Moving Frames and the Geometry of Numerical Analysis

Peter J. Olver

University of Minnesota

http : //www.math.umn.edu/∼olver

Jet Space

Although in use since the time of Lie and Darboux, jet space was first formally defined by Ehresmann in 1950.

Jet space is the proper setting for the geometry of partial differential equations.

In this talk, I will propose a setting, named *multi-space*, for the geometry of numerical approximations to derivatives and differential equations.

I will then show how to apply the method of moving frames on multi-space to systematically construct symmetry–preserving numerical algorithms.

Jet Space

M — smooth m-dimensional manifold

$\mathrm{J}^n = \mathrm{J}^n(M,p)$ — (extended) jet bundle

$\Longrightarrow$ Defined as the space of equivalence classes of submanifolds under the equivalence relation of n^{th} order contact at a single point.

$\Longrightarrow$ Coordinates $(x, u^{(n)})$ given by the derivatives of $u = f(x)$.

$\Longrightarrow$ No bundle structure assumed on M.

Jets and Cartesian Products

Key remark: Every (finite difference) numerical approximation to the derivatives of a function or, geometrically depend on evaluating the function at several points $z_i = (x_i, u_i)$ where $u_i = f(x_i)$.

In other words, we seek to approximate the n^{th} order jet of a submanifold $N \subset M$ by a function $F(z_0, \ldots, z_n)$ defined on the $(n+1)$-fold Cartesian product space $M^{\times(n+1)} = M \times \cdots \times M$, or, more correctly, on the "off-diagonal" part

$$M^{\diamond(n+1)} = \{\, z_i \neq z_j \text{ for all } i \neq j \,\}$$

$$\Longrightarrow \quad \textit{distinct } (n+1)\text{-tuples of points.}$$

Thus, multi-space should contain both the jet space and the off-diagonal Cartesian product space as submanifolds:

$$\left.\begin{array}{c} M^{\diamond(n+1)} \\ \downarrow \\ \mathrm{J}^n(M,p) \end{array}\right\} \quad \subset \quad M^{(n)}$$

Functions $\quad F : M^{(n)} \longrightarrow \mathbb{R} \quad$ are given by

$$F(z_0, \ldots, z_n) \qquad \text{on} \qquad M^{\diamond(n+1)}$$

and extend smoothly to J^n as the points coalesce. In this manner, $F \mid M^{\diamond(n+1)}$ provides a finite difference approximation to the differential function $F \mid \mathrm{J}^n$.

Construction of $M^{(n)}$

Definition. An $(n+1)$-*pointed manifold*

$$\mathbf{M} = (z_0, \ldots, z_n; M)$$

M — smooth manifold

$z_0, \ldots, z_n \in M$ — not necessarily distinct

Given $\mathbf{M}$, let

$$\#i \;=\; \#\left\{\, j \;\middle|\; z_j = z_i \,\right\}$$

denote the number of points which coincide with the i^{th} one.

Multi-contact for Curves

Definition. Two $(n+1)$-pointed curves

$$\mathbf{C} = (z_0, \ldots, z_n; C), \qquad \widetilde{\mathbf{C}} = (\tilde{z}_0, \ldots, \tilde{z}_n; \widetilde{C}),$$

have n^{th} *order multi-contact* if and only if

$$z_i = \tilde{z}_i, \qquad \text{and} \qquad \mathrm{j}_{\#i-1} C|_{z_i} = \mathrm{j}_{\#i-1} \widetilde{C}|_{z_i},$$

for each $i = 0, \ldots, n$.

$$\#i \;=\; \#\left\{\, j \;\middle|\; z_j = z_i \,\right\}$$

Definition. The n^{th} order *multi-space* $M^{(n)}$ is the set of equivalence classes of $(n+1)$-pointed curves in M under the equivalence relation of n^{th} order multi-contact.

