Emmy Noether and Symmetry

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Amalie Emmy Noether
(1882–1935)

Noether's Fundamental Contributions to Analysis and Physics

First Theorem. There is a one-to-one correspondence between symmetry groups of a variational problem and conservation laws of its Euler–Lagrange equations.

Second Theorem. An infinite-dimensional variational symmetry group depending upon an arbitrary function corresponds to a nontrivial differential relation among its Euler–Lagrange equations.

★ The conservation laws associated with the variational symmetries in the Second Theorem are trivial — this resolved Hilbert's original paradox in relativity that was the reason he and Klein invited Noether to Göttingen.

Noether's Three Fundamental Contributions to Analysis and Physics

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Second Theorem. An infinite-dimensional variational symmetry group depending upon an arbitrary function corresponds to a nontrivial differential relation among its Euler–Lagrange equations.

Introduction of higher order generalized symmetries.

$\Longrightarrow$ later (1960's) to play a fundamental role in the discovery and classification of integrable systems and solitons.

Symmetries $\Longrightarrow$ Conservation Laws

- symmetry under space translations
  $\Longrightarrow$ conservation of linear momentum
- symmetry under time translations
  $\Longrightarrow$ conservation of energy
- symmetry under rotations
  $\Longrightarrow$ conservation of angular momentum
- symmetry under boosts (moving coordinates)
  $\Longrightarrow$ linear motion of the center of mass

Precursors

Lagrange (1788) Lagrangian mechanics & conservation laws

Jacobi (1842–43 publ. 1866) Euclidean invariance
— linear and angular momentum

Schütz (1897) time translation — conservation of energy

Herglotz (1911) Poincaré invariance in relativity
— 10 conservation laws

Engel (1916) non-relativistic limit: Galilean invariance
— linear motion of center of mass

A Curious History

- ⋆ Bessel–Hagen (1922) — divergence symmetries
- ♣ Hill (1951) — a very special case
  (first order Lagrangians, geometrical symmetries)
- ♠ 1951–1980 Over 50 papers rediscover and/or prove
  purported generalizations of Noether's First Theorem
- ♠ 2011 Neuenschwander, *Emmy Noether's Wonderful Theorem*
  — back to special cases again!

Continuum mechanics: Rice, Eshelby (1950's),
Günther (1962), Knowles & Sternberg (1972)

Optics: Baker & Tavel (1974)

The Noether Triumvirate

- ⋆ Variational Principle
- ⋆ Symmetry
- ⋆ Conservation Law

A Brief History of Symmetry

Symmetry $\Longrightarrow$ Group Theory!

- Lagrange, Abel, Galois (discrete)

  — polynomials

- Lie (continuous)

  — differential equations and variational principles

- Noether (generalized)

  — conservation laws and higher order symmetries

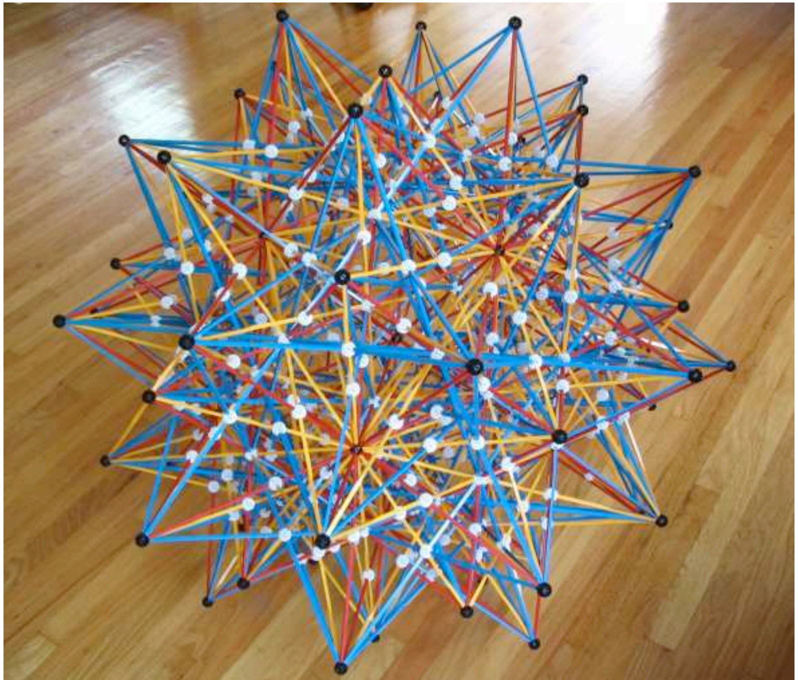
- Weyl, Wigner, etc. — quantum mechanics

  "der Gruppenpest" (J. Slater)

Next to the concept of a function, which is the most important concept pervading the whole of mathematics, the concept of a group is of the greatest significance in the various branches of mathematics and its applications.

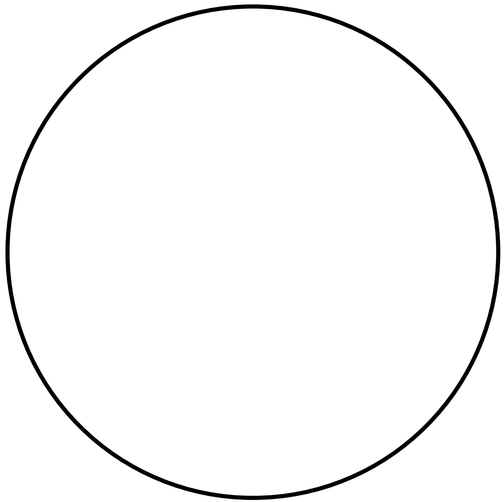
— P.S. Alexandroff

Discrete Symmetry Group



$\Longrightarrow$ crystallography

Continuous Symmetry Group



Symmetry group = *all* rotations

⋆ A continuous group is known as a Lie group
— in honor of Sophus Lie (1842–1899)

A Brief History of Conservation Laws

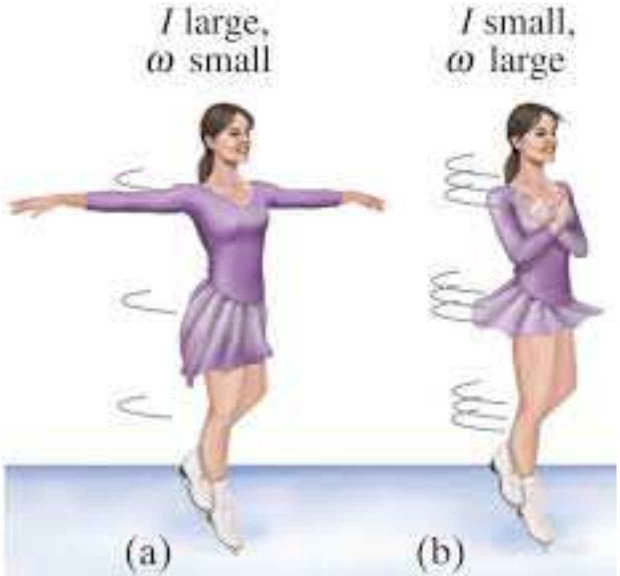
In physics, a conservation law asserts that a particular measurable property P of an isolated physical system does not change as the system evolves.

