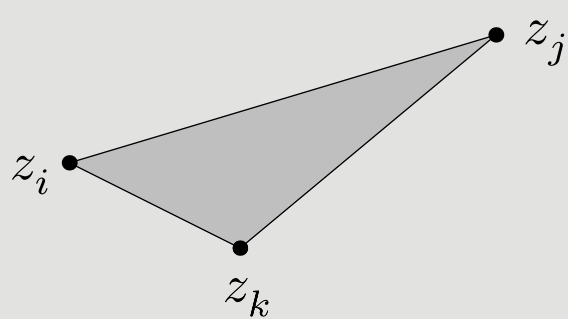
$$d(z_i, z_j) = \| \, z_i - z_j \, \|$$



Joint Equi–Affine Invariants

Theorem. Every planar joint equi–affine invariant is a function of the triangular areas

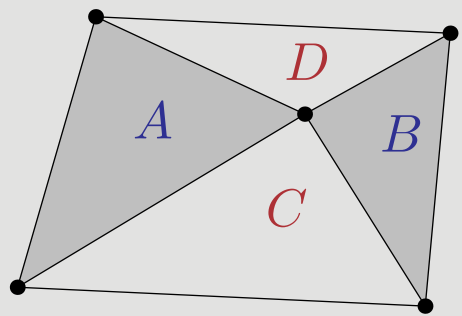
$$[\, i \ j \ k \,] = \tfrac{1}{2}\,(z_i - z_j) \wedge (z_i - z_k)$$



Joint Projective Invariants

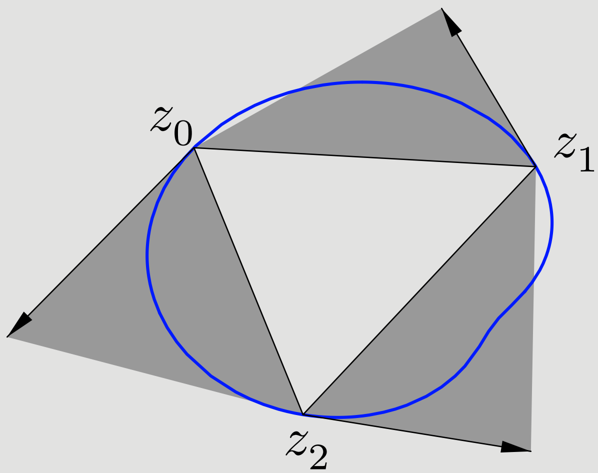
Theorem. Every joint projective invariant is a function of the planar cross-ratios

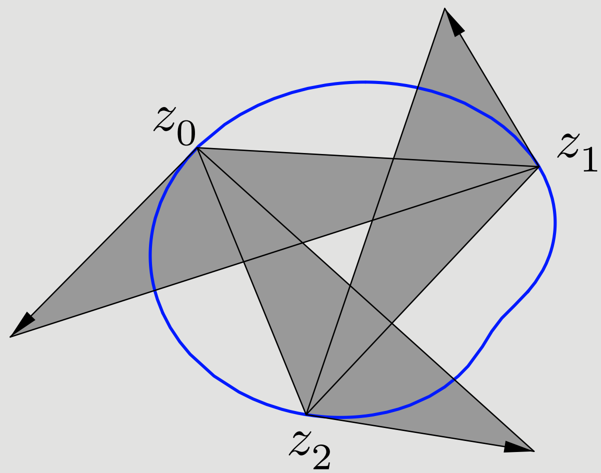
$$[\, z_i, z_j, z_k, z_l, z_m \,] \;=\; \frac{A\,B}{C\,D}$$



- Three–point projective joint differential invariant
  — tangent triangle ratio:

$$\frac{[\,0\;2\;\dot{0}\,]\;[\,0\;1\;\dot{1}\,]\;[\,1\;2\;\dot{2}\,]}{[\,0\;1\;\dot{0}\,]\;[\,1\;2\;\dot{1}\,]\;[\,0\;2\;\dot{2}\,]}$$





Joint Invariant Signatures

If the invariants depend on k points on a p-dimensional submanifold, then you need at least

$$\ell > k\,p$$

distinct invariants $I_1, \ldots, I_\ell$ in order to construct a syzygy. Typically, the number of joint invariants is