The Fundamental Theorem

Theorem. If M is a smooth m-dimensional manifold, then its n^{th} order multi-space $M^{(n)}$ is a smooth manifold of dimension $(n+1)m$, which contains the off-diagonal part $M^{\diamond(n+1)}$ of the Cartesian product space as an open, dense submanifold, and the n^{th} order jet space J^n as a smooth submanifold.

$$\left.\begin{array}{ll} \text{points} & M^{\diamond(n+1)} \\[2em] \text{“multi}-\text{jets”} & \mathrm{J}^{k_1} \diamond \cdots \diamond \mathrm{J}^{k_\nu} \\[2em] \text{jets} & \mathrm{J}^n(M,p) \end{array}\right\} \quad \subset \quad M^{(n)}$$

Example. Let $M = \mathbb{R}^m$

(i) $M^{(1)}$ is the space of two-pointed lines

$$M^{(1)} \simeq \{\, (z_0, z_1; L) \mid \; z_0, z_1 \in L \quad \text{—} \quad \text{line} \,\} .$$

$\Longrightarrow$ Blow-up construction in algebraic geometry

(ii) $M^{(2)}$ is the space of three-pointed circles, i.e.,

$$M^{(2)} \simeq \{\, (z_0, z_1, z_2, C) \mid \; z_0, z_1, z_2 \in C \quad \text{—} \quad \text{circle} \,\} .$$

Straight lines are included as circles of infinite radius, but points are not included (even though they could be viewed as circles of zero radius).

$\Longrightarrow$ Grassmann bundles.

(iii) $M^{(3)}$????

$\star \star \star$ Topology — local and global.

Finite Differences

Local coordinates on J^n are provided by the coefficients of Taylor polynomials

$\Longrightarrow$ derivatives

Local coordinates on $M^{(n)}$ are provided by the coefficients of interpolating polynomials.

$\Longrightarrow$ finite differences

Given $(z_0, \ldots, z_n) \in M^{\diamond(n+1)}$, define the classical *divided differences* by the standard recursive rule

$$[\, z_0 z_1 \ldots z_{k-1} z_k \,] = \frac{[\, z_0 z_1 z_2 \ldots z_{k-2} z_k \,] - [\, z_0 z_1 z_2 \ldots z_{k-2} z_{k-1} \,]}{x_k - x_{k-1}}$$

$$[\, z_j \,] = u_j$$

$\Longrightarrow$ Well-defined provided no two points lie on the same vertical line.

$\Longrightarrow$ Symmetric functions of z_i.

Definition. Given an $(n+1)$-pointed graph $\mathbf{C} = (z_0, \ldots, z_n; C)$, its divided differences are defined by

$$[\, z_j \,]_C = f(x_j)$$

$$[\, z_0 z_1 \ldots z_{k-1} z_k \,]_C = \lim_{z \to z_k} \frac{[\, z_0 z_1 z_2 \ldots z_{k-2} z \,]_C - [\, z_0 z_1 z_2 \ldots z_{k-2} z_{k-1} \,]_C}{x - x_{k-1}}$$

$\Longrightarrow$ When taking the limit, the point $z = (x, f(x))$ must lie on the graph C, and take limiting values $x \to x_k$ and $f(x) \to f(x_k)$.

Theorem. Two $(n+1)$-pointed graphs $\mathbf{C}, \widetilde{\mathbf{C}}$ have n^{th} order multi-contact if and only if they have the same divided differences:

$$[\, z_0 z_1 \ldots z_k \,]_C = [\, z_0 z_1 \ldots z_k \,]_{\widetilde{C}}, \qquad k = 0, \ldots, n.$$

Local coordinates on $M^{(n)}$

They consist of the independent variables along with all the divided differences

$$x_0, \ldots, x_n \qquad \begin{array}{ll} u^{(0)} = u_0 = [\, z_0 \,]_C & u^{(1)} = [\, z_0 z_1 \,]_C \\ u^{(2)} = 2\,[\, z_0 z_1 z_2 \,]_C \quad \ldots & u^{(n)} = n!\,[\, z_0 z_1 \ldots z_n \,]_C \end{array}$$

prescribed by $(n+1)$-pointed graphs

$$\mathbf{C} = (z_0, \ldots, z_n; C)$$

The $n!$ factor is included so that $u^{(n)}$ agrees with the usual derivative coordinate when restricted to J^n.

