Adventures in Imaging

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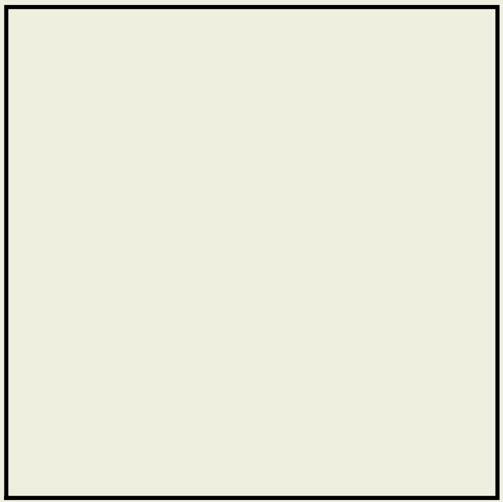
Symmetry

Definition. A symmetry of a set S is a transformation that preserves it:

$$g \cdot S = S$$

$\star\star$ The set of symmetries forms a group, called the symmetry group of the set S.

Discrete Symmetry Group



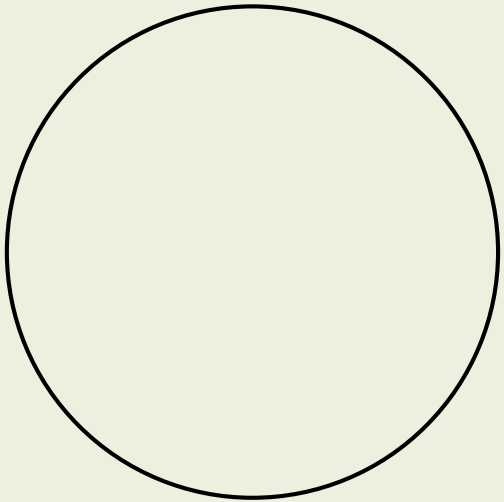
Rotations by 90°:

$$G_S = \mathbb{Z}_4$$

Rotations + reflections:

$$G_S = \mathbb{Z}_2 \ltimes \mathbb{Z}_4$$

Continuous Symmetry Group



Rotations:

$$G_S = \mathrm{SO}(2)$$

Rotations + reflections:

$$G_S = \mathrm{O}(2)$$

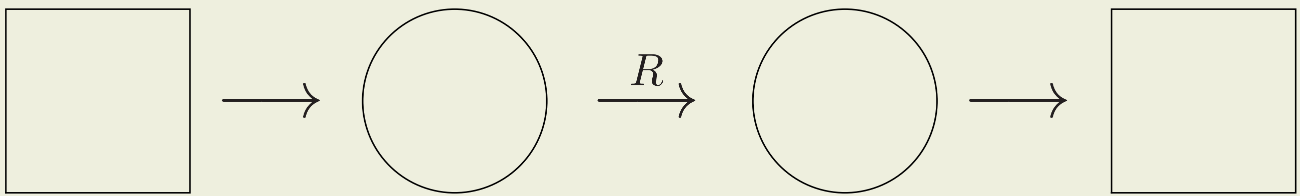
Conformal Inversions:

$$\overline{x} = \frac{x}{x^2 + y^2} \qquad \overline{y} = \frac{y}{x^2 + y^2}$$

⋆ A continuous group is known as a Lie group

— in honor of Sophus Lie.

Continuous Symmetries of a Square



Symmetry

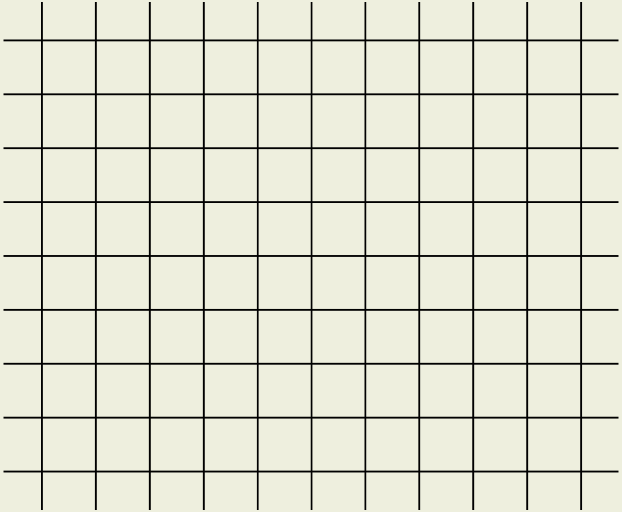
⋆ To define the set of symmetries requires a priori specification of the allowable transformations

G — transformation group containing all allowable transformations of the ambient space M

Definition. A symmetry of a subset $S \subset M$ is an allowable transformation $g \in G$ that preserves it:

$$g \cdot S = S$$

What is the Symmetry Group?



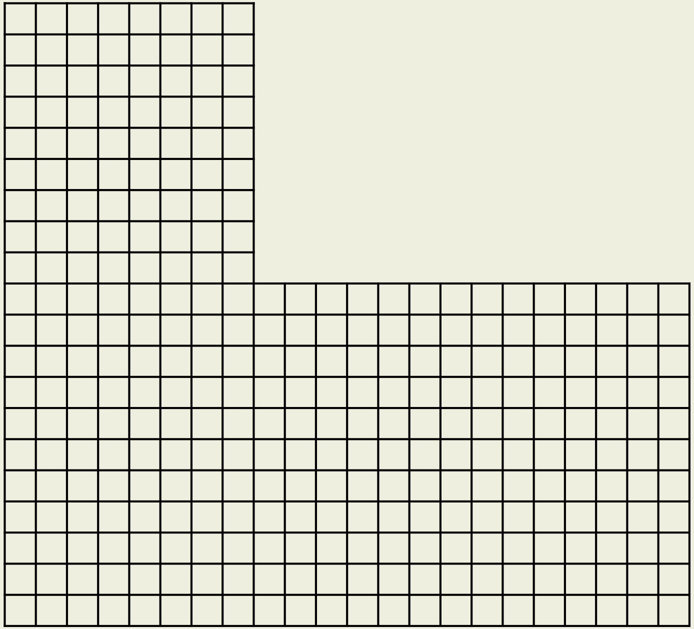
Allowable transformations:

Rigid motions

$G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$

$G_S = \mathbb{Z}_4 \ltimes \mathbb{Z}^2$

What is the Symmetry Group?



Allowable transformations:

Rigid motions

$G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$

$G_S = \{e\}$

Local Symmetries

Definition. $g \in G$ is a local symmetry of $S \subset M$ based at a point $z \in S$ if there is an open neighborhood $z \in U \subset M$ such that

$$g \cdot (S \cap U) = S \cap (g \cdot U)$$

$\star\star$ The set of all local symmetries forms a groupoid!

$\Longrightarrow$ Groupoids form the appropriate framework for studying objects with variable symmetry.

Definition. A groupoid is a small category such that every morphism has an inverse.

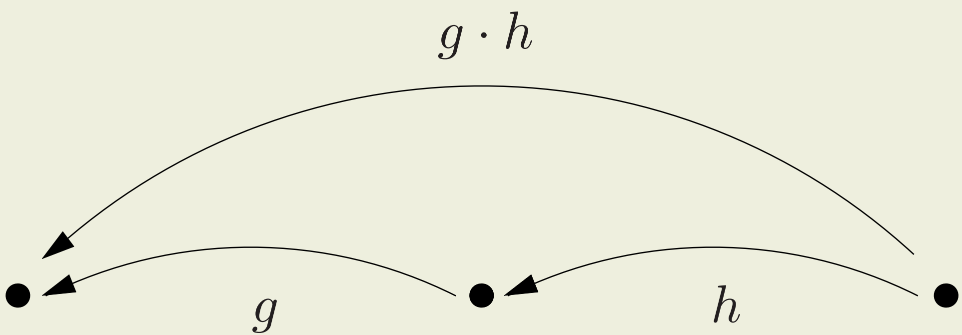
Groupoids

$\Longrightarrow$ In practice you are only allowed to multiply groupoid elements $g \cdot h$ when

source (domain) of g = target (range) of h

Similarly for inverses g^{-1} and the identities e.

A groupoid is a "collection of arrows":



Geometry = Group(oid) Theory

Felix Klein's Erlanger Programm (1872):

Each type of geometry is founded on
an underlying transformation group

Plane Geometries/Groups

Euclidean geometry:

SE(2) — rigid motions (rotations and translations)

$$\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

E(2) — plus reflections?

Equi-affine geometry:

SA(2) — area-preserving affine transformations:

$$\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \qquad \alpha\,\delta - \beta\,\gamma = 1$$

Projective geometry:

PSL(3) — projective transformations:

$$\overline{x} = \frac{\alpha\,x + \beta\,y + \gamma}{\rho\,x + \sigma\,y + \tau} \qquad \overline{y} = \frac{\lambda\,x + \mu\,y + \nu}{\rho\,x + \sigma\,y + \tau}$$

The Equivalence Problem

$\Longrightarrow$ É Cartan

G — transformation group acting on M

Equivalence:

Determine when two subsets

$$S \quad \text{and} \quad \overline{S} \subset M$$

are congruent:

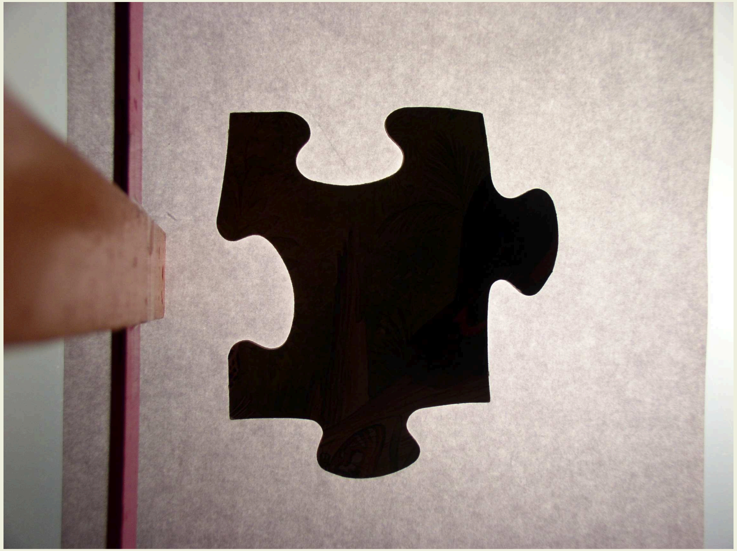
$$\overline{S} = g \cdot S \qquad \text{for} \qquad g \in G$$

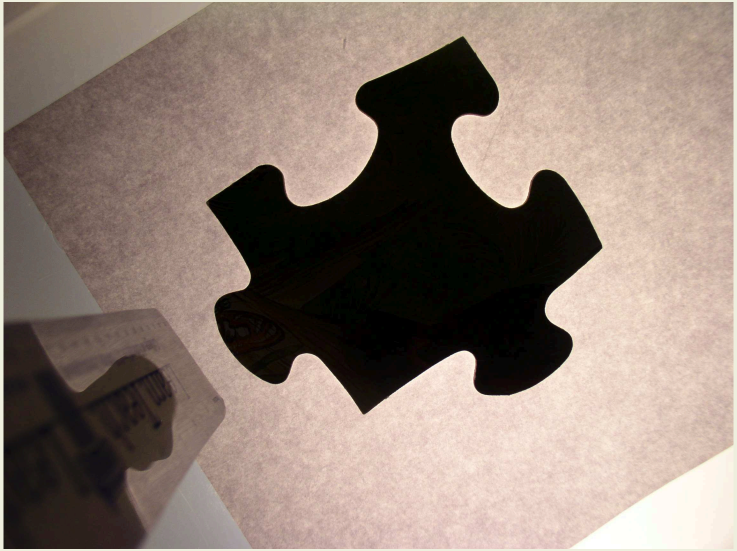
Symmetry:

Find all symmetries or self-congruences:

$$S = g \cdot S$$

Euclidean Equivalence





Projective/Equi-Affine Equivalence

$\Longrightarrow$ Symmetries

Duck = Rabbit?