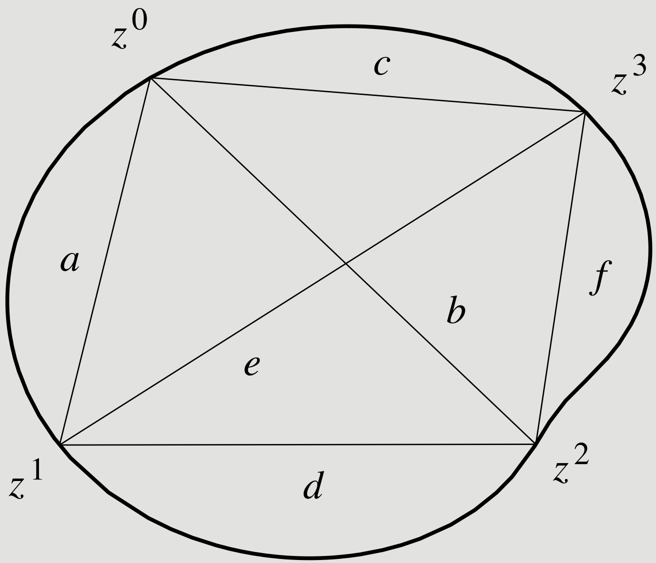
$$\ell = k\,m - r = (\#\text{points})\,(\dim M) - \dim G$$

Therefore, a purely joint invariant signature requires at least

$$k \;\geq\; \frac{r}{m-p} + 1$$

points on our p-dimensional submanifold $N \subset M$.

Joint Euclidean Signature



Joint signature map:

$$\Sigma : \mathcal{C}^{\times 4} \longrightarrow \Sigma \subset \mathbb{R}^6$$

$$a = \| z_0 - z_1 \| \qquad b = \| z_0 - z_2 \| \qquad c = \| z_0 - z_3 \|$$

$$d = \| z_1 - z_2 \| \qquad e = \| z_1 - z_3 \| \qquad f = \| z_2 - z_3 \|$$

$\Longrightarrow$ six functions of four variables

Syzygies:

$$\Phi_1(a, b, c, d, e, f) = 0 \qquad \Phi_2(a, b, c, d, e, f) = 0$$

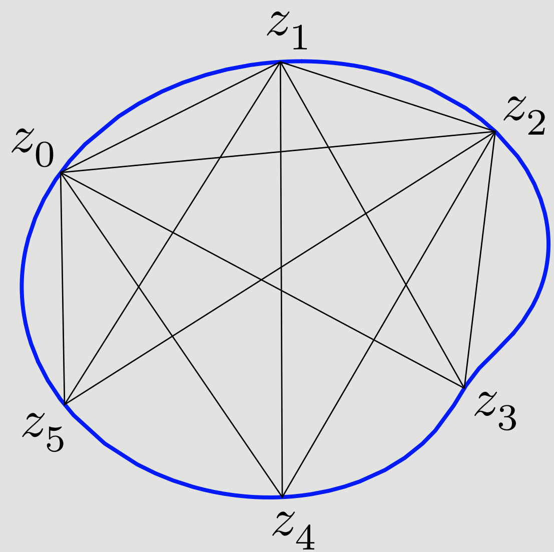
Universal Cayley–Menger syzygy $\quad \Longleftrightarrow \quad \mathcal{C} \subset \mathbb{R}^2$

$$\det \begin{vmatrix} 2\,a^2 & a^2 + b^2 - d^2 & a^2 + c^2 - e^2 \\ a^2 + b^2 - d^2 & 2\,b^2 & b^2 + c^2 - f^2 \\ a^2 + c^2 - e^2 & b^2 + c^2 - f^2 & 2\,c^2 \end{vmatrix} = 0$$

Joint Equi–Affine Signature

Requires 7 triangular areas:

$[\,0\ 1\ 2\,],\ [\,0\ 1\ 3\,],\ [\,0\ 1\ 4\,],\ [\,0\ 1\ 5\,],\ [\,0\ 2\ 3\,],\ [\,0\ 2\ 4\,],\ [\,0\ 2\ 5\,]$