Numerical Approximations

$\Delta(x, u^{(n)})$ — differential function

$$\Delta \colon \mathrm{J}^n \to \mathbb{R}$$

System of differential equations:

$$\Delta_1(x, u^{(n)}) = \cdots = \Delta_k(x, u^{(n)}) = 0.$$

Definition. An $(n+1)$*-point numerical approximation of order* k to a differential function $\Delta \colon \mathrm{J}^n \to \mathbb{R}$ is a k^{th} order extension $F \colon M^{(n)} \to \mathbb{R}$ of Δ to multi-space, based on the inclusion $\mathrm{J}^n \subset M^{(n)}$.

$$F(x_0, \ldots, x_n, u^{(0)}, \ldots, u^{(n)}) \longrightarrow F(x, \ldots, x, u^{(0)}, \ldots, u^{(n)}) = \Delta(x, u^{(n)})$$

Invariant Numerical Approximations

G — Lie group acting on M

Basic Idea:

Every invariant finite difference approximation to a differential invariant must expressible in terms of the joint invariants of the transformation group.

Differential Invariant

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

Joint Invariant

$$J(g \cdot P_1, \ldots, g \cdot P_k) = J(P_1, \ldots, P_k)$$

Semi-differential invariant =

Joint differential invariant

$\Longrightarrow$ *Approximate differential invariants by joint invariants*

Euclidean Invariants

Joint Euclidean invariant:

$$\mathbf{d}(z, w) = \| \, z - w \, \|$$

Euclidean curvature:

$$\kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}}$$

Euclidean arc length:

$$ds = \sqrt{1 + u_x^2} \; dx$$

Higher order differential invariants:

$$\kappa_s = \frac{d\kappa}{ds} \qquad \kappa_{ss} = \frac{d^2\kappa}{ds^2} \qquad \ldots$$

Euclidean–invariant differential equation:

$$F(\kappa, \kappa_s, \kappa_{ss}, \ldots) = 0$$

Three point approximation

Heron's formula

$$\widetilde{\kappa}(A,B,C) = 4\,\frac{\Delta}{abc} = 4\,\frac{\sqrt{s(s-a)(s-b)(s-c)}}{abc}$$

$$s = \frac{a+b+c}{2} \qquad — \quad \text{semi-perimeter}$$

Expansion:

$$\begin{aligned}\widetilde{\kappa} = \kappa &+ \frac{1}{3}(b-a)\frac{d\kappa}{ds} + \frac{1}{12}(b^2-ab+a^2)\frac{d^2\kappa}{ds^2} + \\ &+ \frac{1}{60}(b^3-ab^2+a^2b-a^3)\frac{d^3\kappa}{ds^3} + \\ &+ \frac{1}{120}(b-a)(3b^2+5ab+3a^2)\kappa^2\frac{d\kappa}{ds} + \cdots.\end{aligned}$$

Multi-Invariants

G — Lie group which acts smoothly on M

$\Longrightarrow$ G preserves the multi-contact equivalence relation

---

$G^{(n)}$ — n^{th} *multi-prolongation* to $M^{(n)}$

$\Longrightarrow$ On $\mathrm{J}^n \subset M^{(n)}$ it coincides with the usual jet space prolongation

$\Longrightarrow$ On $M^{\diamond(n+1)} \subset M^{(n)}$ it coincides with the $(n+1)$-fold Cartesian product action.

$K : M^{(n)} \to \mathbb{R}$ — *multi-invariant*

$$K(g^{(n)} \cdot z^{(n)}) = K(z^{(n)})$$

$\Longrightarrow$ $K \mid \mathrm{J}^n$ — differential invariant

$\Longrightarrow$ $K \mid M^{\diamond(n+1)}$ — joint invariant

$\Longrightarrow$ $K \mid \mathrm{J}^{k_1} \diamond \cdots \diamond \mathrm{J}^{k_\nu}$ — joint diff. invariant

---

The theory of multi-invariants *is* the theory of invariant numerical approximations!