Conservation of momentum: Wallis (1670), Newton (1687)

Conservation of mass: Lomonosov (1748), Lavoisier (1774)

Conservation of energy: Lagrange (1788), Helmholtz (1847), Rankine (1850), also: Mohr, Mayer, Joule, Faraday, . . .

Conservation Laws



In Summary . . .

Noether's (First) Theorem states that to each continuous symmetry group of the action functional there is a corresponding conservation law of the physical equations and (via generalized symmetries) vice versa.

Noether and Symmetry

Noether realized that, to make a one-to-one correspondence between symmetries of variational problems and conservation laws of their corresponding field equations, one needs to generalize Lie's concept of continuous symmetry group to include higher order generalized symmetries.

This remarkable innovation was completely ignored (or not understood) until the 1960's when generalized symmetries were rediscovered and subsequently seen to play a fundamental role in the modern theory of integrable systems.

Noether and Symmetry

Note: Some modern authors ascribe the discovery of generalized symmetries to Lie and Bäcklund, and then give them the misnomer "Lie–Bäcklund transformations".

However, a careful study of their work has failed to find anything beyond contact transformations, which are an extremely special case.

Integrable Systems

The second half of the twentieth century saw two revolutionary discoveries in the field of nonlinear systems:

- ⋆ chaos
- ⋆ integrability

Both have their origins in the classical mechanics of the nineteenth century:

chaos:	Poincaré
integrability:	Hamilton, Jacobi, Liouville, Kovalevskaya

Sofia Vasilyevna Kovalevskaya (1850–1891)

⋆⋆ Doctorate in mathematics, summa cum laude,
1874, University of Göttingen

Integrable Systems

In the 1960's, the discovery of the soliton in Kruskal and Zabusky's numerical studies of the Korteweg–deVries equation, a model for nonlinear water waves, which was motivated by the Fermi–Pasta–Ulam problem, provoked a revolution in the study of nonlinear dynamics.

The theoretical justification of their observations came through the study of the associated symmetries and conservation laws.

Indeed, integrable systems like the Korteweg–deVries equation, nonlinear Schrödinger equation, sine-Gordon equation, KP equation, etc. are characterized by their admitting an infinite number of higher order symmetries – as first defined by Noether — and, through Noether's theorem, higher order conservation laws!

And, the most powerful means of classifying integrable systems is through higher order generalized symmetry analysis.

The Calculus of Variations

[Leibniz] conceives God in the creation of the world like a mathematician who is solving a minimum problem, or rather, in our modern phraseology, a problem in the calculus of variations — the question being to determine among an infinite number of possible worlds, that for which the sum of necessary evil is a minimum.

— Paul du Bois-Reymond

The Calculus of Variations

Nature is Leibnizian (Panglossian):

⋆ A physical system in equilibrium chooses "the best of all possible worlds" by minimizing some overall cost: energy or force or time or money or . . .

Principle of least action:

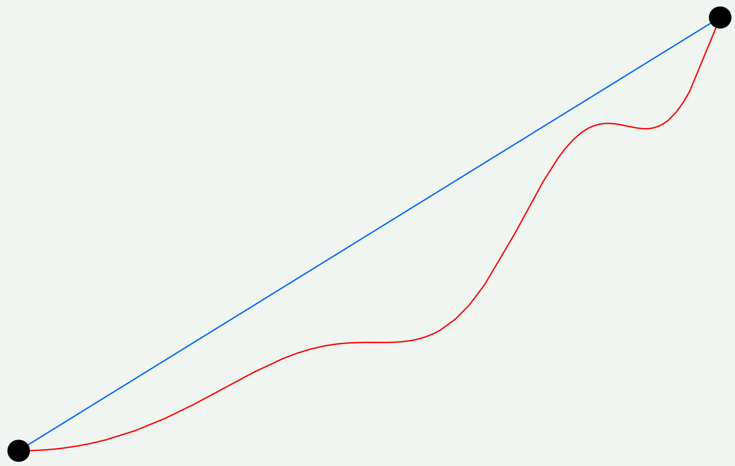
"Nature is thrifty in all its actions."

$\Longrightarrow$ Pierre Louis Maupertuis

⋆ Analysis developed by Johann & Jakob Bernoullis, Euler, Lagrange, Hamilton, Jacobi, Weierstrass, Dirichlet, Hilbert, . . .

Examples of Variational Problems:

The shortest path between two points in space is a straight line.



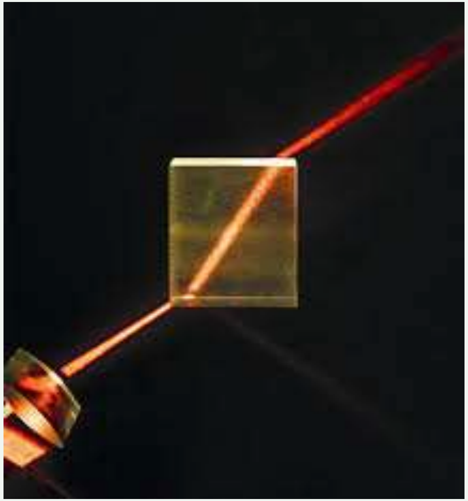
Geodesics

The shortest path between two points on a sphere is a great circular arc.

The shortest path between two points on a curved surface is a geodesic arc.

Fermat's Principle in Optics

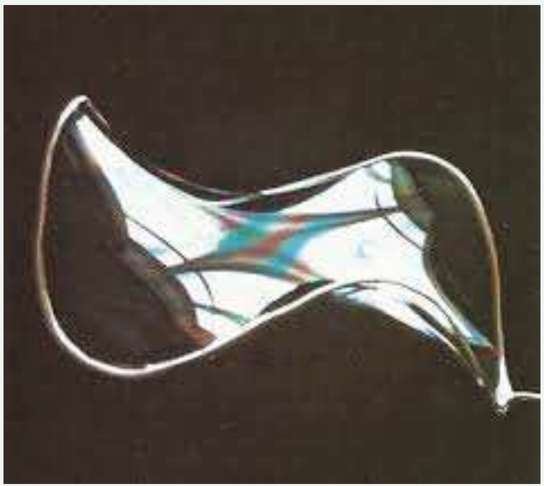
Light travels along the path that takes the least time:



$\Longrightarrow$ Snell's Law = Loi de Descartes

Plateau’s Problem

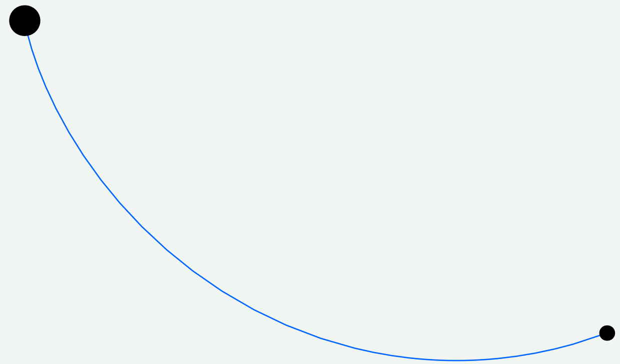
The surface of least area spanning a space curve
is a minimal surface.