Joint Invariant Signatures

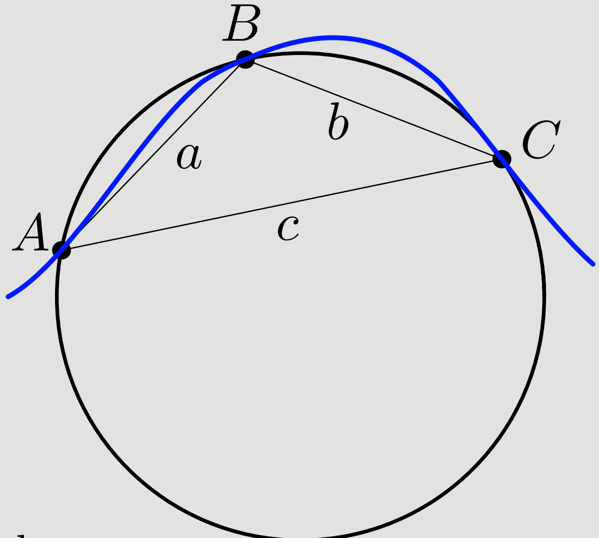
- The joint invariant signature subsumes other signatures, but resides in a higher dimensional space and contains a lot of redundant information.
- Identification of landmarks can significantly reduce the redundancies (Boutin)
- It includes the differential invariant signature and semi-differential invariant signatures as its "coalescent boundaries".
- Invariant numerical approximations to differential invariants and semi-differential invariants are constructed (using moving frames) near these coalescent boundaries.

Symmetry–Preserving Numerical Methods

- Invariant numerical approximations to differential invariants.
- Invariantization of numerical integration methods.

$\Longrightarrow$ Structure-preserving algorithms

Numerical approximation to curvature



Heron's formula

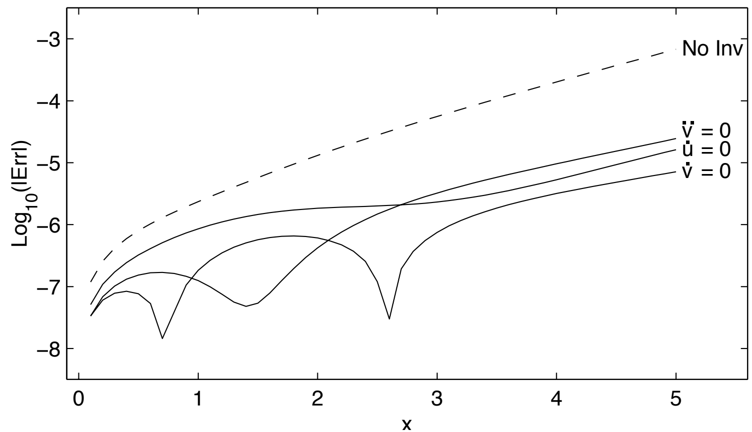
$$\widetilde{\kappa}(A,B,C) = 4\,\frac{\Delta}{abc} = 4\,\frac{\sqrt{s(s-a)(s-b)(s-c)}}{abc}$$

$$s = \frac{a+b+c}{2} \quad \text{— semi-perimeter}$$

Invariantization of Numerical Schemes

Suppose we are given a numerical scheme for integrating a differential equation, e.g., a Runge–Kutta Method for ordinary differential equations, or the Crank–Nicolson method for parabolic partial differential equations.

If G is a symmetry group of the differential equation, then one can use an appropriately chosen moving frame to invariantize the numerical scheme, leading to an invariant numerical scheme that preserves the symmetry group. In challenging regimes, the resulting invariantized numerical scheme can, with an inspired choice of moving frame, perform significantly better than its progenitor.