Limitations of Projective Equivalence

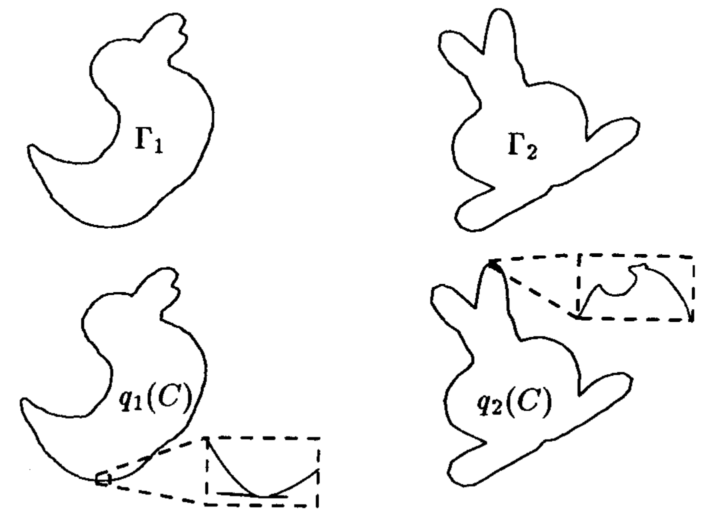


Fig. 3. The upper two curves are not projectively equivalent, but the lower two curves are. The lower curves are constructed by introducing small ripples along the convex hull, these are illustrated in the magnified pictures.

$\Longrightarrow$ K. Åström (1995)

Thatcher Illusion

$\Longrightarrow$ Groupoid equivalence?

Invariants

The solution to an equivalence problem rests on understanding its invariants.

Definition. If G is a group acting on M, then an invariant is a real-valued function $I : M \to \mathbb{R}$ that does not change under the action of G:

$$I(g \cdot z) = I(z) \qquad \text{for all} \qquad g \in G, \quad z \in M$$

★ If G acts transitively, there are no (non-constant) invariants.

Differential Invariants

Given a submanifold (curve, surface, ...)

$$S \subset M$$

a differential invariant is an invariant of the prolonged action of G on its Taylor coefficients (jets):

$$\boxed{I(g \cdot z^{(k)}) \;=\; I(z^{(k)})}$$

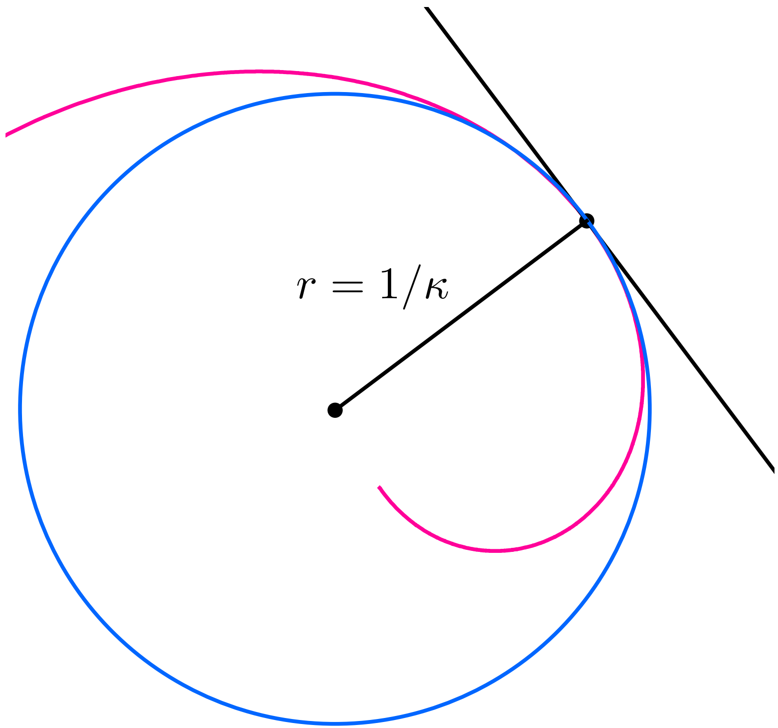
Euclidean Plane Curves

$G = \mathrm{SE}(2)$ acts on curves $C \subset M = \mathbb{R}^2$

The simplest differential invariant is the curvature, defined as the reciprocal of the radius of the osculating circle:

$$\kappa = \frac{1}{r}$$

Curvature



Euclidean Plane Curves: $G = \mathrm{SE}(2)$

Differentiation with respect to the Euclidean-invariant arc length element ds is an invariant differential operator, meaning that it maps differential invariants to differential invariants.

Thus, starting with curvature κ, we can generate an infinite collection of higher order Euclidean differential invariants:

$$\kappa, \quad \frac{d\kappa}{ds}, \quad \frac{d^2\kappa}{ds^2}, \quad \frac{d^3\kappa}{ds^3}, \quad \cdots$$

Theorem. All Euclidean differential invariants are functions of the derivatives of curvature with respect to arc length: $\kappa, \ \kappa_s, \ \kappa_{ss}, \ \cdots$

Euclidean Plane Curves: $G = \mathrm{SE}(2)$

Assume the curve $C \subset M$ is a graph: $\quad y = u(x)$

Differential invariants:

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}, \qquad \frac{d\kappa}{ds} = \frac{(1+u_x^2)u_{xxx} - 3u_x u_{xx}^2}{(1+u_x^2)^3}, \qquad \frac{d^2\kappa}{ds^2} = \cdots$$

Arc length (invariant one-form):

$$ds = \sqrt{1+u_x^2}\; dx, \qquad \frac{d}{ds} = \frac{1}{\sqrt{1+u_x^2}}\,\frac{d}{dx}$$

Equi-affine Plane Curves: $G = \mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$

Equi-affine curvature:

$$\kappa = \frac{5\,u_{xx}u_{xxxx} - 3\,u_{xxx}^2}{9\,u_{xx}^{8/3}} \qquad \frac{d\kappa}{ds} = \cdots$$

Equi-affine arc length:

$$ds = \sqrt[3]{u_{xx}}\; dx \qquad \frac{d}{ds} = \frac{1}{\sqrt[3]{u_{xx}}}\,\frac{d}{dx}$$

Theorem. All equi-affine differential invariants are functions of the derivatives of equi-affine curvature with respect to equi-affine arc length: $\kappa, \quad \kappa_s, \quad \kappa_{ss}, \quad \cdots$

Projective Plane Curves: $G = \mathrm{PSL}(2)$

Projective curvature:

$$\kappa = K(u^{(7)}, \cdots) \qquad \frac{d\kappa}{ds} = \cdots \qquad \frac{d^2\kappa}{ds^2} = \cdots$$

Projective arc length:

$$ds = L(u^{(5)}, \cdots)\, dx \qquad \frac{d}{ds} = \frac{1}{L}\frac{d}{dx}$$

Theorem. All projective differential invariants are functions of the derivatives of projective curvature with respect to projective arc length:

$$\kappa, \quad \kappa_s, \quad \kappa_{ss}, \quad \cdots$$

Moving Frames

The equivariant method of moving frames provides a systematic and algorithmic calculus for determining complete systems of differential invariants, joint invariants, invariant differential operators, invariant differential forms, invariant variational problems, invariant numerical algorithms, etc., etc.

Equivalence & Invariants

- Equivalent submanifolds $S \approx \overline{S}$
  must have the same invariants: $I = \bar{I}$.

Constant invariants provide immediate information:

$$\text{e.g.} \qquad \kappa = 2 \quad \Longleftrightarrow \quad \overline{\kappa} = 2$$

Non-constant invariants are not useful in isolation, because an equivalence map can drastically alter the dependence on the submanifold parameters:

$$\text{e.g.} \qquad \kappa = x^3 \qquad \text{versus} \qquad \overline{\kappa} = \sinh x$$

Syzygies

However, a functional dependency or syzygy among the invariants *is* intrinsic:

$$\text{e.g.} \qquad \kappa_s = \kappa^3 - 1 \quad \Longleftrightarrow \quad \overline{\kappa}_{\overline{s}} = \overline{\kappa}^3 - 1$$

- Universal syzygies — Gauss–Codazzi
- Distinguishing syzygies.

Theorem. (Cartan)
Two regular submanifolds are locally equivalent if and only if they have identical syzygies among *all* their differential invariants.

Finiteness of Generators and Syzygies

♠ There are, in general, an infinite number of differential invariants and hence an infinite number of syzygies must be compared to establish equivalence.

♡ But the higher order differential invariants are always generated by invariant differentiation from a finite collection of basic differential invariants, and the higher order syzygies are all consequences of a finite number of low order syzygies!

Example — Plane Curves

If non-constant, both κ and κ_s depend on a single parameter, and so, locally, are subject to a syzygy:

$$\kappa_s = H(\kappa) \qquad (*)$$

But then

$$\kappa_{ss} = \frac{d}{ds} H(\kappa) = H'(\kappa)\,\kappa_s = H'(\kappa)\,H(\kappa)$$

and similarly for κ_{sss}, etc.