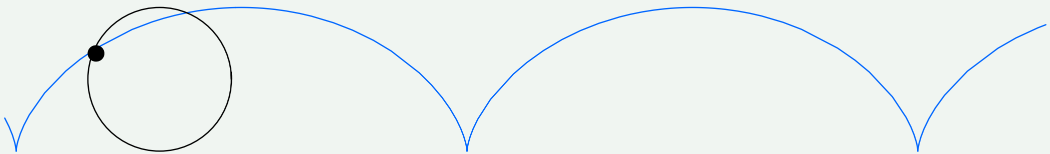
$\Longrightarrow$ soap films

The Brachistochrone

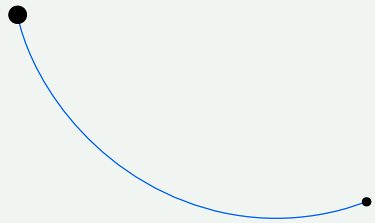
A bead slides down a wire fastest



when it has the shape of a cycloid



The Brachistochrone Cycloid



Galileo (1638) says it is a circular arc

Tautochrone problem — solved by Huygens (1659)

★ produces great increase in accuracy of time-keeping, leading to the solution to the Problem of the Longitude

Johann Bernoulli's contest (1696)

$\Longrightarrow$ Correct entries by Newton, Leibniz, Jakob Bernoulli, l'Hôpital, Tschirnhaus

Thus began the calculus of variations!

Variational Problems

A variational problem requires minimizing a functional

$$F[u] = \int L(x, u^{(n)})\, dx$$

The integrand is known as the Lagrangian.

The Lagrangian $L(x, u^{(n)})$ can depend upon the space/time coordinates x, the function(s) or field(s) $u = f(x)$ and their derivatives up to some order n

The Euler–Lagrange Equations

The minima of the functional

$$F[u] = \int L(x, u^{(n)})\, dx$$

must occur where the functional gradient vanishes: $\delta F[u] = 0$

This is a system of differential equations $\qquad \Delta = E(L) = 0$

known as the Euler–Lagrange equations.

The (smooth) minimizers $u(x)$ of the functional are solutions to the Euler–Lagrange equations — as are any maximizers and, in general, all "critical functions".

E — Euler operator (variational derivative):

$$E^\alpha(L) = \frac{\delta L}{\delta u^\alpha} = \sum_J (-D)^J \frac{\partial L}{\partial u^\alpha_J} = 0$$

$\Longrightarrow$ integration by parts

$$F[u] = \text{arc length functional} = \int ds$$

Euler–Lagrange equation: $\quad \text{curvature} = \kappa = 0$

Solutions: geodesics

$$F[u] = \text{surface area functional} = \iint \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}\; dx\, dy$$

Euler–Lagrange equation =

minimal surface equation ($\mathbb{R}^3$ version):

$$(1 + u_y^2)\, u_{xx} - 2\, u_x\, u_y\, u_{xy} + (1 + u_x^2)\, u_{yy} = 0$$

Solutions: minimal surfaces

$$F[u] = \text{Hilbert action functional} = \frac{c^4}{16\pi G} \int (R + L_m) \sqrt{-g}\, d^4x$$

Euler–Lagrange equations =

Einstein equations of general relativity:

$$R_{\mu\nu} = \frac{1}{2} R\, g_{\mu\nu} + \frac{8\pi G}{c^4} T_{\mu\nu}$$

Solutions: Einstein space–time manifolds

The Modern Manual for Physics

as envisioned by E. Noether (1918)

As Hilbert expresses his assertion, the lack of a proper law of energy constitutes a characteristic of the "general theory of relativity." For that assertion to be literally valid, it is necessary to understand the term "general relativity" in a wider sense than is usual, and to extend it to the aforementioned groups that depend on n arbitrary functions.[27]

[27] This confirms once more the accuracy of Klein's remark that the term "relativity" as it is used in physics should be replaced by "invariance with respect to a group."

The Modern Manual for Physics

♠ To construct a physical theory:

Step 1: Determine the allowed group of symmetries:

- translations
- rotations
- conformal (angle-preserving) transformations
- Galilean boosts
- Poincaré transformations: $\mathrm{SO}(4,2)$ (special relativity)
- gauge transformations
- CPT (charge, parity, time reversal) symmetry
- supersymmetry
- $\mathrm{SU}(3)$, G_2, $\mathrm{E}_8 \times \mathrm{E}_8$, $\mathrm{SO}(32)$, $\ldots$
- etc., etc.

Step 2: Construct a variational principle ("energy") that admits the given symmetry group.

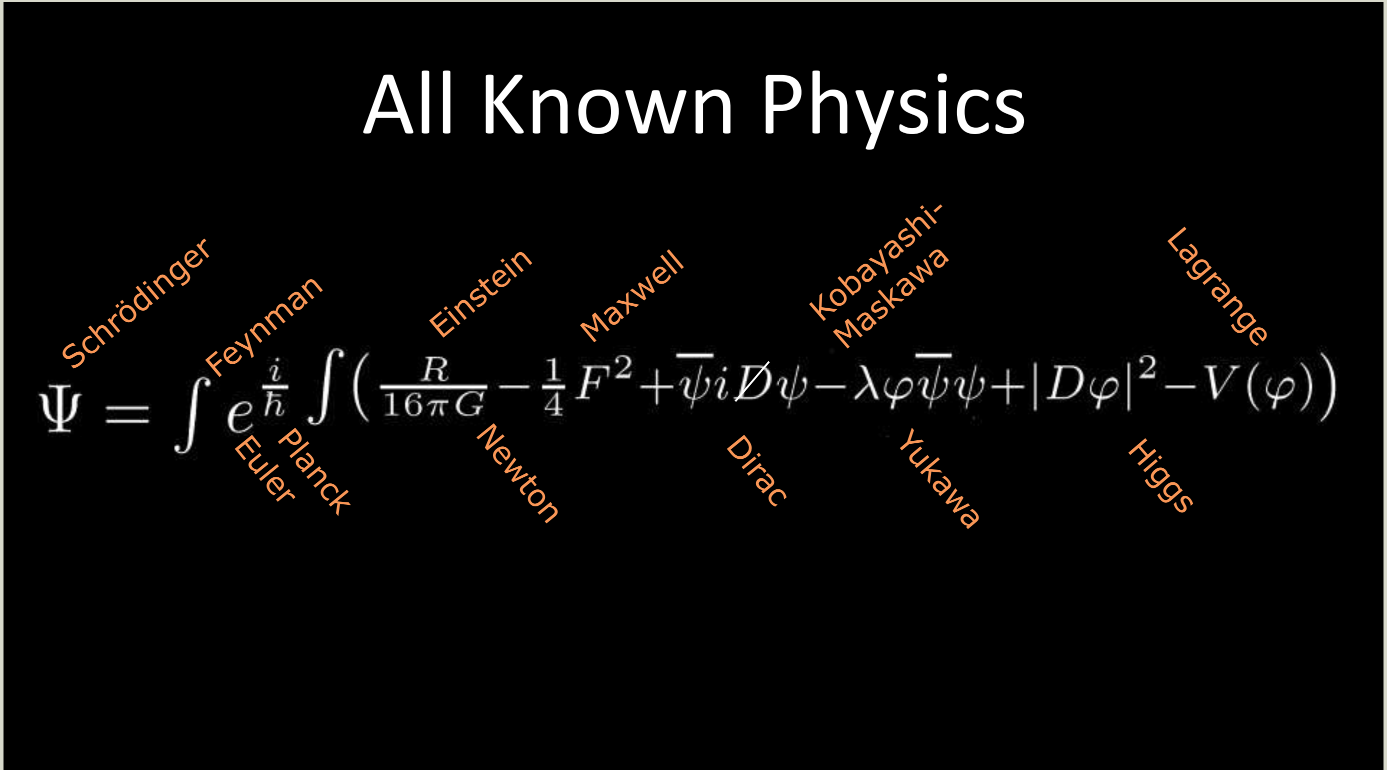
Step 3: Invoke Nature's obsession with minimization to construct the corresponding field equations (Euler–Lagrange equations) associated with the variational principle.