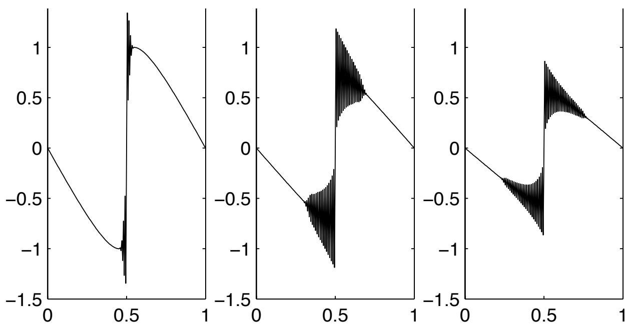


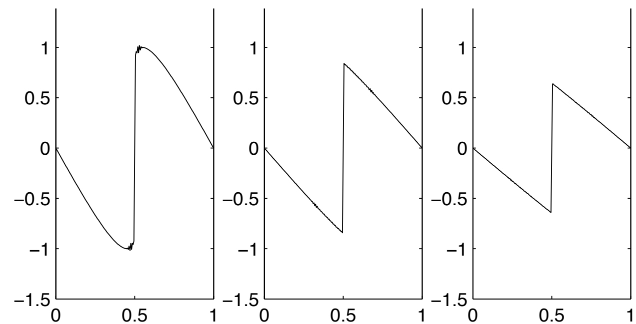
Invariant Runge–Kutta schemes

$$u_{xx} + x\,u_x - (x+1)u = \sin x, \qquad u(0) = u_x(0) = 1.$$

Invariantization of Crank–Nicolson for Burgers' Equation

$$u_t = \varepsilon\, u_{xx} + u\, u_x$$





$\Longrightarrow$ Pilwon Kim

Morphological PDEs

Hamilton–Jacobi partial differential equation:

$$u_t = \pm \, |\nabla u|$$

Symmetry Group:

$$u \quad \longmapsto \quad \varphi(u)$$

Here, we focus on the one-parameter subgroup

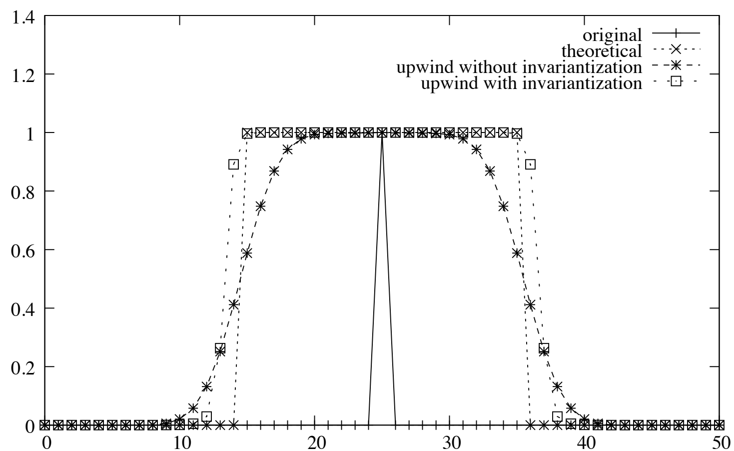
$$u \longmapsto \frac{\lambda u}{1 + (\lambda - 1)u}$$

Invariantization of 1D Morphology

$$u_t = |u_x|$$

Upwind scheme:

$$u_i^{k+1} = u_i^k + \frac{\Delta t}{\Delta x} \max\{u_{i+1}^k - u_i^k, u_{i-1}^k - u_i^k, 0\} \ .$$

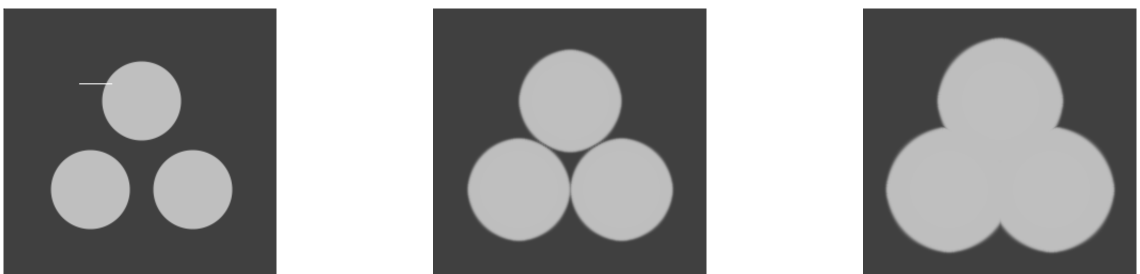


1D dilation of a single peak, 20 iterations,
$\Delta t = \Delta x = 0.5$, without and with invariantization.