Consequently, all the higher order syzygies are generated by the fundamental first order syzygy $(*)$.

Thus, for Euclidean (or equi-affine or projective or ...) plane curves we need only know a single syzygy between κ and κ_s in order to establish equivalence!

Signature Curves

Definition. The signature curve $\Sigma \subset \mathbb{R}^2$ of a plane curve $C \subset \mathbb{R}^2$ is parametrized by the two lowest order differential invariants

$$\chi \,:\, C \longrightarrow \Sigma = \left\{ \left(\kappa \,,\, \frac{d\kappa}{ds} \right) \right\} \subset \mathbb{R}^2$$

$\Longrightarrow$ *Calabi, PJO, Shakiban, Tannenbaum, Haker*

Theorem. Two regular curves C and $\overline{C}$ are locally equivalent:

$$\overline{C} = g \cdot C$$

if and only if their signature curves are identical:

$$\overline{\Sigma} = \Sigma$$

$\Longrightarrow$ regular: $(\kappa_s, \kappa_{ss}) \neq 0$.

Continuous Symmetries of Curves

Theorem. For a connected curve, the following are equivalent:

- All the differential invariants are constant on C:

$$\kappa = c, \qquad \kappa_s = 0, \qquad \ldots$$

- The signature Σ degenerates to a point: $\dim \Sigma = 0$
- C admits a one-dimensional (local) symmetry group
- C is a piece of an orbit of a 1-dimensional subgroup $H \subset G$

Discrete Symmetries of Curves

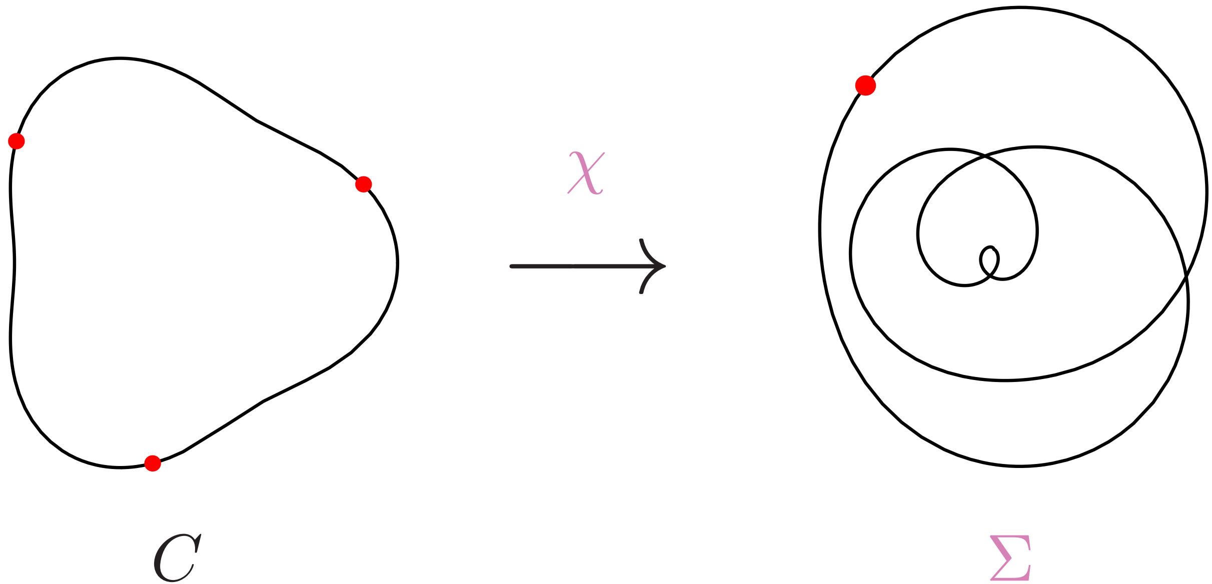
Definition. The index of a completely regular point $\zeta \in \Sigma$ equals the number of points in C which map to it:

$$i_\zeta = \# \, \chi^{-1}\{\zeta\}$$

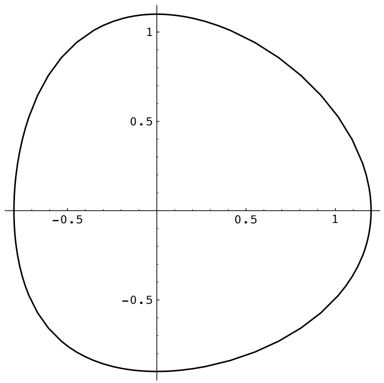
Regular means that, in a neighborhood of ζ, the signature is an embedded curve — no self-intersections.

Theorem. If $\chi(z) = \zeta$ is completely regular, then its index counts the number of discrete local symmetries of C.

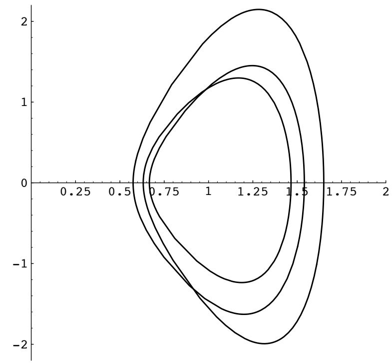
The Index



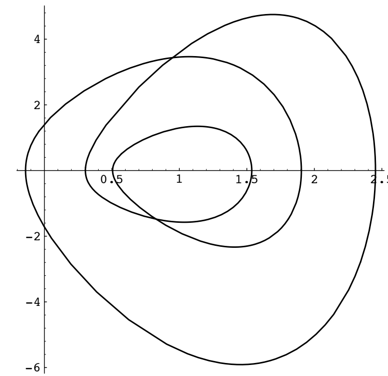
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \sin t + \frac{1}{10}\sin^2 t$



The Original Curve

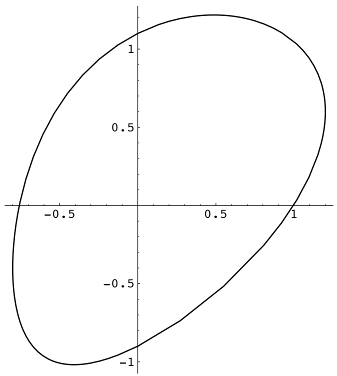


Euclidean Signature

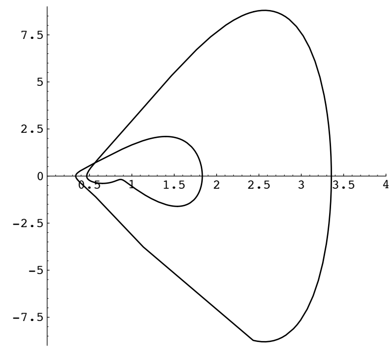


Equi-affine Signature

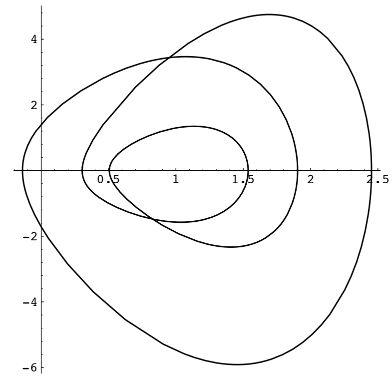
The Curve $x = \cos t + \frac{1}{5}\cos^2 t, \quad y = \frac{1}{2}x + \sin t + \frac{1}{10}\sin^2 t$