Moving frames provide a

systematic algorithm for

constructing multi-invariants!

A moving frame on multi-space

$$\rho \colon M^{(n)} \quad \longrightarrow \quad G$$

is called a *multi-frame*.

Example. $G = \mathbb{R}^2 \ltimes \mathbb{R}$

$$(x, u) \quad \longmapsto \quad (\lambda^{-1} x + a, \lambda u + b)$$

Multi-prolonged action: compute the divided differences of the basic lifted invariants

$$y_k = \lambda^{-1} x_k + a, \qquad v_k = \lambda\, u_k + b.$$

We find

$$v^{(1)} = [\, w_0 w_1 \,] = \frac{v_1 - v_0}{y_1 - y_0}$$

$$= \lambda^2 \, \frac{u_1 - u_0}{x_1 - x_0} = \lambda^2 \, [\, z_0 z_1 \,] = \lambda^2 \, u^{(1)},$$

$$v^{(n)} = \lambda^{n+1} \, u^{(n)}.$$

Moving frame cross-section

$$y_0 = 0, \qquad v_0 = 0, \qquad v^{(1)} = 1.$$

Solve for the group parameters

$$a = -\sqrt{u^{(1)}} \; x_0, \qquad b = -\,\frac{u_0}{\sqrt{u^{(1)}}}, \qquad \lambda = \frac{1}{\sqrt{u^{(1)}}}\,.$$

$\Longrightarrow$ multi-frame $\rho\colon M^{(n)} \to G$.

Multi-invariants:

$$y_k: \quad H_k = (x_k - x_0)\sqrt{u^{(1)}} = (x_k - x_0)\sqrt{\frac{u_1 - u_0}{x_1 - x_0}}$$

$$u_k: \quad K_k = \frac{u_k - u_0}{\sqrt{u^{(1)}}} = (u_k - u_0)\sqrt{\frac{x_1 - x_0}{u_1 - u_0}}$$

$$u^{(n)}: \quad K^{(n)} = \frac{u^{(n)}}{(u^{(1)})^{(n+1)/2}} = \frac{n!\,[\,z_0 z_1 \ldots z_n\,]}{[\,z_0 z_1 z_2\,]^{(n+1)/2}}\,.$$

$$K^{(0)} = K_0 = 0 \qquad K^{(1)} = 1$$

Coalescent limit

$$K^{(n)} \quad \longrightarrow \quad I^{(n)} = \frac{u^{(n)}}{(u^{(1)})^{(n+1)/2}}$$

$\Longrightarrow$ $K^{(n)}$ is a first order invariant numerical approximation to the differential invariant $I^{(n)}$.

$\Longrightarrow$ Higher order invariant numerical approximations are obtained by invariantization of higher order divided difference approximations.

$$F(\ldots, x_k, \ldots, u^{(n)}, \ldots) \quad \longrightarrow \quad F(\ldots, H_k, \ldots, K^{(n)}, \ldots)$$

To construct an invariant numerical scheme for any similarity-invariant ordinary differential equation

$$F(x, u, u^{(1)}, u^{(2)}, \ldots u^{(n)}) = 0,$$

we merely invariantize the defining differential function, leading to the general similarity–invariant numerical approximation

$$F(0, 0, 1, K^{(2)}, \ldots, K^{(n)}) = 0.$$

Example. Euclidean group SE(2)

$$y = x\cos\theta - u\sin\theta + a \qquad v = x\sin\theta + u\cos\theta + b$$

Multi-prolonged action on $M^{(1)}$:

$$y_0 = x_0\cos\theta - u_0\sin\theta + a \qquad v_0 = x_0\sin\theta + u_0\cos\theta + b$$

$$y_1 = x_1\cos\theta - u_1\sin\theta + a \qquad v^{(1)} = \frac{\sin\theta + u^{(1)}\cos\theta}{\cos\theta - u^{(1)}\sin\theta}$$