Invariantization of 2D Morphology

Non-invariant upwind scheme:

Invariantized upwind scheme:



The Calculus of Variations

$\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x}$ — variational problem

$L(x, u^{(n)})$ — Lagrangian

To construct the Euler-Lagrange equations: $\mathbf{E}(L) = 0$

- Take the first variation:

$$\delta(L\, d\mathbf{x}) = \sum_{\alpha, J} \frac{\partial L}{\partial u_J^\alpha}\, \delta u_J^\alpha\, d\mathbf{x}$$

- Integrate by parts:

$$\delta(L\, d\mathbf{x}) = \sum_{\alpha, J} \frac{\partial L}{\partial u_J^\alpha}\, D_J(\delta u^\alpha)\, d\mathbf{x}$$

$$\equiv \sum_{\alpha, J} (-D)^J \frac{\partial L}{\partial u_J^\alpha}\, \delta u^\alpha\, d\mathbf{x} = \sum_{\alpha=1}^{q} \mathbf{E}_\alpha(L)\, \delta u^\alpha\, d\mathbf{x}$$

Invariant Variational Problems

According to Lie, any G–invariant variational problem can be written in terms of the differential invariants:

$$\boxed{\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\,\ldots\;\mathcal{D}_K I^\alpha\;\ldots\,)\;\boldsymbol{\omega}}$$

$I^1, \ldots, I^\ell$ — fundamental differential invariants

$\mathcal{D}_1, \ldots, \mathcal{D}_p$ — invariant differential operators

$\mathcal{D}_K I^\alpha$ — differentiated invariants

$\boldsymbol{\omega} = \omega^1 \wedge \cdots \wedge \omega^p$ — invariant volume form

If the variational problem is G-invariant, so

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ )\ \boldsymbol{\omega}$$

then its Euler–Lagrange equations admit G as a symmetry group, and hence can also be expressed in terms of the differential invariants:

$$\mathbf{E}(L) \simeq F(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ ) = 0$$

Main Problem:

Construct F directly from P.

(P. Griffiths, I. Anderson)

Planar Euclidean group $\quad G = \mathrm{SE}(2)$

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}} \qquad — \quad \text{curvature (differential invariant)}$$

$$ds = \sqrt{1+u_x^2}\,dx \qquad — \quad \text{arc length}$$

$$\mathcal{D} = \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx} \qquad — \quad \text{arc length derivative}$$

Euclidean–invariant variational problem

$$\mathcal{I}[u] = \int L(x, u^{(n)})\,dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\,ds$$

Euler-Lagrange equations

$$\mathbf{E}(L) \simeq F(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ ) = 0$$

Euclidean Curve Examples

Minimal curves (geodesics):

$$\mathcal{I}[u] = \int ds = \int \sqrt{1 + u_x^2}\; dx$$

$$\mathbf{E}(L) = -\,\kappa = 0$$

$\Longrightarrow$ straight lines

The Elastica (Euler):

$$\mathcal{I}[u] = \int \tfrac{1}{2}\,\kappa^2\, ds = \int \frac{u_{xx}^2\, dx}{(1 + u_x^2)^{5/2}}$$

$$\mathbf{E}(L) = \kappa_{ss} + \tfrac{1}{2}\,\kappa^3 = 0$$

$\Longrightarrow$ elliptic functions

General Euclidean–invariant variational problem

$$\mathcal{I}[u] = \int L(x, u^{(n)})\, dx = \int P(\kappa, \kappa_s, \kappa_{ss}, \ \ldots\ )\, ds$$

To construct the invariant Euler-Lagrange equations:

Take the first variation:

$$\delta(P\, ds) = \sum_j \frac{\partial P}{\partial \kappa_j}\, \delta\kappa_j\, ds + P\, \delta(ds)$$

Invariant variation of curvature:

$$\delta\kappa = \mathcal{A}_\kappa(\delta u) \qquad\qquad \mathcal{A}_\kappa = \mathcal{D}^2 + \kappa^2$$

Invariant variation of arc length:

$$\delta(ds) = \mathcal{B}(\delta u)\, ds \qquad\qquad \mathcal{B} = -\,\kappa$$

$\Longrightarrow$ moving frame recurrence formulae

Integrate by parts:

$$\begin{aligned}\delta(P\,ds) &\equiv [\,\mathcal{E}(P)\,\mathcal{A}(\delta u) - \mathcal{H}(P)\,\mathcal{B}(\delta u)\,]\,ds \\ &\equiv [\,\mathcal{A}^*\mathcal{E}(P) - \mathcal{B}^*\mathcal{H}(P)\,]\,\delta u\,ds = \mathbf{E}(L)\,\delta u\,ds\end{aligned}$$

Invariantized Euler–Lagrange expression

$$\mathcal{E}(P) = \sum_{n=0}^{\infty} (-\mathcal{D})^n \frac{\partial P}{\partial \kappa_n} \qquad\qquad \mathcal{D} = \frac{d}{ds}$$