The Original Curve

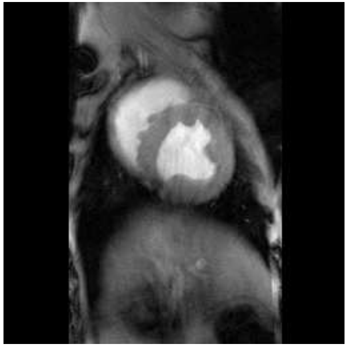


Euclidean Signature

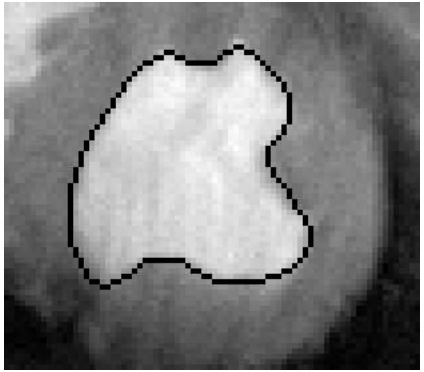


Equi-affine Signature

Canine Left Ventricle Signature

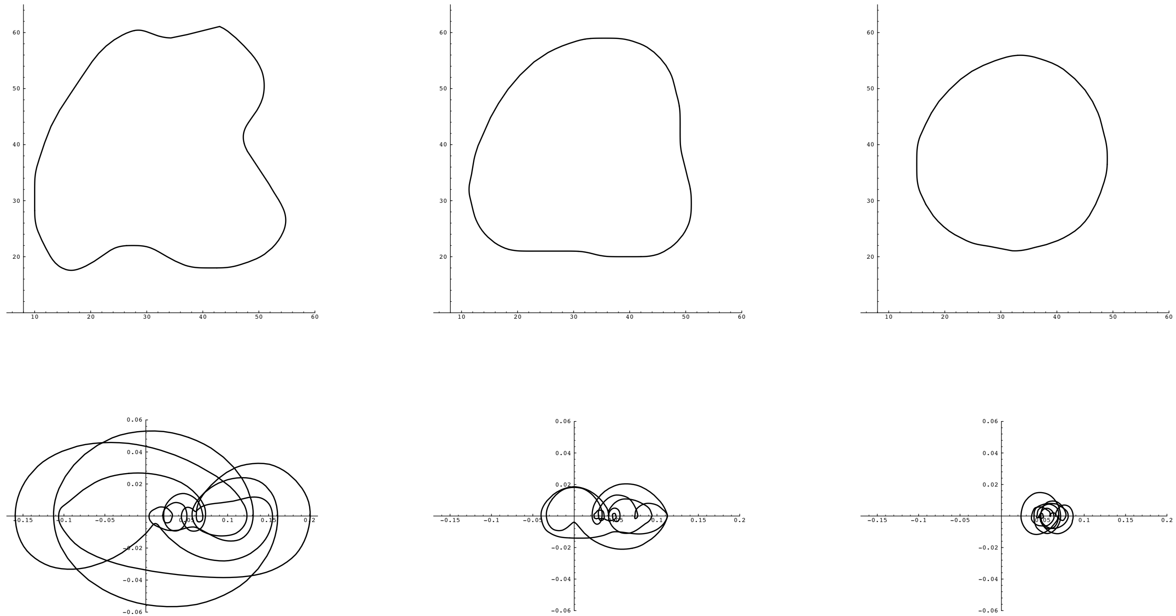


Original Canine Heart
MRI Image



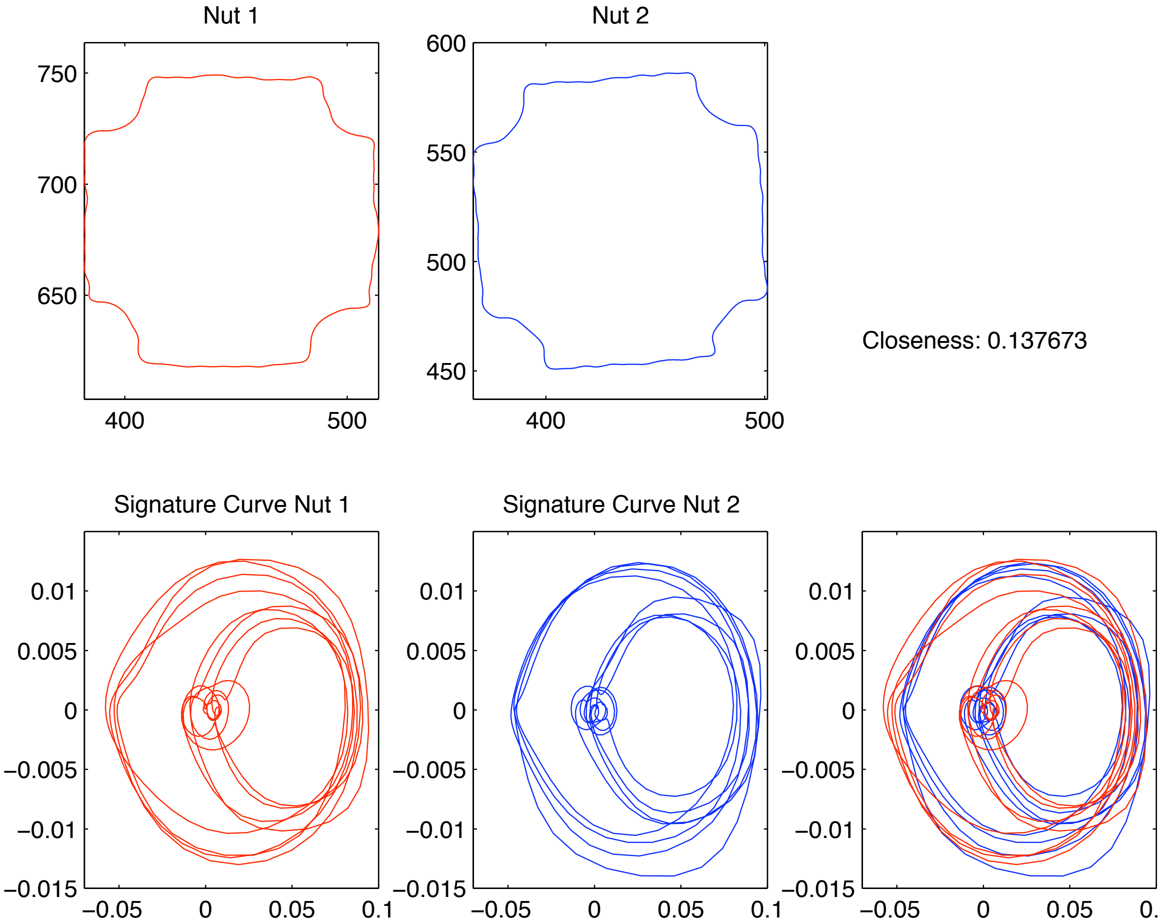
Boundary of Left Ventricle

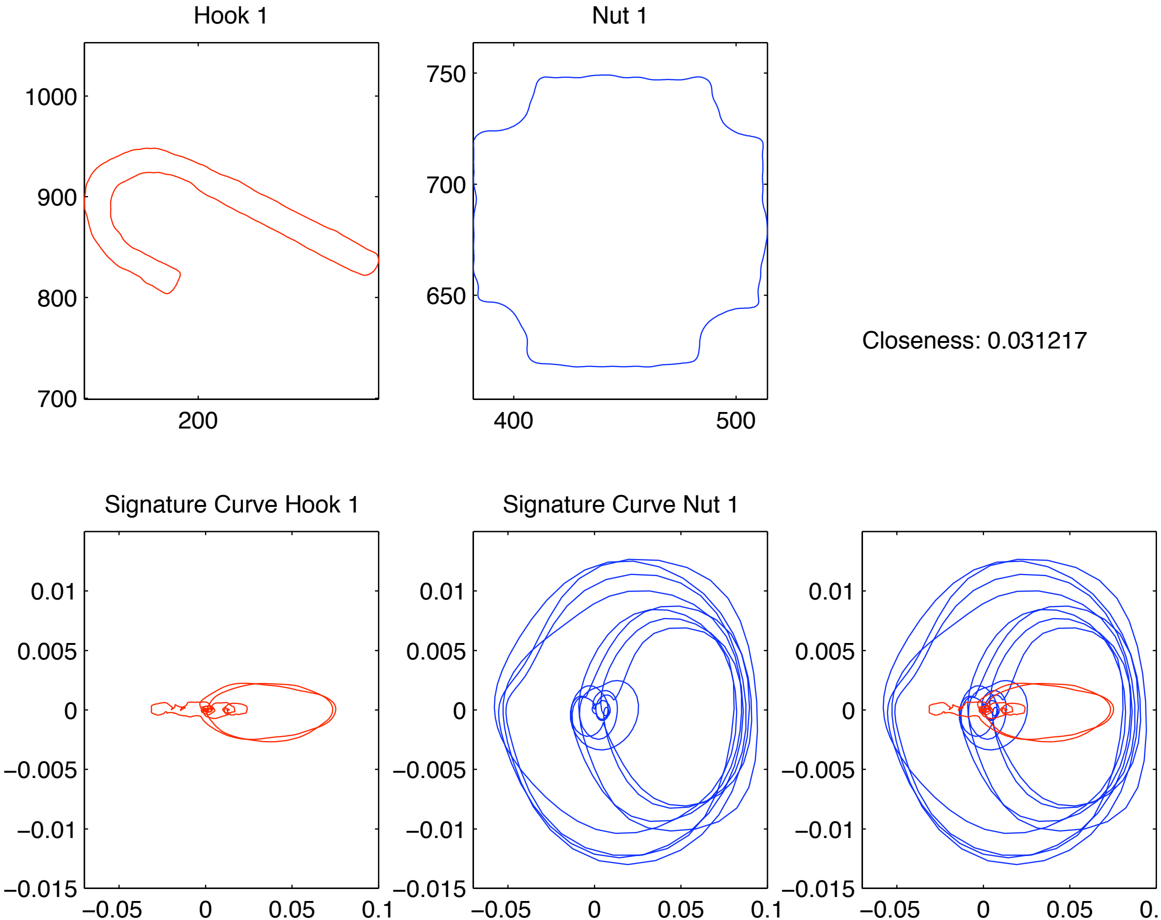
Smoothed Ventricle Signature



Object Recognition

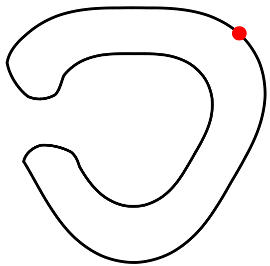
⟹ Steve Haker


Nut 1
Nut 2
Closeness: 0.137673
Signature Curve Nut 1
Signature Curve Nut 2

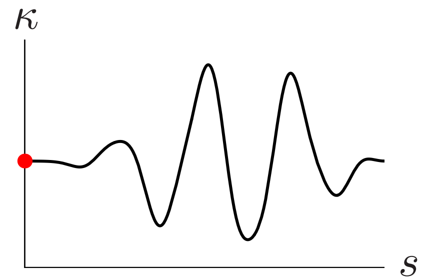


Hook 1
Nut 1
Closeness: 0.031217
Signature Curve Hook 1
Signature Curve Nut 1
1000
900
800
700
200
750
700
650
400
500
0.01
0.005
0
–0.005
–0.01
–0.015
–0.05
0
0.05
0.1

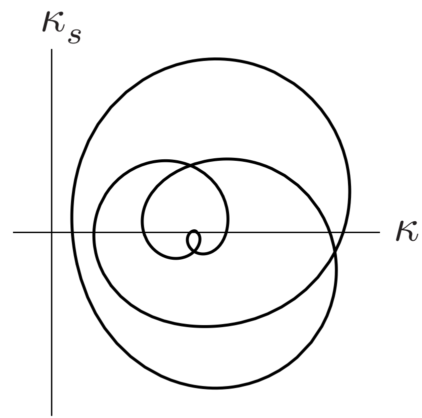
Signatures



Original curve

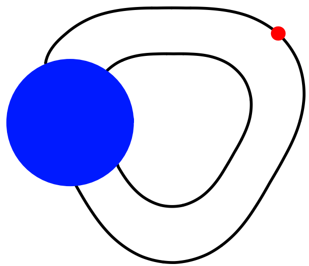


Classical Signature

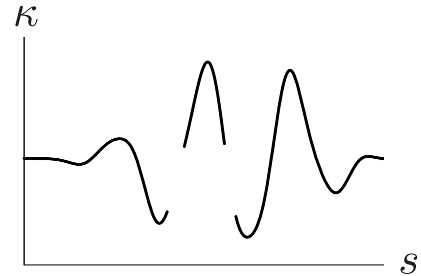


Differential invariant signature

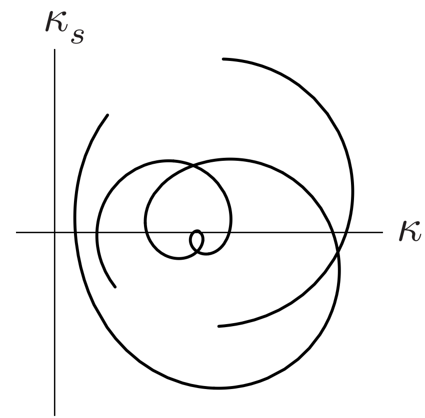
Occlusions



Original curve



Classical Signature



Differential invariant signature

3D Differential Invariant Signatures

Euclidean space curves: $C \subset \mathbb{R}^3$

$$\Sigma = \{ \, (\kappa \, , \, \kappa_s \, , \, \tau \,) \, \} \subset \mathbb{R}^3$$