Cross-section

$$y_0 = v_0 = v^{(1)} = 0$$

Right moving frame

$$a = -\,x_0\cos\theta + u_0\sin\theta = -\frac{x_0 + u^{(1)}\,u_0}{\sqrt{1+(u^{(1)})^2}}$$

$$b = -x_0\sin\theta - u_0\cos\theta = \frac{x_0\,u^{(1)} - u_0}{\sqrt{1+(u^{(1)})^2}}$$

$$\tan\theta = -\,u^{(1)}\,.$$

Euclidean multi-invariants

$$(y_k, v_k) \quad \longrightarrow \quad I_k = (H_k, K_k)$$

$$H_k = \frac{(x_k - x_0) + u^{(1)}\,(u_k - u_0)}{\sqrt{1 + (u^{(1)})^2}} = (x_k - x_0)\,\frac{1 + [\,z_0 z_1\,]\,[\,z_0 z_k\,]}{\sqrt{1 + [\,z_0 z_1\,]^2}}$$

$$K_k = \frac{(u_k - u_0) - u^{(1)}\,(x_k - x_0)}{\sqrt{1 + (u^{(1)})^2}} = (x_k - x_0)\,\frac{[\,z_0 z_k\,] - [\,z_0 z_1\,]}{\sqrt{1 + [\,z_0 z_1\,]^2}}$$

Difference quotients

$$[\,I_0 I_k\,] = \frac{K_k - K_0}{H_k - H_0} = \frac{K_k}{H_k} = \frac{(x_k - x_1)[\,z_0 z_1 z_k\,]}{1 + [\,z_0 z_k\,]\,[\,z_0 z_1\,]}$$

$$I^{(1)} = [\,I_0 I_1\,] = 0$$

$$I^{(2)} = 2\,[\,I_0 I_1 I_2\,] = 2\,\frac{[\,I_0 I_2\,] - [\,I_0 I_1\,]}{H_2 - H_1}$$

$$= \frac{2\,[\,z_0 z_1 z_2\,]\sqrt{1 + [\,z_0 z_1\,]^2}}{(\,1 + [\,z_0 z_1\,]\,[\,z_1 z_2\,]\,)(\,1 + [\,z_0 z_1\,]\,[\,z_0 z_2\,]\,)}$$

$$= \frac{u^{(2)}\sqrt{1 + (u^{(1)})^2}}{\left[\,1 + (u^{(1)})^2 + \frac{1}{2}u^{(1)}u^{(2)}(x_2 - x_0)\,\right]\left[\,1 + (u^{(1)})^2 + \frac{1}{2}u^{(1)}u^{(2)}(x_2 - x_1)\,\right]}$$

Euclidean–invariant numerical approximation to the Euclidean curvature:

$$\lim_{z_1, z_2 \to z_0} I^{(2)} = \kappa = \frac{u^{(2)}}{(1 + (u^{(1)})^2)^{3/2}}$$

Similarly, the third order multi-invariant

$$I^{(3)} = 6\,[\,I_0 I_1 I_2 I_3\,] = 6\,\frac{[\,I_0 I_1 I_3\,] - [\,I_0 I_1 I_2\,]}{H_3 - H_2}$$

will form a Euclidean–invariant approximation for the normalized differential invariant

$$\kappa_s = \iota(u_{xxx})$$

Higher Dimensional Submanifolds

$T^{(n)} M|_z$ — n^{th} order tangent space

Proposition. Two p-dimensional submanifolds $N, \widetilde{N}$ have n^{th} order *contact* at a common point $z \in N \cap \widetilde{N}$ if and only if

$$T^{(n)} N|_z = T^{(n)} \widetilde{N}|_z$$

$\Longrightarrow$ Requires $\binom{p+n}{n}$ coalescing points to approximate

Surfaces $p = 2$

n	$\binom{p+n}{n}$
0	1
1	3
2	6
3	10
$\vdots$	$\vdots$

Definition. A subspace $V \subset T^{(n)}M|_z$ is called *admissible* if for every vector

$\mathbf{v} \in V \cap T^{(k)}M|_z$, $1 \leq k \leq n$,

there exists a submanifold $N \subset M$ such that $\mathbf{v} \in T^{(k)}N|_z \subset V$.