Invariantized Hamiltonian

$$\mathcal{H}(P) = \sum_{i>j} \kappa_{i-j}\,(-\mathcal{D})^j \frac{\partial P}{\partial \kappa_i} - P$$

Euclidean–invariant Euler-Lagrange formula

$$\mathbf{E}(L) = \mathcal{A}^*\mathcal{E}(P) - \mathcal{B}^*\mathcal{H}(P) = (\mathcal{D}^2 + \kappa^2)\,\mathcal{E}(P) + \kappa\,\mathcal{H}(P) = 0.$$

The Elastica:

$$\mathcal{I}[u] = \int \tfrac{1}{2}\,\kappa^2\,ds \qquad P = \tfrac{1}{2}\,\kappa^2$$

$$\mathcal{E}(P) = \kappa \qquad\qquad \mathcal{H}(P) = -\,P = -\,\tfrac{1}{2}\,\kappa^2$$

$$\mathbf{E}(L) = (\mathcal{D}^2 + \kappa^2)\,\kappa + \kappa\,(\,-\,\tfrac{1}{2}\,\kappa^2\,) = \kappa_{ss} + \tfrac{1}{2}\,\kappa^3 = 0$$

The shape of a Möbius strip

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The Möbius strip, obtained by taking a rectangular strip of plastic or paper, twisting one end through 180°, and then joining the ends, is the canonical example of a one-sided surface. Finding its characteristic developable shape has been an open problem ever since its first formulation in refs 1,2. Here we use the invariant variational bicomplex formalism to derive the first equilibrium equations for a wide developable strip undergoing large deformations, thereby giving the first non-trivial demonstration of the potential of this approach. We then formulate the boundary-value problem for the Möbius strip and solve it numerically. Solutions for increasing width show the formation of creases bounding nearly flat triangular regions, a feature also familiar from fabric draping[3] and paper crumpling[4,5]. This could give new insight into energy localization phenomena in unstretchable sheets[6], which might help to predict points of onset of tearing. It could also aid our understanding of the relationship between geometry and physical properties of nano- and microscopic Möbius strip structures[7–9].

It is fair to say that the Möbius strip is one of the few icons of mathematics that have been absorbed into wider culture. It has mathematical beauty and inspired artists such as Escher[10]. In engineering, pulley belts are often used in the form of Möbius strips to wear 'both' sides equally. At a much smaller scale, Möbius strips have recently been formed in ribbon-shaped $NbSe_3$ crystals under certain growth conditions involving a large temperature gradient[7,8].

Figure 1 Photo of a paper Möbius strip of aspect ratio 2π. The strip adopts a characteristic shape. Inextensibility of the material causes the surface to be developable. Its straight generators are drawn and the colouring varies according to the bending energy density.