- κ — curvature, τ — torsion

Euclidean surfaces: $S \subset \mathbb{R}^3$ (generic)

$$\Sigma = \left\{ \left(H \, , \, K \, , \, H_{,1} \, , \, H_{,2} \, , \, K_{,1} \, , \, K_{,2} \right) \right\} \subset \mathbb{R}^6$$

or $$\widehat{\Sigma} = \left\{ \left(H \, , \, H_{,1} \, , \, H_{,2} \, , \, H_{,11} \right) \right\} \subset \mathbb{R}^4$$

- H — mean curvature, K — Gauss curvature

Equi–affine surfaces: $S \subset \mathbb{R}^3$ (generic)

$$\Sigma = \left\{ \left(P \, , \, P_{,1} \, , \, P_{,2} , \, P_{,11} \right) \right\} \subset \mathbb{R}^4$$

- P — Pick invariant

Vertices of Euclidean Curves

Ordinary vertex: local extremum of curvature

Generalized vertex: $\kappa_s \equiv 0$

- critical point
- circular arc
- straight line segment

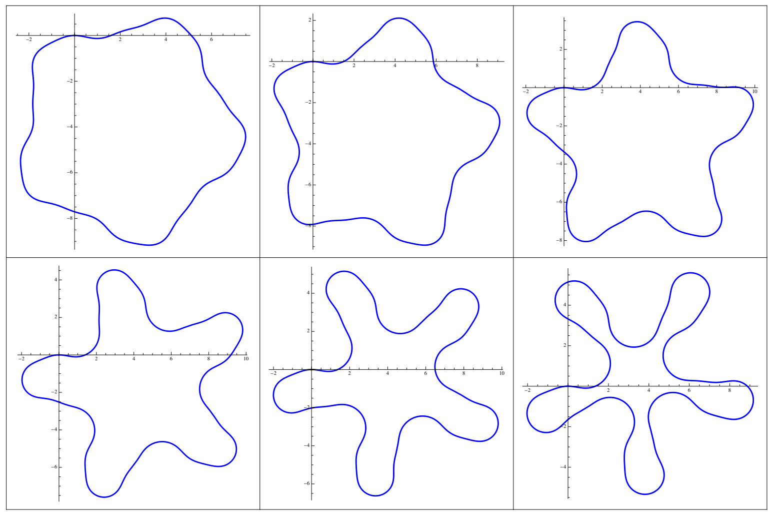
Mukhopadhya's Four Vertex Theorem:

A simple closed, non-circular plane curve has $n \geq 4$ generalized vertices.

“Counterexamples”

★ Generalized vertices map to a single point of the signature.

Hence, the (degenerate) curves obtained by replace ordinary vertices with circular arcs of the same radius all have *identical* signature:

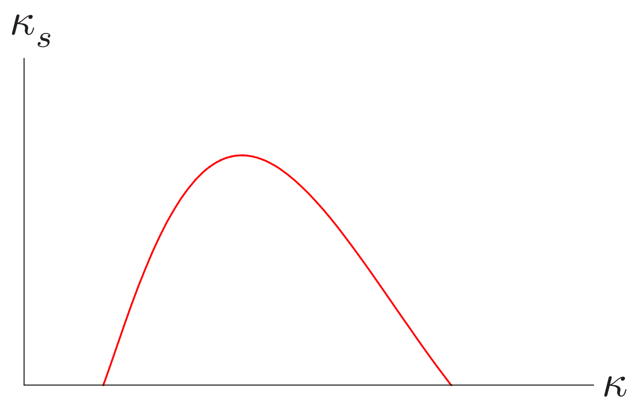


$\Longrightarrow$ Musso–Nicoldi

Bivertex Arcs

Bivertex arc: $\kappa_s \neq 0$ everywhere on the arc $B \subset C$
except $\kappa_s = 0$ at the two endpoints

The signature $\Sigma = \chi(B)$ of a bivertex arc is a single arc that starts and ends on the κ–axis.



Bivertex Decomposition

v-regular curve — finitely many generalized vertices

$$C = \bigcup_{j=1}^{m} B_j \ \cup \ \bigcup_{k=1}^{n} V_k$$

$B_1, \ldots, B_m$ — bivertex arcs

$V_1, \ldots, V_n$ — generalized vertices: $n \geq 4$

Main Idea: Compare individual bivertex arcs, and then decide whether the rigid equivalences are (approximately) the same.

D. Hoff & PJO, Extensions of invariant signatures for object recognition, *J. Math. Imaging Vision* **45** (2013), 176–185.

Signature Metrics

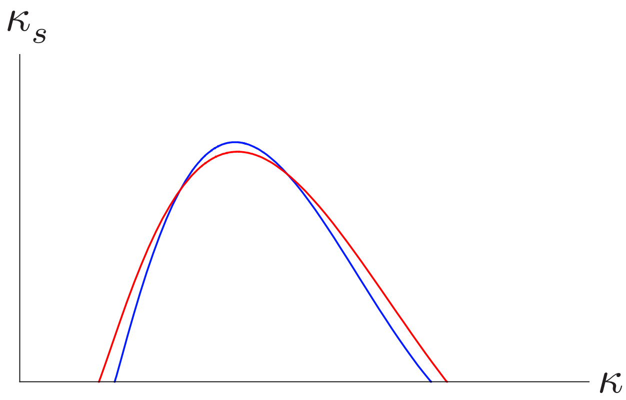
Used to compare signatures:

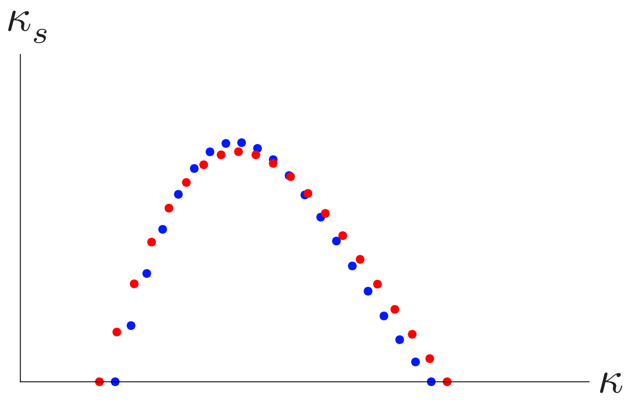
- Hausdorff
- Monge–Kantorovich transport
- Electrostatic/gravitational attraction
- Latent semantic analysis
- Histograms
- Geodesic distance
- Diffusion metric
- Gromov–Hausdorff & Gromov–Wasserstein

Gravitational/Electrostatic Attraction

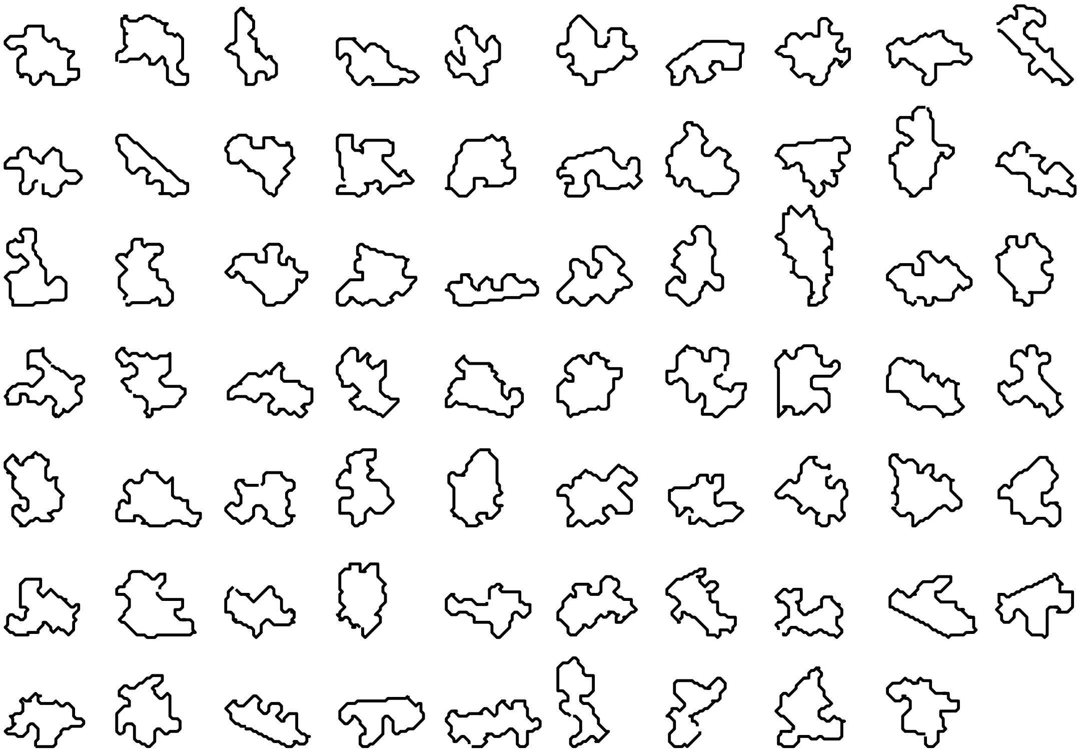
♡ Treat the two signature curves as masses or as oppositely charged wires. The higher their mutual attraction, the closer they are together.

♠ In practice, we are dealing with discrete data (pixels) and so treat the curves and signatures as point masses/charges.