Definition. Two submanifolds $N, \widetilde{N}$ have r^{th} *order subcontact* at a common point if and only if for some n, there exists an admissible common r-dimensional subspace

$$S \subset T^{(n)}N|_z \cap T^{(n)}\widetilde{N}|_z \subset T^{(n)}M|_z$$

Example. Surfaces: $S, \widetilde{S} \subset M$

order	Conditions
0	$z \in S \cap \widetilde{S}$ — common point
1	tangent curves: $TC\|_z = T\widetilde{C}\|_z$
2	$\begin{cases} \text{tangent surfaces: } TS\|_z = T\widetilde{S}\|_z \\ \text{osculating curves: } T^{(2)}C\|_z = T^{(2)}\widetilde{C}\|_z \end{cases}$
3	$\begin{cases} TS\|_z = T\widetilde{S}\|_z \quad \text{and} \quad T^{(2)}C\|_z = T^{(2)}\widetilde{C}\|_z \\ T^{(3)}C\|_z = T^{(3)}\widetilde{C}\|_z \end{cases}$
⋮	⋮
5	$\begin{cases} T^{(2)}S\|_z = T^{(2)}\widetilde{S}\|_z \\ TS\|_z = T\widetilde{S}\|_z,\ T^{(3)}C\|_z = T^{(3)}\widetilde{C}\|_z,\ T^{(2)}C'\|_z = T^{(2)}\widetilde{C}'\|_z \\ TS\|_z = T\widetilde{S}\|_z,\ T^{(4)}C\|_z = T^{(4)}\widetilde{C}\|_z \\ T^{(5)}C\|_z = T^{(5)}\widetilde{C}\|_z \end{cases}$

Multi-space and Multi-Variate Interpolation

Definition. Let M be a smooth manifold. The n^{th} order *multi-space* $M^{(n)}$ is the set of all *n-point interpolant data*

$$\mathbf{Z} = (z_0, \ldots, z_{n-1}; V_0, \ldots, V_{n-1}),$$

consisting of

(a) an ordered set of n points $z_0, \ldots, z_{n-1} \in M$.

$$\#i = \#\left\{\, j \;\middle|\; z_j = z_i \,\right\}$$

(b) an ordered collection of admissible subspaces $V_i \subset T^{(n)}M|_{z_i}$ such that

$$\begin{cases} V_i = V_j \quad \text{if} \quad z_i = z_j \\ \dim V_i = \#i - 1 \end{cases}$$

In particular, if $\#i = 1$, and so z_i only appears once in $\mathbf{Z}$, then $V_i = \{0\}$ is trivial.

Multivariate Hermite Interpolation

Definition. An *interpolant* to $\mathbf{Z}$ is a submanifold $N \subset M$ such that $z_i \in N$ and $V_i \subset T^{(n)}N|_{z_i}$.

Conjecture. The multispace $M^{(n)}$ is a manifold of dimension nm. It contains

- $M^{\diamond n}$ as an open, dense submanifold
- *all* $\mathrm{J}^k(M,p)$ that have dimension $\leq nm$ as submanifolds
- various off-diagonal copies of multi-jet spaces $\mathrm{J}^{i_1}(M,p)\diamond\cdots\diamond \mathrm{J}^{i_k}(M,p)$ for $i_1+\cdots+i_k = n-k$ as submanifolds.

$\Longrightarrow$ smooth or analytic

Difficulties

♠ Multi-variate interpolation theory.

♠ Multi-variate divided differences.

♠ Coordinates at coalescent points.