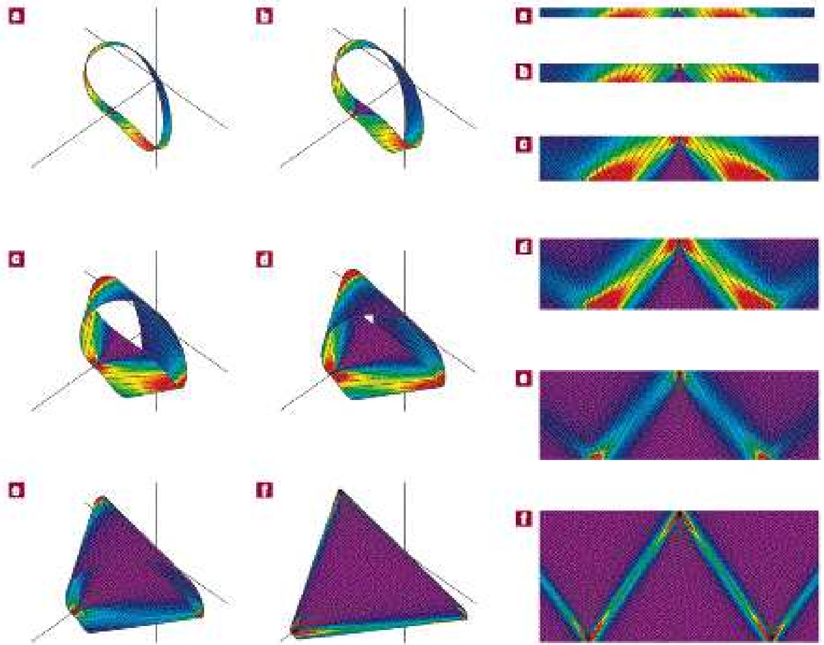

Figure 2 Computed Möbius strips. The left panel shows their three-dimensional shapes for $w = 0.1$ (**a**), 0.2 (**b**), 0.5 (**c**), 0.8 (**d**), 1.0 (**e**) and 1.5 (**f**), and the right panel the corresponding developments on the plane. The colouring changes according to the local bending energy density, from violet for regions of low bending to red for regions of high bending (scales are individually adjusted). Solution **c** may be compared with the paper model in Fig. 1 on which the generator field and density colouring have been printed.

Evolution of Invariants and Signatures

G — Lie group acting on $\mathbb{R}^2$

$C(t)$ — parametrized family of plane curves

G–invariant curve flow:

$$\frac{dC}{dt} = \mathbf{V} = I\,\mathbf{t} + J\,\mathbf{n}$$

- I, J — differential invariants
- $\mathbf{t}$ — "unit tangent"
- $\mathbf{n}$ — "unit normal"
- The tangential component $I\,\mathbf{t}$ only affects the underlying parametrization of the curve. Thus, we can set I to be anything we like without affecting the curve evolution.

Normal Curve Flows

$$C_t = J\,\mathbf{n}$$

Examples — Euclidean–invariant curve flows

- $C_t = \mathbf{n}$ — geometric optics or grassfire flow;
- $C_t = \kappa\,\mathbf{n}$ — curve shortening flow;
- $C_t = \kappa^{1/3}\,\mathbf{n}$ — equi-affine invariant curve shortening flow:
$$C_t = \mathbf{n}_{\text{equi-affine}}\,;$$
- $C_t = \kappa_s\,\mathbf{n}$ — modified Korteweg–deVries flow;
- $C_t = \kappa_{ss}\,\mathbf{n}$ — thermal grooving of metals.

Intrinsic Curve Flows

Theorem. The curve flow generated by

$$\mathbf{v} = I\,\mathbf{t} + J\,\mathbf{n}$$

preserves arc length if and only if

$$\mathcal{B}(J) + \mathcal{D}\,I = 0.$$

$\mathcal{D}$ — invariant arc length derivative

$\mathcal{B}$ — invariant arc length variation

$$\delta(ds) = \mathcal{B}(\delta u)\,ds$$

Normal Evolution of Differential Invariants

Theorem. Under a normal flow $C_t = J\,\mathbf{n}$,

$$\frac{\partial \kappa}{\partial t} = \mathcal{A}_\kappa(J), \qquad \frac{\partial \kappa_s}{\partial t} = \mathcal{A}_{\kappa_s}(J).$$

Invariant variations:

$$\delta\kappa = \mathcal{A}_\kappa(\delta u), \qquad \delta\kappa_s = \mathcal{A}_{\kappa_s}(\delta u).$$

$\mathcal{A}_\kappa = \mathcal{A}$ — invariant variation of curvature;

$\mathcal{A}_{\kappa_s} = \mathcal{D}\,\mathcal{A} + \kappa\,\kappa_s$ — invariant variation of κ_s.

Euclidean–invariant Curve Evolution

Normal flow: $\quad C_t = J\,\mathbf{n}$

$$\frac{\partial \kappa}{\partial t} = \mathcal{A}_\kappa(J) = (\mathcal{D}^2 + \kappa^2)\,J,$$

$$\frac{\partial \kappa_s}{\partial t} = \mathcal{A}_{\kappa_s}(J) = (\mathcal{D}^3 + \kappa^2\mathcal{D} + 3\,\kappa\,\kappa_s)\,J.$$

Warning: For non-intrinsic flows, ∂_t and ∂_s *do not commute*!

Euclidean–invariant Curve Evolution

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Grassfire flow: $J = 1$

$$\frac{\partial \kappa}{\partial t} = \kappa^2, \qquad \frac{\partial \kappa_s}{\partial t} = 3\,\kappa\,\kappa_s, \qquad \ldots$$

$\Longrightarrow$ caustics

Euclidean–invariant Curve Evolution

Normal flow: $C_t = J\,\mathbf{n}$

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$\Longrightarrow$ caustics

$\star$ Signature evolution: $\Sigma_t = \cdots$

Intrinsic Evolution of Differential Invariants

Theorem.