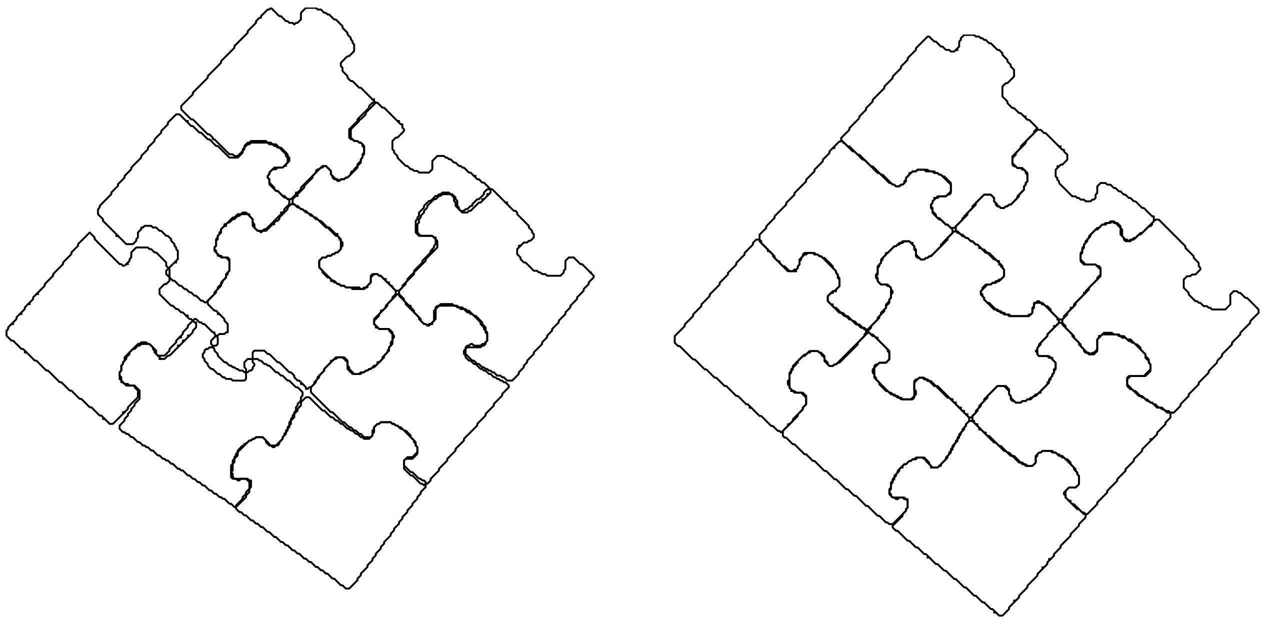




The Baffler Jigsaw Puzzle

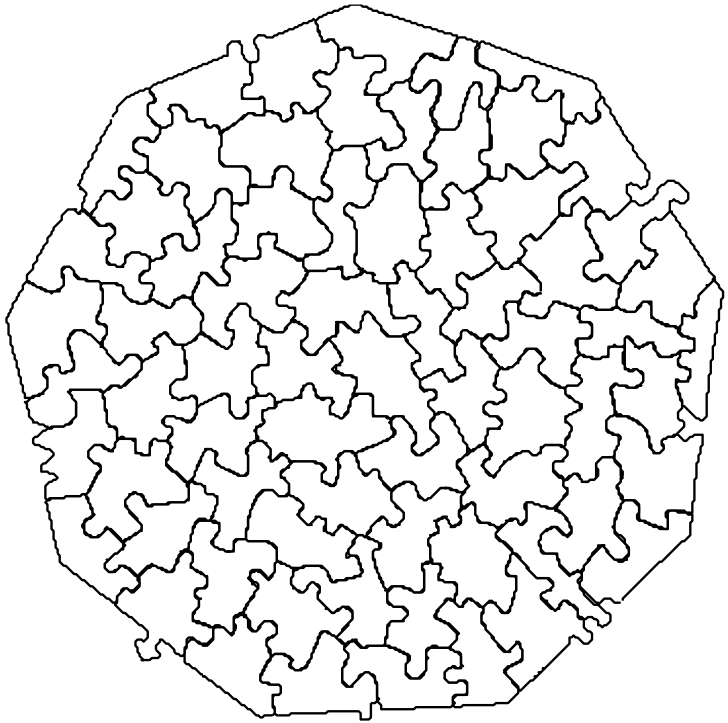


Piece Locking

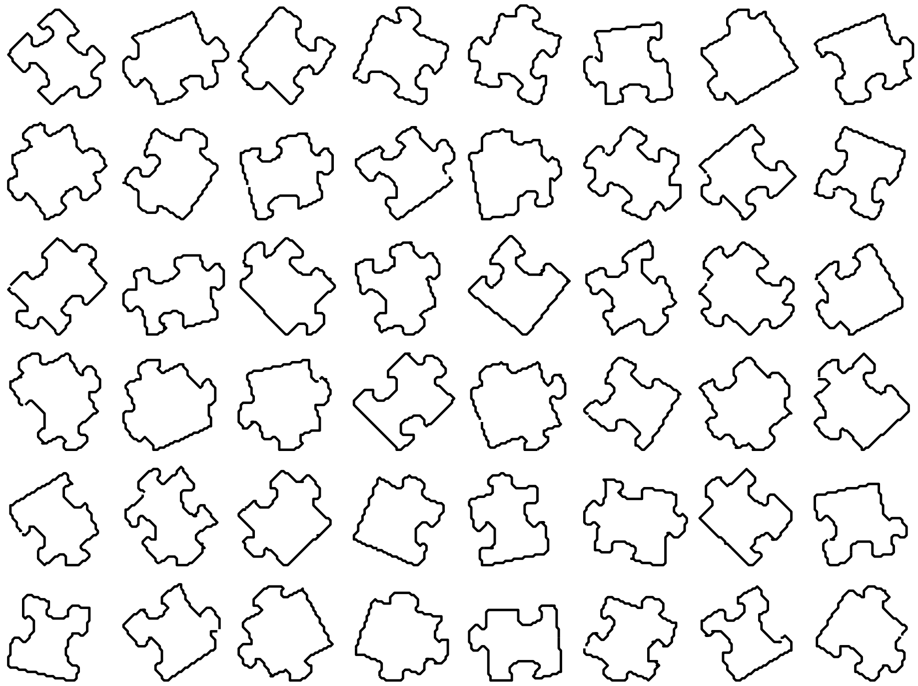


★★ Minimize force and torque based on gravitational attraction of the two matching edges.

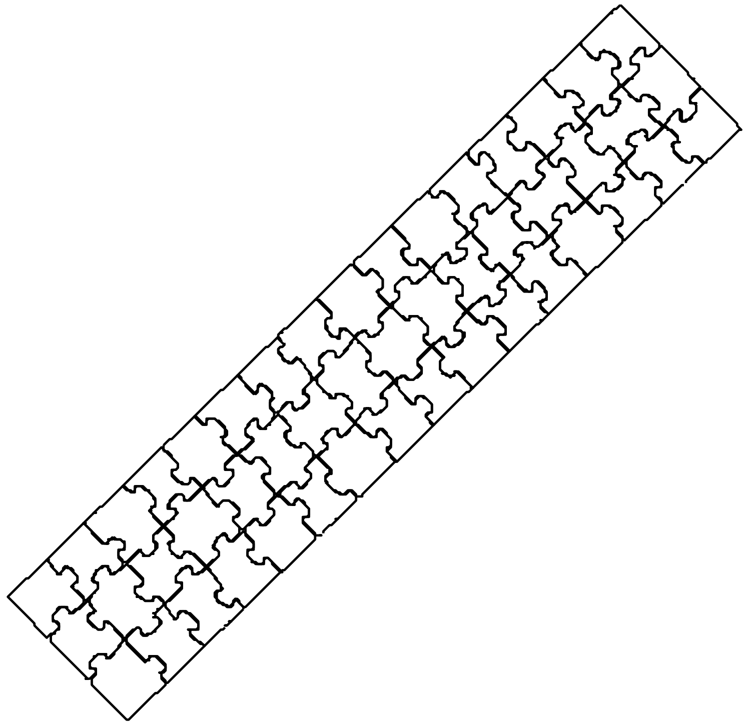
The Baffler Solved



The Rain Forest Giant Floor Puzzle

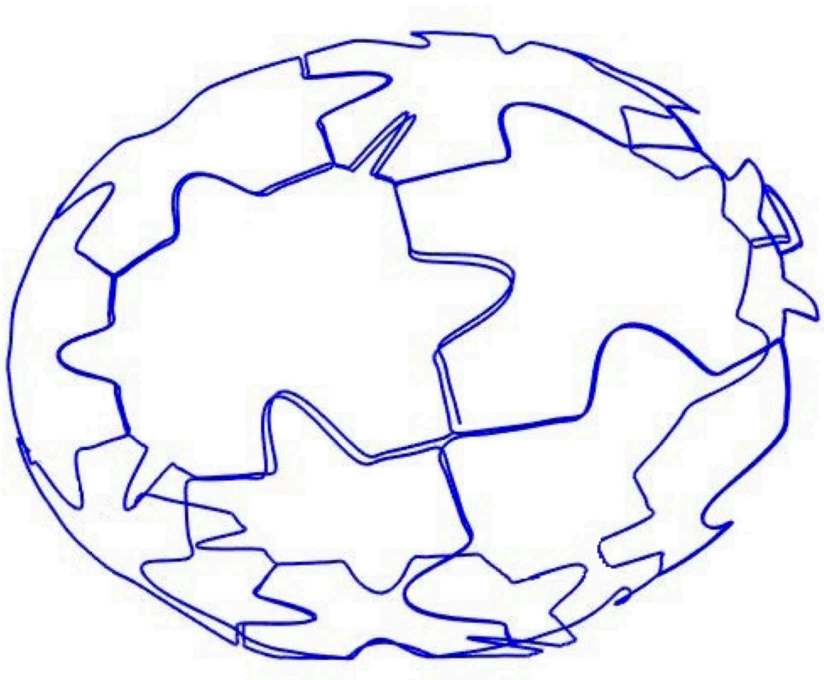


The Rain Forest Puzzle Solved



⟹ D. Hoff & PJO, Automatic solution of jigsaw puzzles, *J. Math. Imaging Vision* **49** (2014) 234–250.

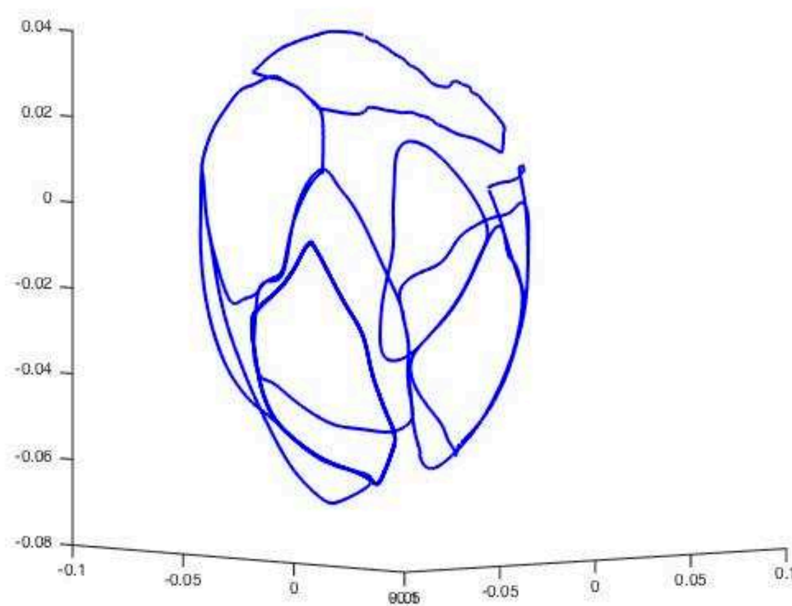
3D Jigsaw Puzzles



$\Longrightarrow$ Anna Grim, Tim O'Connor, Ryan Schlecta
Cheri Shakiban, Rob Thompson, PJO