♠ Topological structure — local and global

The Simplest Case

Three points

$$w_0 = (0,0,0),\ \ w_1 = (x_1, y_1, z_1),\ \ w_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$$

can viewed as interpolating either

- A quadratic curve C, or
- A linear surface

Curve: Newton's form

$$y = a\,x + b\,x(x - x_1), \qquad z = c\,x + d\,x(x - x_1).$$

Divided differences

$$a = [\,y_0 y_1\,] = \frac{y_1}{x_1}\,, \qquad b = [\,y_0 y_1 y_2\,] = \frac{x_1 y_2 - x_2 y_1}{x_1 x_2 (x_1 - x_2)}\,,$$

$$c = [\,z_0 z_1\,] = \frac{z_1}{x_1}\,, \qquad d = [\,z_0 z_1 z_2\,] = \frac{x_1 z_2 - x_2 z_1}{x_1 x_2 (x_1 - x_2)}\,.$$

Surface:

$$z = p\,x + q\,y$$

Interpolation formulae

$$p = \frac{y_1 z_2 - y_2 z_1}{x_1 y_2 - x_2 y_1}\,, \qquad q = \frac{x_1 z_2 - x_2 z_1}{x_1 y_2 - x_2 y_1}\,,$$

$\Longrightarrow$ poised

Connecting formula:

$$p = c - \frac{d}{b}\,a = [\,z_0 z_1\,] - \frac{[\,z_0 z_1 z_2\,]}{[\,y_0 y_1 y_2\,]}\,[\,y_0 y_1\,],$$

$$q = \frac{d}{b} = \frac{[\,z_0 z_1 z_2\,]}{[\,y_0 y_1 y_2\,]}\,.$$

Coalescent limit

$$w_1, w_2 \longrightarrow 0 = w_0.$$

Curve:

$$y = y(x), \qquad z = z(x).$$

$$a \longrightarrow \frac{dy}{dx} \qquad b \longrightarrow \frac{1}{2}\,\frac{d^2y}{dx^2}$$

$$c \longrightarrow \frac{dz}{dx} \qquad d \longrightarrow \frac{1}{2}\,\frac{d^2z}{dx^2}$$

Surface:

$$z = z(x, y)$$

$$p \longrightarrow \frac{\partial z}{\partial x} \qquad q \longrightarrow \frac{\partial z}{\partial y}$$

Connecting formula:

$$\frac{\partial z}{\partial x} = z_x - y_x\,\frac{z_{xx}}{y_{xx}} = \frac{z_{yy}}{x_{yy}}$$

$$\frac{\partial z}{\partial y} = \frac{z_{xx}}{y_{xx}} \qquad = z_y - x_y\,\frac{z_{yy}}{x_{yy}}$$

A Simple Calculus

$$z = p\,x + q\,y + \mathrm{O}(2)$$

$$\frac{dz}{dx} = p + q\ \frac{dy}{dx}$$

$$\frac{d^2z}{dx^2} = q\ \frac{d^2y}{dx^2}$$

Solution:

$$\frac{\partial z}{\partial x} = z_x - y_x\,\frac{z_{xx}}{y_{xx}} = \frac{z_{yy}}{x_{yy}}$$

$$\frac{\partial z}{\partial y} = \frac{z_{xx}}{y_{xx}} \qquad\qquad = z_y - x_y\,\frac{z_{yy}}{x_{yy}}$$

Infinite Curvature Limit

When

$$C = \{\, y = y(x), z = z(x)\,\} \;\subset\; S = \{\, z = z(x,y)\,\}$$

then

$$z(x, y(x)) = z(x)$$

Then

$$z_x = p + q y_x$$

$$z_{xx} = q y_{xx} + r + 2 s y_x + t y_x^2$$

Solving for p, q:

$$\frac{\partial z}{\partial x} = p = z_x - q y_x,$$

$$\frac{\partial z}{\partial y} = q = \frac{z_{xx}}{y_{xx}} - \frac{r + 2 s y_x + t y_x^2}{y_{xx}}\,.$$

Infinite curvature limit $y_{xx}, z_{xx} \to \infty$

$$\frac{\partial u}{\partial x} \;\longrightarrow\; z_x - q y_x = \frac{z_x y_{xx} - z_{xx} y_x}{y_{xx}}$$

$$\frac{\partial u}{\partial y} \;\longrightarrow\; \frac{z_{xx}}{y_{xx}}$$

$\Longrightarrow$ *Surfaces are limiting cases of curves as the curvature becomes infinite*!