Under an arc-length preserving flow,

$$\kappa_t = \mathcal{R}(J) \qquad \text{where} \qquad \mathcal{R} = \mathcal{A} - \kappa_s \mathcal{D}^{-1}\mathcal{B} \qquad (*)$$

In surprisingly many situations,
(*) is a well-known integrable evolution equation,
and $\mathcal{R}$ is (closely related to) its recursion operator!

$\Longrightarrow$ Hasimoto

$\Longrightarrow$ Langer, Singer, Perline

$\Longrightarrow$ Marí–Beffa, Sanders, Wang, Qu, Chou, Anco,

$\Longrightarrow$ Benson, and many more ...

Euclidean plane curves

$$G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$$

$$\mathcal{A} = \mathcal{D}^2 + \kappa^2 \qquad\qquad \mathcal{B} = -\,\kappa$$

$$\mathcal{R} = \mathcal{A} - \kappa_s \mathcal{D}^{-1} \mathcal{B} = \mathcal{D}^2 + \kappa^2 + \kappa_s \mathcal{D}^{-1} \cdot \kappa$$

$$\boxed{\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{sss} + \tfrac{3}{2}\kappa^2 \kappa_s}$$

$\Longrightarrow$ modified Korteweg-deVries equation

Equi-affine plane curves

$$G = \mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$$

$$\mathcal{A} = \mathcal{D}^4 + \tfrac{5}{3}\,\kappa\,\mathcal{D}^2 + \tfrac{5}{3}\,\kappa_s\mathcal{D} + \tfrac{1}{3}\,\kappa_{ss} + \tfrac{4}{9}\,\kappa^2$$

$$\mathcal{B} = \tfrac{1}{3}\,\mathcal{D}^2 - \tfrac{2}{9}\,\kappa$$

$$\mathcal{R} = \mathcal{A} - \kappa_s\mathcal{D}^{-1}\mathcal{B}$$

$$= \mathcal{D}^4 + \tfrac{5}{3}\,\kappa\,\mathcal{D}^2 + \tfrac{4}{3}\,\kappa_s\mathcal{D} + \tfrac{1}{3}\,\kappa_{ss} + \tfrac{4}{9}\,\kappa^2 + \tfrac{2}{9}\,\kappa_s\mathcal{D}^{-1}\cdot\kappa$$

$$\boxed{\kappa_t = \mathcal{R}(\kappa_s) = \kappa_{5s} + \tfrac{5}{3}\,\kappa\,\kappa_{sss} + \tfrac{5}{3}\,\kappa_s\kappa_{ss} + \tfrac{5}{9}\,\kappa^2\kappa_s}$$

$\Longrightarrow$ Sawada–Kotera equation

Recursion operator: $\widehat{\mathcal{R}} = \mathcal{R}\cdot(\mathcal{D}^2 + \tfrac{1}{3}\kappa + \tfrac{1}{3}\kappa_s\mathcal{D}^{-1})$

Euclidean space curves

$$G = \mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$$

$$\mathcal{A} = \begin{pmatrix} D_s^2 + (\kappa^2 - \tau^2) & -2\tau D_s - \tau_s \\ \frac{2\tau}{\kappa} D_s^2 + \frac{3\kappa\tau_s - 2\kappa_s\tau}{\kappa^2} D_s + \frac{\kappa\tau_{ss} - \kappa_s\tau_s + 2\kappa^3\tau}{\kappa^2} & \frac{1}{\kappa} D_s^3 - \frac{\kappa_s}{\kappa^2} D_s^2 + \frac{\kappa^2 - \tau^2}{\kappa} D_s + \frac{\kappa_s\tau^2 - 2\kappa\tau\tau_s}{\kappa^2} \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} -\kappa & 0 \end{pmatrix}$$

$$\mathcal{R} = \mathcal{A} - \begin{pmatrix} \kappa_s \\ \tau_s \end{pmatrix} \mathcal{D}^{-1} \mathcal{B}$$

$$\boxed{\begin{pmatrix} \kappa_t \\ \tau_t \end{pmatrix} = \mathcal{R} \begin{pmatrix} 0 \\ \kappa \end{pmatrix}}$$

$\Longrightarrow$ vortex filament flow (Hasimoto)