Reassembling Humpty Dumpty

Broken ostrich egg shell — Marshall Bern

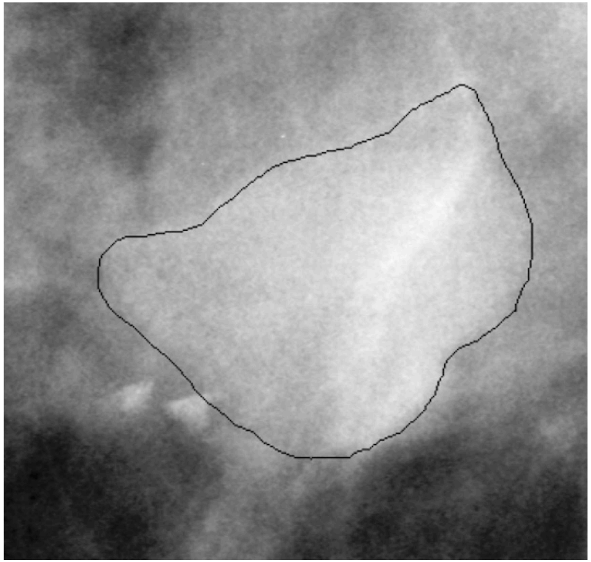


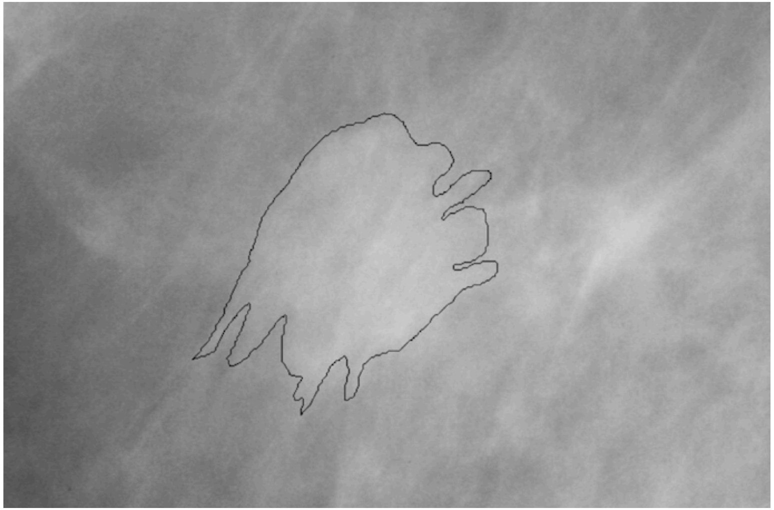
Archaeology

$\Longrightarrow$ **Virtual Archaeology**

Surgery

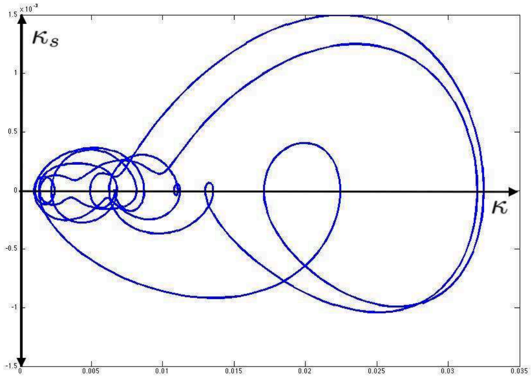
Benign vs. Malignant Tumors

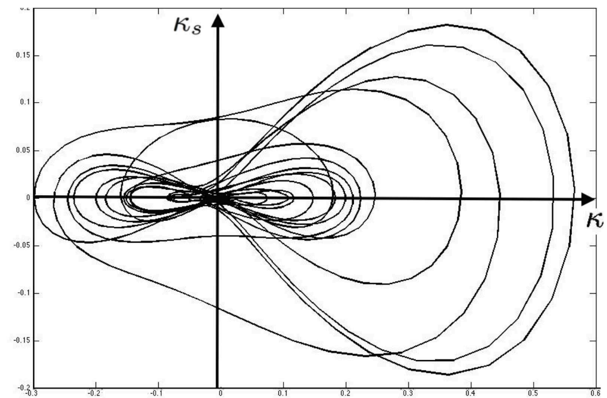




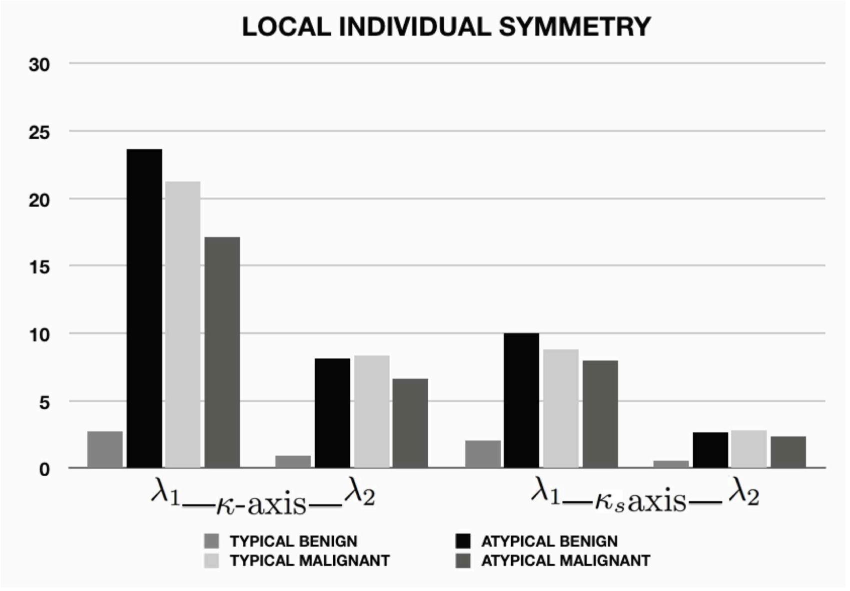
$\Longrightarrow$ A. Grim, C. Shakiban

Benign vs. Malignant Tumors





Benign vs. Malignant Tumors



Noise Resistant Signatures

Use lower order invariants to construct a signature:

- joint invariants
- joint differential invariants
- integral invariants
- topological invariants
- ...

Invariant Histograms

Euclidean geometry:

Definition. The distance histogram of a finite set of points $P = \{z_1, \ldots, z_n\} \subset V$ is the function

$$\eta_P(r) = \#\left\{ (i,j) \;\middle|\; 1 \leq i < j \leq n,\ d(z_i, z_j) = r \right\}.$$

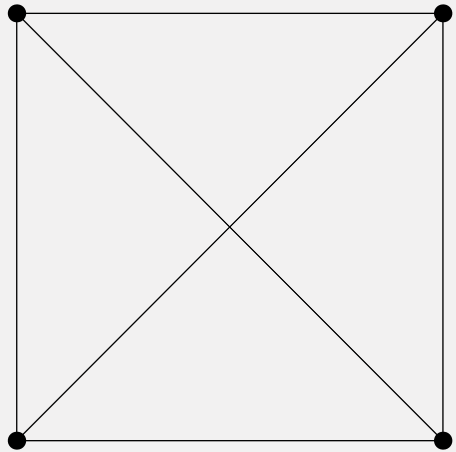
Characterization of Point Sets

⋆ ⋆ If $\widetilde{P} = g \cdot P$ is obtained from $P \subset \mathbb{R}^m$ by a rigid motion $g \in \mathrm{E}(n)$, then they have the same distance histogram: $\eta_P = \eta_{\widetilde{P}}$.

Question: Can one uniquely characterize, up to rigid motion, a set of points $P\{z_1, \ldots, z_n\} \subset \mathbb{R}^m$ by its distance histogram?

$\Longrightarrow$ Tinkertoy problem.

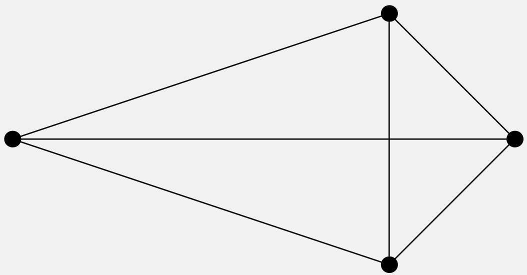
Yes:



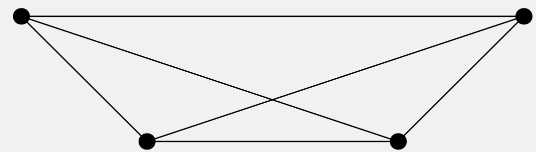
$$\eta = 1,\ 1,\ 1,\ 1,\ \sqrt{2},\ \sqrt{2}.$$

No:

Kite



Trapezoid



$$\eta = \sqrt{2},\ \ \sqrt{2},\ \ 2,\ \ \sqrt{10},\ \ \sqrt{10},\ \ 4.$$

No:

$$\left.\begin{aligned} P &= \{0, 1, 4, 10, 12, 17\} \\ Q &= \{0, 1, 8, 11, 13, 17\} \end{aligned}\right. \subset \mathbb{R}$$

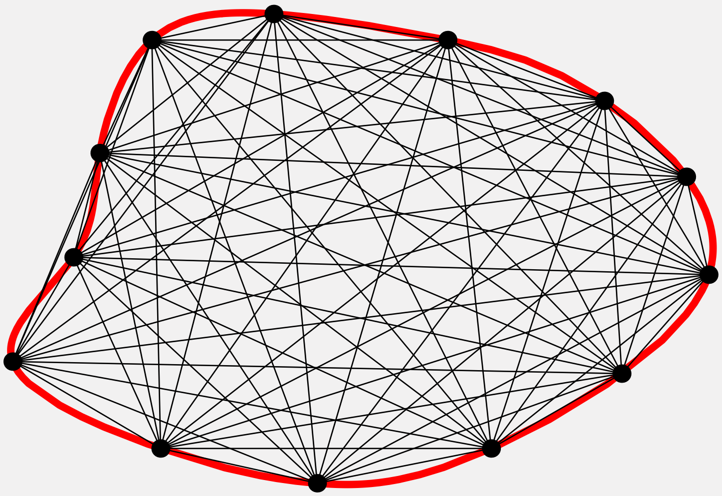
$$\eta = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17$$

$\implies$ G. Bloom, *J. Comb. Theory, Ser. A* **22** (1977) 378–379

Theorem. (*Boutin–Kemper*) Suppose $n \leq 3$ or $n \geq m+2$. Then there is a Zariski dense open subset in the space of n point configurations in $\mathbb{R}^m$ that are uniquely characterized, up to rigid motion, by their distance histograms.