Step 4: Use Noether's First and Second Theorems to write down (*a*) conservation laws, and (*b*) differential identities satisfied by the field equations.

Step 5: Try to solve the field equations.

Even special solutions are of immense interest

$\Longrightarrow$ black holes.



$\Longrightarrow$ Neil Turok (Perimeter Institute)

Characterization of Invariant Variational Problems

According to Lie, any G–invariant variational problem can be written in terms of the differential invariants:

$$\boxed{\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ )\ \boldsymbol{\omega}}$$

$I^1, \ldots, I^\ell$ — fundamental differential invariants

$\mathcal{D}_1, \ldots, \mathcal{D}_p$ — invariant differential operators

$\mathcal{D}_K I^\alpha$ — differentiated invariants

$\boldsymbol{\omega} = \omega^1 \wedge \cdots \wedge \omega^p$ — invariant volume form

If the variational problem is G-invariant, so

$$\mathcal{I}[u] = \int P(\ \ldots\ \mathcal{D}_K I^\alpha\ \ldots\ )\ \boldsymbol{\omega}$$

then its Euler–Lagrange equations admit G as a symmetry group, and hence can also be expressed in terms of the differential invariants:

$$E(L) \simeq F(\ \ldots\ \mathcal{D}_K I^\alpha\ \ldots\ ) = 0$$

Problem: Construct F directly from P.

$\Longrightarrow$ Solved in general by Irina Kogan & PJO (2001) using moving frames

A Physical Conundrum

Since all Lie groups and most Lie pseudo-groups admit infinitely many differential invariants, there are an infinite number of distinct invariant variational principles

$$\boxed{\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots\ \mathcal{D}_K I^\alpha\ \ldots\ )\ \boldsymbol{\omega}}$$

$\star\star$ Physicists are very talented finding the "simplest" such invariant variational principle, even for very complicated physical symmetry groups, which then forms the basis of the consequential physics.

A Physical Conundrum

$$\boxed{\mathcal{I}[u] = \int L(x, u^{(n)})\, d\mathbf{x} = \int P(\ \ldots\ \mathcal{D}_K I^\alpha\ \ldots\ )\ \boldsymbol{\omega}}$$

On the other hand, physicists seem to be mostly unaware of the theory of differential invariants and the consequent existence of infinitely many alternative invariant variational principles, hence:

Does the underlying physics depend upon which of these invariant variational principles is used and, if so, how does one select the "correct" physical variational principle?

Symmetry Groups of Differential Equations

$\Longrightarrow$ Sophus Lie (1842–1899)

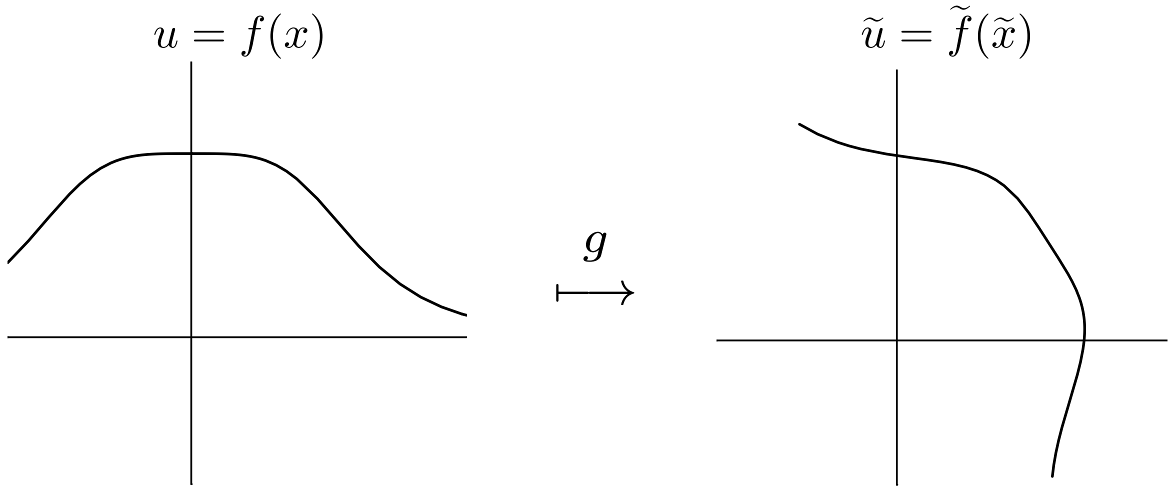
System of differential equations

$$\Delta(x, u^{(n)}) = 0$$

G — Lie group or Lie pseudo-group acting on the space of independent and dependent variables:

$$(\widetilde{x}, \widetilde{u}) = g \cdot (x, u)$$

G acts on functions by transforming their graphs:



Definition. G is a symmetry group of the system $\Delta = 0$ if $\widetilde{f} = g \cdot f$ is a solution whenever f is.

One–Parameter Groups

A Lie group whose transformations depend upon a single parameter $\varepsilon \in \mathbb{R}$ is called a one-parameter group.