$\Longrightarrow$ M. Boutin, G. Kemper, *Adv. Appl. Math.* **32** (2004) 709–735

Limiting Curve Histogram



D. Brinkman & PJO, Invariant histograms,

Amer. Math. Monthly **118** (2011) 2–24.

Limiting Curve Histogram Functions

Length of a curve

$$l(C) = \int_C ds < \infty$$

Local curve distance histogram function $\quad z \in V$

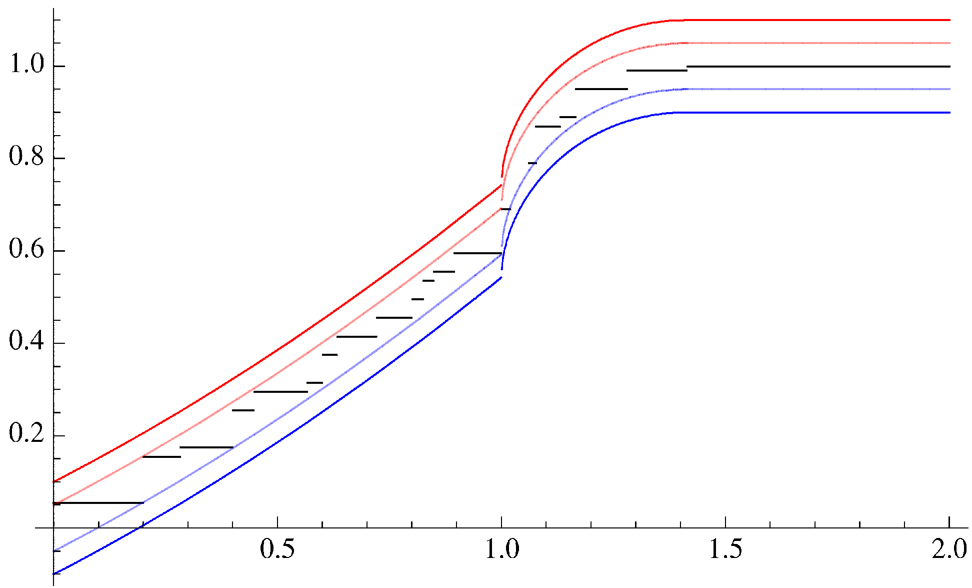
$$h_C(r, z) = \frac{l(C \cap B_r(z))}{l(C)}$$

$\Longrightarrow$ The fraction of the curve contained in the ball of radius r centered at z.

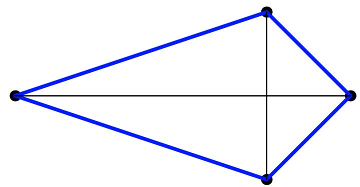
Global curve distance histogram function:

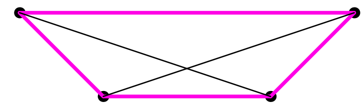
$$H_C(r) = \frac{1}{l(C)} \int_C h_C(r, z(s))\, ds.$$

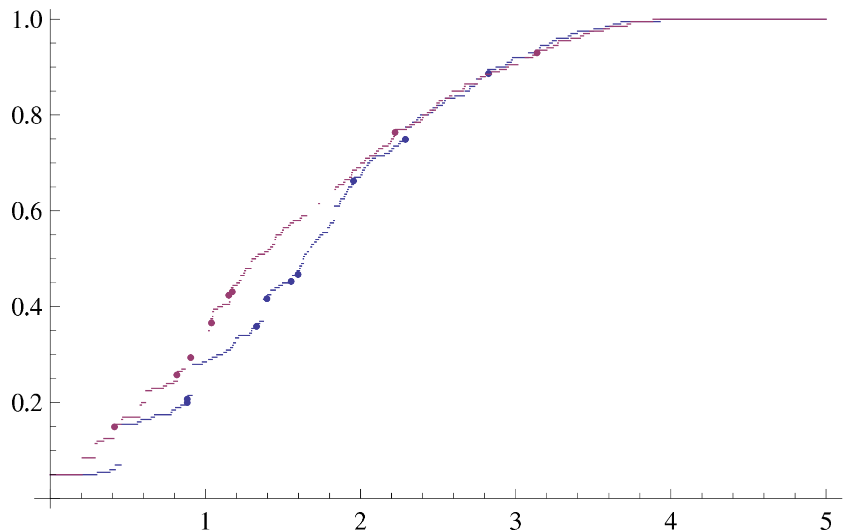
Square Curve Histogram with Bounds



Kite and Trapezoid Curve Histograms





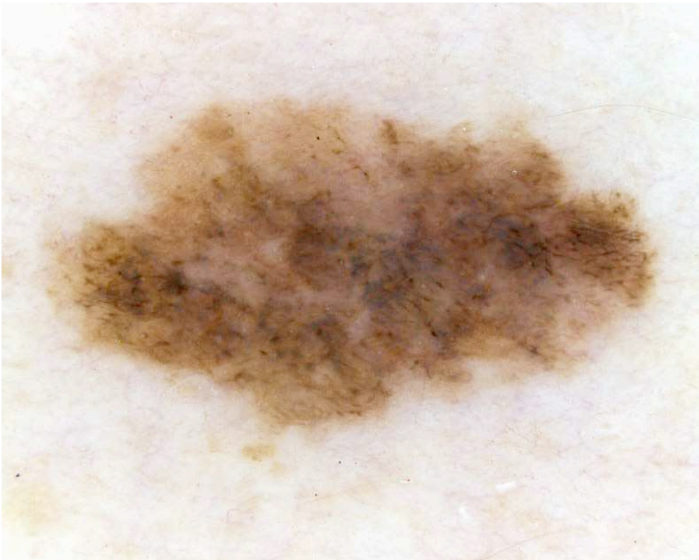


Histogram–Based Shape Recognition

500 sample points

Shape	(a)	(b)	(c)	(d)	(e)	(f)
(a) triangle	2.3	20.4	66.9	81.0	28.5	76.8
(b) square	28.2	.5	81.2	73.6	34.8	72.1
(c) circle	66.9	79.6	.5	137.0	89.2	138.0
(d) 2×3 rectangle	85.8	75.9	141.0	2.2	53.4	9.9
(e) 1×3 rectangle	31.8	36.7	83.7	55.7	4.0	46.5
(f) star	81.0	74.3	139.0	9.3	60.5	.9

Distinguishing Melanomas from Moles

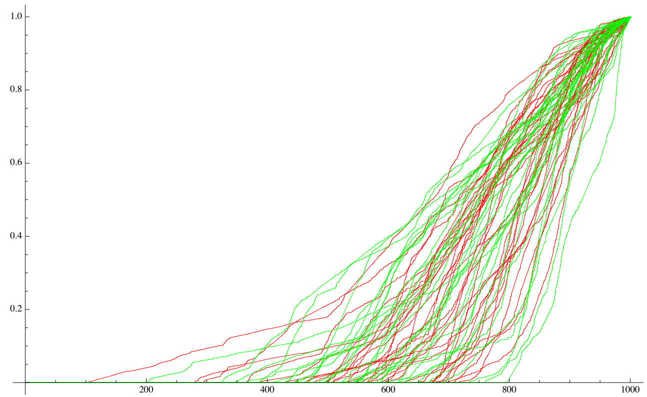


Melanoma

Mole

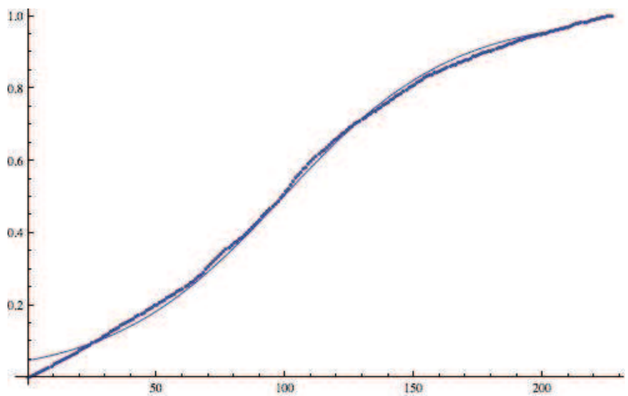
$\Longrightarrow$ A. Rodriguez, J. Stangl, C. Shakiban

Cumulative Global Histograms

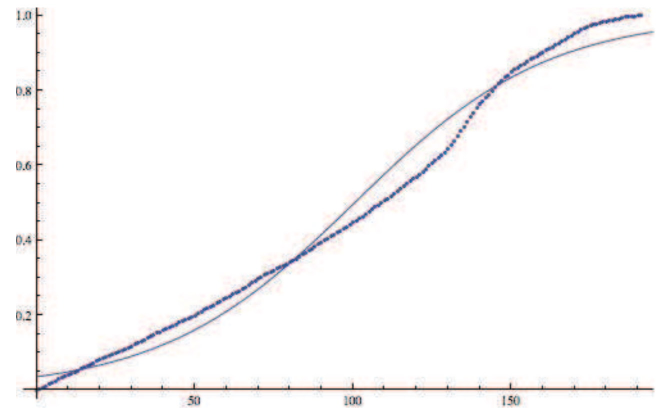


Red: melanoma Green: mole

Logistic Function Fitting

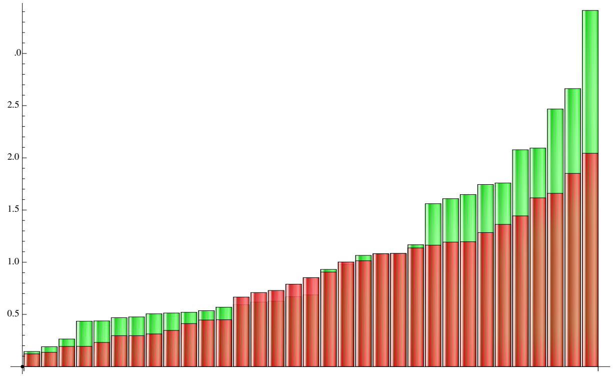


Melanoma



Mole

Logistic Function Fitting — Residuals



$$\text{Melanoma} = 17.1336 \pm 1.02253$$

$$\text{Mole} = 19.5819 \pm 1.42892$$

58.7% Confidence

Curve Histogram Conjecture

Two sufficiently regular plane curves C and $\widetilde{C}$ have identical global distance histogram functions, so $H_C(r) = H_{\widetilde{C}}(r)$ for all $r \geq 0$, if and only if they are rigidly equivalent: $C \simeq \widetilde{C}$.