Translations in a single direction:

$$(x, y, z) \quad \longmapsto \quad (x + \varepsilon, y + 2\varepsilon, z - \varepsilon)$$

Rotations around a fixed axis:

$$(x, y, z) \quad \longmapsto \quad (x \cos \varepsilon - z \sin \varepsilon, y, x \sin \varepsilon + z \cos \varepsilon)$$

Screw motions:

$$(x, y, z) \quad \longmapsto \quad (x \cos \varepsilon - y \sin \varepsilon, x \sin \varepsilon + y \cos \varepsilon, z + \varepsilon)$$

Scaling transformations:

$$(x, y, z) \quad \longmapsto \quad (\lambda x, \lambda y, \lambda^{-1} z)$$

Infinitesimal Generators

Every one-parameter group can be viewed as the flow of a vector field $\mathbf{v}$, known as its infinitesimal generator.

In other words, the one-parameter group is realized as the solution to the system of ordinary differential equations governing the vector field's flow:

$$\frac{dz}{d\varepsilon} = \mathbf{v}(z)$$

Equivalently, if one expands the group transformations in powers of the group parameter ε, the infinitesimal generator comes from the linear terms:

$$z(\varepsilon) = z + \varepsilon\,\mathbf{v}(z) + \cdots$$

Infinitesimal Generators = Vector Fields

In differential geometry, it has proven to be very useful to identify a vector field with a first order differential operator

In local coordinates $(\ldots x^i \ldots u^\alpha \ldots)$, the vector field

$$\mathbf{v} = (\ \ldots\ \xi^i(x,u)\ \ldots\ \varphi^\alpha(x,u)\ \ldots\ )$$

that generates the one-parameter group (flow)

$$\frac{dx^i}{d\varepsilon} = \xi^i(x,u) \qquad \frac{du^\alpha}{d\varepsilon} = \varphi^\alpha(x,u)$$

is identified with the differential operator

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

Invariance

A function $F : M \to \mathbb{R}$ is invariant if it is not affected by the group transformations:

$$F(g \cdot z) = F(z)$$

for all $g \in G$ and $z \in M$.

Infinitesimal Invariance

Theorem. (Lie) A function is invariant under a one-parameter group with infinitesimal generator $\mathbf{v}$ (viewed as a differential operator) if and only if

$$\mathbf{v}(F) = 0$$

Prolongation

Since G acts on functions, it acts on their derivatives $u^{(n)}$,
leading to the prolonged group action:

$$(\widetilde{x}, \widetilde{u}^{(n)}) = \mathrm{pr}^{(n)}\, g \cdot (x, u^{(n)})$$

$\Longrightarrow$ formulas provided by implicit differentiation

Prolonged infinitesimal generator:

$$\mathrm{pr}\, \mathbf{v} = \mathbf{v} + \sum_{\alpha, J} \varphi_J^\alpha(x, u^{(n)})\, \frac{\partial}{\partial u_J^\alpha}$$

The Prolongation Formula

The coefficients of the prolonged vector field are given by the explicit prolongation formula:

$$\boxed{\varphi_J^\alpha = D_J\, Q^\alpha + \sum_{i=1}^{p} \xi^i\, u_{J,i}^\alpha}$$

where $$Q^\alpha(x, u^{(1)}) = \varphi^\alpha - \sum_{i=1}^{p} \xi^i \frac{\partial u^\alpha}{\partial x^i}$$

$Q = (Q^1, \ldots, Q^q)$ — characteristic of $\mathbf{v}$

⋆ Invariant functions are solutions to

$$Q(x, u^{(1)}) = 0.$$

Example. The vector field

$$\mathbf{v} = \xi \frac{\partial}{\partial x} + \varphi \frac{\partial}{\partial u} = -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u}$$

generates the rotation group

$$(x, u) \quad \longmapsto \quad (x \cos \varepsilon - u \sin \varepsilon, x \sin \varepsilon + u \cos \varepsilon)$$

The prolonged action is (implicit differentiation)

$$\begin{aligned}
u_x &\longmapsto \frac{\sin \varepsilon + u_x \cos \varepsilon}{\cos \varepsilon - u_x \sin \varepsilon} \\
u_{xx} &\longmapsto \frac{u_{xx}}{(\cos \varepsilon - u_x \sin \varepsilon)^3} \\
u_{xxx} &\longmapsto \frac{(\cos \varepsilon - u_x \sin \varepsilon)\, u_{xxx} - 3\, u_{xx}^2 \sin \varepsilon}{(\cos \varepsilon - u_x \sin \varepsilon)^5} \\
&\vdots
\end{aligned}$$

$$\mathbf{v} = \xi \frac{\partial}{\partial x} + \varphi \frac{\partial}{\partial u} = -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u}$$

Characteristic:

$$Q(x, u, u_x) = \varphi - u_x \xi = x + u\, u_x$$

By the prolongation formula, the infinitesimal generator is

$$\operatorname{pr} \mathbf{v} = -u \frac{\partial}{\partial x} + x \frac{\partial}{\partial u} + (1 + u_x^2) \frac{\partial}{\partial u_x} + 3 u_x u_{xx} \frac{\partial}{\partial u_{xx}} + \cdots$$

★ The solutions to the characteristic equation

$$Q(x, u, u_x) = x + u\, u_x = 0$$

are circular arcs — rotationally invariant curves.

Lie's Infinitesimal Symmetry Criterion for Differential Equations

Theorem. A connected group of transformations G is a symmetry group of a nondegenerate system of differential equations $\Delta = 0$ if and only if

$$\boxed{\operatorname{pr} \mathbf{v}(\Delta) = 0 \qquad \text{whenever} \qquad \Delta = 0}$$

for every infinitesimal generator $\mathbf{v}$ of G.

Calculation of Symmetries

$$\boxed{\operatorname{pr}\mathbf{v}(\Delta) = 0 \qquad \text{whenever} \qquad \Delta = 0}$$

These are the determining equations of the symmetry group to $\Delta = 0$. They form an overdetermined system of elementary partial differential equations for the coefficients ξ^i, φ^α of $\mathbf{v}$ that can (usually) be explicitly solved — there are even Maple and Mathematica packages that do this automatically — thereby producing the most general infinitesimal symmetry and hence the (continuous) symmetry group of the system of partial differential equations.

★ For systems arising in applications, many symmetries are evident from physical intuition, but there are significant examples where the Lie method produces new symmetries.

Variational Symmetries

Definition. A (strict) variational symmetry is a transformation $(\widetilde{x}, \widetilde{u}) = g \cdot (x, u)$ which leaves the variational problem invariant:

$$\int_{\widetilde{\Omega}} L(\widetilde{x}, \widetilde{u}^{(n)})\, d\widetilde{x} = \int_{\Omega} L(x, u^{(n)})\, dx$$

Infinitesimal invariance criterion:

$$\operatorname{pr} \mathbf{v}(L) + L \operatorname{Div} \xi = 0$$

Divergence symmetry (Bessel–Hagen):

$$\operatorname{pr} \mathbf{v}(L) + L \operatorname{Div} \xi = \operatorname{Div} B$$

$\Longrightarrow$ Every divergence symmetry has an equivalent strict variational symmetry

Conservation Laws

A conservation law of a discrete dynamical system of ordinary differential equations is a function

$$T(t, u, u_t, \dots)$$

depending on the time t, the field variables u, and their derivatives, that is constant on solutions, or, equivalently,

$$D_t T = 0$$

on all solutions to the field equations.

Conservation Laws — Dynamics

In continua, a conservation law states that the temporal rate of change of a quantity T in a region of space D is governed by the associated flux through its boundary:

$$\frac{\partial}{\partial t} \int_D T \, dx = \oint_{\partial D} X$$

or, in differential form,

$$\mathrm{D}_t T = \operatorname{Div} X$$

- In particular, if the flux X vanishes on the boundary ∂D, then the total density $\int_D T \, dx$ is conserved — constant.

Conservation Laws — Statics

In statics, a conservation law corresponds to a path- or surface-independent integral $\oint_C X = 0$ — in differential form,

$$\operatorname{Div} X = 0$$

Thus, in fracture mechanics, one can measure the conserved quantity near the tip of a crack by evaluating the integral at a safe distance.

Conservation Laws in Analysis

- ⋆ In modern mathematical analysis of partial differential equations, most existence theorems, stability results, scattering theory, etc., rely on the existence of suitable conservation laws.

- ⋆ Completely integrable systems can be characterized by the existence of infinitely many higher order conservation laws.

- ⋆ In the absence of symmetry, Noether's Identity is used to construct divergence identities that take the place of conservation laws in analysis.

Trivial Conservation Laws

Let $\Delta = 0$ be a system of differential equations.

Type I If $P = 0$ for all solutions to $\Delta = 0$,
then $\operatorname{Div} P = 0$ on solutions

Type II (Null divergences) If $\operatorname{Div} P \equiv 0$ for *all* functions $u = f(x)$, then it trivially vanishes on solutions.

Examples:

$$D_x(u_y) + D_y(-u_x) \equiv 0$$

$$D_x \frac{\partial(u,v)}{\partial(y,z)} + D_y \frac{\partial(u,v)}{\partial(z,x)} + D_z \frac{\partial(u,v)}{\partial(x,y)} \equiv 0$$

$\Longrightarrow$ (generalized) curl: $P = \operatorname{Curl} Q$

Two conservation laws P and $\widetilde{P}$ are equivalent if they differ by a sum of trivial conservation laws:

$$P = \widetilde{P} + P_I + P_{II}$$

where

$$P_I = 0 \quad \text{on solutions} \qquad \operatorname{Div} P_{II} \equiv 0.$$

Theorem. Every conservation law of a (nondegenerate) system of differential equations $\Delta = 0$ is equivalent to one in characteristic form

$$\operatorname{Div} P = Q\,\Delta$$

Proof: — integration by parts

$\Longrightarrow \quad Q = (Q_1, \ldots, Q_q)$ is called the characteristic of the conservation law.

Noether's First Theorem

Theorem. If $\mathbf{v}$ generates a one-parameter group of variational symmetries of a variational problem, then the characteristic Q of $\mathbf{v}$ is the characteristic of a conservation law of the Euler-Lagrange equations:

$$\operatorname{Div} P = Q\, E(L)$$

Proof: Noether's Identity = Integration by Parts

$$\boxed{\operatorname{pr} \mathbf{v}(L) + L \operatorname{Div} \xi = Q\, E(L) - \operatorname{Div} P}$$

$\operatorname{pr} \mathbf{v}$ — prolonged vector field (infinitesimal generator)

Q — characteristic of $\mathbf{v}$

P — boundary terms resulting from the integration by parts computation

Symmetry $\implies$ Conservation Law

$$\boxed{\mathrm{pr}\,\mathbf{v}(L) + L\,\mathrm{Div}\,\xi = Q\,E(L) - \mathrm{Div}\,P}$$

Thus, if $\mathbf{v}$ is a variational symmetry, then by infinitesimal invariance of the variational principle, the left hand side of Noether's Identity vanishes and hence

$$\mathrm{Div}\,P = Q\,E(L)$$

is a conservation law with characteristic Q.

More generally, if $\mathbf{v}$ is a divergence symmetry

$$\mathrm{pr}\,\mathbf{v}(L) + L\,\mathrm{Div}\,\xi = \mathrm{Div}\,B$$

then the conservation law is

$$\mathrm{Div}(P + B) = Q\,E(L)$$

Conservation of Energy

Group:

$$(t, u) \quad \longmapsto \quad (t + \varepsilon, u)$$

Infinitesimal generator and characteristic:

$$\mathbf{v} = \frac{\partial}{\partial t} \qquad\qquad Q = -\,u_t$$

Invariant variational problem

$$F[u] = \int L(u, u_t, u_{tt}, \ldots)\, dt \qquad\qquad \frac{\partial L}{\partial t} = 0$$

Euler–Lagrange equations:

$$E(L) = \frac{\partial L}{\partial u} - D_t \frac{\partial L}{\partial u_t} + D_t^2 \frac{\partial L}{\partial u_{tt}} - \ \cdots \ = 0$$

Conservation of Energy

Infinitesimal generator and characteristic:

$$\mathbf{v} = \frac{\partial}{\partial t} \qquad\qquad Q = -\,u_t$$

Euler–Lagrange equations:

$$E(L) = \frac{\partial L}{\partial u} - D_t \frac{\partial L}{\partial u_t} + D_t^2 \frac{\partial L}{\partial u_{tt}} - \;\cdots\; = 0$$

Conservation law:

$$0 = Q\,E(L) = -\,u_t \left(\frac{\partial L}{\partial u} - D_t \frac{\partial L}{\partial u_t} + D_t^2 \frac{\partial L}{\partial u_{tt}} - \;\cdots\; \right)$$

$$= D_t \left(-\,L + u_t \frac{\partial L}{\partial u_t} - \;\cdots\; \right)$$

Conservation Law $\Longrightarrow$ Symmetry

$$\boxed{\operatorname{pr}\mathbf{v}(L) + L\,\mathrm{Div}\,\xi = Q\,E(L) - \mathrm{Div}\,P}$$

Conversely, if

$$\mathrm{Div}\,A = Q\,E(L)$$

is any conservation law, assumed, without loss of generality, to be in characteristic form, and Q is the characteristic of the vector field $\mathbf{v}$, then

$$\operatorname{pr}\mathbf{v}(L) + L\,\mathrm{Div}\,\xi = \mathrm{Div}(A - P) = \mathrm{Div}\,B$$

and hence $\mathbf{v}$ generates a divergence symmetry group.

What's the catch?

How do we know the characteristic Q of the conservation law is the characteristic of a vector field $\mathbf{v}$?

Answer: it's *not* if we restrict our attention to ordinary, geometrical symmetries, but it is if we allow the vector field $\mathbf{v}$ to depend on derivatives of the field variable!

★ One needs higher order generalized symmetries

— first defined by Noether!

Generalized Symmetries of Differential Equations

Determining equations :

$$\text{pr}\,\mathbf{v}(\Delta) = 0 \qquad \text{whenever} \qquad \Delta = 0$$

A generalized symmetry is trivial if its characteristic vanishes on solutions to Δ. This means that the corresponding group transformations acts trivially on solutions.

Two symmetries are equivalent if their characteristics differ by a trivial symmetry.

The Kepler Problem

$$\ddot{x} + \frac{m\,x}{r^3} = 0 \qquad\qquad L = \tfrac{1}{2}\,\dot{x}^2 - \frac{m}{r}$$

Generalized symmetries (three-dimensional):

$$\mathbf{v} = (x \cdot \ddot{x})\partial_x + \dot{x}(x \cdot \partial_x) - 2\,x(\dot{x} \cdot \partial_x)$$

Conservation laws

$$\mathrm{pr}\ \mathbf{v}(L) = D_t R$$

where

$$R = \dot{x} \wedge (x \wedge \dot{x}) - \frac{m\,x}{r}$$

are the components of the Runge-Lenz vector

$\Longrightarrow$ Super-integrability

The Strong Version

Noether's First Theorem. Let $\Delta = 0$ be a normal system of Euler-Lagrange equations. Then there is a one-to-one correspondence between nontrivial conservation laws and nontrivial variational symmetries.

⋆ A system of partial differential equations is normal if, under a change of variables, it can be written in Cauchy–Kovalevskaya form.

⋆ Abnormal systems are either over- or under-determined.

Example: Einstein's field equations in general relativity.

$\Longrightarrow$ Noether's Second Theorem and the Bianchi identities

Generalized Symmetries

★ Due to Noether (1918)

★ *NOT* Lie or Bäcklund, who only got as far as contact transformations.

Key Idea: Allow the coefficients of the infinitesimal generator to depend on derivatives of u, but drop the requirement that the (prolonged) vector field define a geometrical transformation on any finite order jet space:

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u^{(k)}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x, u^{(k)}) \frac{\partial}{\partial u^\alpha}$$

Characteristic :

$$Q_\alpha(x, u^{(k)}) = \varphi^\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha$$

Evolutionary vector field:

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} Q_\alpha(x, u^{(k)}) \frac{\partial}{\partial u^\alpha}$$

Prolongation formula:

$$\mathrm{pr}\ \mathbf{v} = \mathrm{pr}\ \mathbf{v}_Q + \sum_{i=1}^{p} \xi^i D_i$$

$$\mathrm{pr}\ \mathbf{v}_Q = \sum_{\alpha, J} D_J Q_\alpha \frac{\partial}{\partial u_J^\alpha}$$

$$D_i = \sum_{\alpha, J} u_{J,i}^\alpha \frac{\partial}{\partial u_J^\alpha} \qquad D_J = D_{j_1} \cdots D_{j_k}$$

$\Longrightarrow$ total derivative

Generalized Flows

- The one-parameter group generated by the evolutionary vector field

$$\mathbf{v}_Q = \sum_{\alpha=1}^{q} Q_\alpha(x, u^{(k)}) \frac{\partial}{\partial u^\alpha}$$

is found by solving the Cauchy problem for an associated system of evolution equations

$$\frac{\partial u^\alpha}{\partial \varepsilon} = Q_\alpha(x, u^{(n)}) \qquad u|_{\varepsilon=0} = f(x)$$

♠ Existence/uniqueness?

♣ Ill–posedness?

Example. $\mathbf{v} = \dfrac{\partial}{\partial x}$ generates the one-parameter group of translations:

$$(x, y, u) \quad \longmapsto \quad (x + \varepsilon, y, u)$$

Evolutionary form:

$$\mathbf{v}_Q = -u_x \frac{\partial}{\partial x}$$

Corresponding group:

$$\frac{\partial u}{\partial \varepsilon} = -u_x$$

Solution

$$u = f(x, y) \quad \longmapsto \quad u = f(x - \varepsilon, y)$$

★ $\mathbf{v}$ is a generalized symmetry of a differential equation if and only if its evolutionary form $\mathbf{v}_Q$ is.

Example. Burgers' equation.

$$u_t = u_{xx} + u u_x$$

Characteristics of generalized symmetries:

u_x	space translations
$u_{xx} + u\,u_x$	time translations

$$u_{xxx} + \tfrac{3}{2}\,u\,u_{xx} + \tfrac{3}{2}\,u_x^2 + \tfrac{3}{4}\,u^2 u_x$$

$$u_{xxxx} + 2\,u\,u_{xxx} + 5\,u_x u_{xx} + \tfrac{3}{2}\,u^2 u_x + 3\,u\,u_x^2 + \tfrac{1}{2}\,u^3 u_x$$

$$\vdots$$

Equations with one higher order symmetry almost always have infinitely many.

Equations with higher order generalized symmetries are called "integrable".

Linearizable "C integrable" — Burgers'

Solvable by inverse scattering "S integrable" — KdV

See Mikhailov–Shabat and J.P. Wang's thesis for long lists of equations with higher order symmetries.

Sanders–Wang use number theoretic techniques to completely classify all integrable evolution equations, of a prescribed type, e.g. polynomial with linear leading term in which every example is in a small collection of well-known equations of order ≤ 5 (linear, Burgers', KdV, mKdV, Sawada–Kotera, Kaup–Kupershmidt, Kupershmidt, Ibragimov–Shabat & potential versions) or a higher order symmetry thereof.

Bakirov's Example:

The "triangular system" of evolution equations

$$u_t = u_{xxxx} + v^2 \qquad v_t = \tfrac{1}{5}\, v_{xxxx}$$

has one sixth order generalized symmetry,
but no further higher order symmetries.

- Bakirov (1991)
- Beukers–Sanders–Wang (1998)
- van der Kamp–Sanders (2002)

Recursion operators

$\Longrightarrow$ Olver (1977)

Definition. An operator $\mathcal{R}$ is called a recursion operator for the system $\Delta = 0$ if it maps symmetries to symmetries, i.e., if $\mathbf{v}_Q$ is a generalized symmetry (in evolutionary form), and $\widetilde{Q} = \mathcal{R}Q$, then $\mathbf{v}_{\widetilde{Q}}$ is also a generalized symmetry.

$\Longrightarrow$ A recursion operator generates infinitely many symmetries with characteristics

$$Q, \quad \mathcal{R}Q, \quad \mathcal{R}^2 Q, \quad \mathcal{R}^3 Q, \quad \ldots$$

Theorem. Given the system $\Delta = 0$ with Fréchet derivative (linearization) D_Δ, if

$$[D_\Delta, \mathcal{R}] = 0$$

on solutions, then $\mathcal{R}$ is a recursion operator.

Example. Burgers' equation.

$$u_t = u_{xx} + uu_x \qquad \mathcal{R} = D_x + \tfrac{1}{2}u + \tfrac{1}{2}uD_x^{-1}$$

$$D_\Delta = D_t - D_x^2 - uD_x - u_x$$

$$D_\Delta \cdot \mathcal{R} = D_tD_x - D_x^3 - \tfrac{3}{2}uD_x^2 - \tfrac{1}{2}(5u_x + u^2)D_x + \tfrac{1}{2}u_t - \\ - \tfrac{3}{2}u_{xx} - \tfrac{3}{2}uu_x + \tfrac{1}{2}(u_{xt} - u_{xxx} - uu_{xx} - u_x^2)D_x^{-1},$$

$$\mathcal{R} \cdot D_\Delta = D_tD_x - D_x^3 - \tfrac{3}{2}uD_x^2 - \tfrac{1}{2}(5u_x + u^2)D_x - u_{xx} - uu_x$$

hence

$$[D_\Delta, \mathcal{R}] = \tfrac{1}{2}(u_t - u_{xx} - uu_x) + \tfrac{1}{2}(u_{xt} - u_{xxx} - uu_{xx} - u_x^2)D_x^{-1}$$

which vanishes on solutions.

Linear Equations

Theorem. Let

$$\Delta[u] = 0$$

be a linear system of partial differential equations. Then any symmetry $\mathbf{v}_Q$ with linear characteristic $Q = \mathcal{D}[u]$ determines a recursion operator $\mathcal{D}$.

$$[\mathcal{D}, \Delta] = \widetilde{\mathcal{D}} \cdot \Delta$$

If $\mathcal{D}_1, \ldots, \mathcal{D}_m$ determine linear symmetries $\mathbf{v}_{Q_1}, \ldots, \mathbf{v}_{Q_m}$, then any polynomial in the $\mathcal{D}_j$'s also gives a linear symmetry.

Question 1: Given a linear system, when are all symmetries

a) linear? b) generated by first order symmetries?

Question 2: What is the structure of the non-commutative symmetry algebra?

Bi–Hamiltonian systems

$\Longrightarrow$ Magri (1978)

Theorem. Suppose

$$\frac{du}{dt} = F_1 = J_1 \nabla H_1 = J_2 \nabla H_0$$

is a biHamiltonian system, where J_1, J_2 form a compatible pair of Hamiltonian operators. Assume that J_1 is nondegenerate. Then

$$\mathcal{R} = J_2 J_1^{-1}$$

is a recursion operator that generates an infinite hierarch of biHamiltonian symmetries

$$\frac{du}{dt} = F_n = J_1 \nabla H_n = J_2 \nabla H_{n-1} = \mathcal{R}\, F_{n-1}$$

whose Hamiltonian function(al)s $H_0, H_1, H_2, \ldots$ are in involution with respect to either Poisson bracket:

$$\{\, H_n, \, H_m \,\}_1 = 0 = \{\, H_n, \, H_m \,\}_2$$

and hence define conservation laws for every system.

The Korteweg–deVries Equation

$$\frac{\partial u}{\partial t} = u_{xxx} + u\,u_x = J_1 \frac{\delta \mathcal{H}_1}{\delta u} = J_2 \frac{\delta \mathcal{H}_0}{\delta u}$$

$$J_1 = D_x \qquad\qquad \mathcal{H}_1[u] = \int \left(\tfrac{1}{6} u^3 - \tfrac{1}{2} u_x^2 \right) dx$$

$$J_2 = D_x^3 + \tfrac{2}{3} u\, D_x + \tfrac{1}{3} u_x \qquad\qquad \mathcal{H}_0[u] = \int \tfrac{1}{2} u^2 \, dx$$

★ ★ Bi–Hamiltonian system with recursion operator (Lenard)

$$\mathcal{R} = J_2 \cdot J_1^{-1} = D_x^2 + \tfrac{2}{3} u + \tfrac{1}{3} u_x D_x^{-1}$$

Hierarchy of generalized symmetries and higher order conservation laws:

$$\frac{\partial u}{\partial t} = u_{xxxxx} + \tfrac{5}{3} u\, u_{xxx} + \tfrac{10}{3} u_x u_{xx} + \tfrac{5}{6} u^2 u_x = J_1 \frac{\delta \mathcal{H}_2}{\delta u} = J_2 \frac{\delta \mathcal{H}_1}{\delta u}$$

$$\mathcal{H}_2[u] = \int \left(\tfrac{1}{2} u_{xx}^2 - \tfrac{5}{6} u_x^2 + \tfrac{5}{72} u^4 \right) dx$$

and so on . . . (Gardner, Green, Kruskal, Miura, Lax)

Noether's Second Theorem

Theorem. A system of Euler-Lagrange equations is under-determined, and hence admits a nontrivial differential relation if and only if it admits an infinite dimensional variational symmetry group depending on an arbitrary function.
The associated conservation laws are trivial.

Proof — Integration by parts:

For any linear differential operator $\mathcal{D}$ and any function F:

$$F\,\mathcal{D}\,E(L) = \mathcal{D}^*(F)\,E(L) + \mathrm{Div}\,P[F, E(L)].$$

where $\mathcal{D}^*$ is the formal adjoint of $\mathcal{D}$. Now apply Noether's Identity using the symmetry/conservation law characteristic

$$Q = \mathcal{D}^*(F).$$

Noether's Second Theorem

Theorem. A system of Euler-Lagrange equations is under-determined, and hence admits a nontrivial differential relation if and only if it admits an infinite dimensional variational symmetry group depending on an arbitrary function.

$\Longrightarrow$ The associated conservation laws are trivial.

Open Question: Are there over-determined systems of Euler–Lagrange equations for which trivial symmetries give non-trivial conservation laws?

A Very Simple Example:

Variational problem:

$$I[u,v] = \iint (u_x + v_y)^2 \, dx \, dy$$

Variational symmetry group:

$$(u,v) \longmapsto (u + \varphi_y, v - \varphi_x)$$

Euler-Lagrange equations:

$$\Delta_1 = E_u(L) = u_{xx} + v_{xy} = 0$$

$$\Delta_2 = E_v(L) = u_{xy} + v_{yy} = 0$$

Differential relation:

$$D_y \Delta_1 - D_x \Delta_2 \equiv 0$$

Relativity

Noether's Second Theorem effectively resolved Hilbert's dilemma regarding the law of conservation of energy in Einstein's field equations for general relativity.

Namely, the time translational symmetry that ordinarily leads to conservation of energy in fact belongs to an infinite-dimensional symmetry group, and thus, by Noether's Second Theorem, the corresponding conservation law is trivial, meaning that it vanishes on all solutions.

References

[1] E. Noether, Invariante Variationsprobleme, *Nachr. König. Gesell. Wissen. Göttingen, Math.–Phys. Kl.* (1918) 235–257.

[2] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Second Edition, Graduate Texts in Math., vol. 107, Springer–Verlag, New York, 1993.

[3] Y. Kosmann-Schwarzbach, *The Noether Theorems. Invariance and Conservation Laws in the Twentieth Century*, Springer, New York, 2011.

Includes English translation of [1]