# *Moving Frames for Pseudo–Groups*

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## Pseudogroups in Action

- Lie — Medolaghi — Vessiot
- Cartan ... Guillemin, Sternberg
- Kuranishi, Spencer, Goldschmidt, Kumpera, ...
- Relativity
- Gauge theory and field theories
  Maxwell, Yang–Mills, conformal, string, ...
- Noether's Second Theorem
- Fluid Mechanics, Metereology
  Euler, Navier–Stokes,
  boundary layer, quasi-geostropic , ...
- Solitons (in $2+1$)
  K–P, Davey-Stewartson, ...
- Kac–Moody
- *Lie groups*!

## What's New?

- Invariant Maurer–Cartan forms and
  structure equations
- Moving frames
  $\Longrightarrow$ direct method
  $\Longrightarrow$ Taylor series method
- Differential invariants
- Invariant differential forms
- Invariant differential operators
- Recurrence formulae
- Tresse–Kumpera Basis Theorem
- Applications
  $\Longrightarrow$ Symmetries of differential equations
  $\Longrightarrow$ Vessiot group splitting
  $\Longrightarrow$ Calculus of variations

## Key Ingredients

- Groupoids
- Variational bicomplex
- Equivariant moving frames
- Gröbner basis methods
- *No* Spencer machinery

# Diffeomorphism Pseudogroups

$M$ — smooth $m$-dimensional manifold

$\mathcal{D} = \mathcal{D}(M)$ — local diffeomorphism pseudo-group

$$Z = \varphi(z)$$

$z$ — source coordinates
$Z$ — target coordinates

$\mathcal{D}^{(n)} = \mathcal{D}^{(n)}(M) \subset \mathrm{J}^n(M, M)$ — $n^{\text{th}}$ order jets

$$Z^{(n)} = \varphi^{(n)}(z)$$

$Z^a_J$ — target jets (Taylor coordinates)

Local coordinates on $\mathcal{D}^{(n)}$:

$$g^{(n)} = (z, Z^{(n)})$$

Frame bundle:

$$\mathcal{F}^{(n)} = \{ z = Z \} \subset \mathcal{D}^{(n)}$$

# Groupoid Structure

Double fibration:

$$\boxed{\begin{array}{ccccc} & & \mathcal{D}^{(n)} & & \\ & \boldsymbol{\sigma}^{(n)} \swarrow & & \searrow \boldsymbol{\tau}^{(n)} & \\ M & & & & M \end{array}}$$

$\boldsymbol{\sigma}^{(n)}$ — source map

$\boldsymbol{\tau}^{(n)}$ — target map

$\Longrightarrow$ Ehresmann

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You are only allowed to multiply $h^{(n)} \cdot g^{(n)}$ if

$$\boldsymbol{\tau}^{(n)}(g^{(n)}) = \boldsymbol{\sigma}^{(n)}(h^{(n)})$$

In other words, composing Taylor series is only well-defined if the target $Z$ of the first Taylor series is the source of the second.

## One-dimensional case: $M = \mathbb{R}$

Source coordinate: $x$ Target coordinate: $X$

Local coordinates on $\mathcal{D}^{(n)}(\mathbb{R})$

$$g^{(n)} = (x, X, X_x, X_{xx}, X_{xxx}, \ \ldots \ , X_n)$$

Infinite jet: $g^{(\infty)} = (x, X^{(\infty)})$

$$X[\![\, h \,]\!] = X + X_x\, h + \tfrac{1}{2}\, X_{xx}\, h^2 + \tfrac{1}{6}\, X_{xxx}\, h^3 + \ \cdots$$

$\Longrightarrow$ Taylor series at a source point $x$

Groupoid multiplication of diffeomorphism jets:

$$(X, \mathbf{X}, \mathbf{X}_X, \mathbf{X}_{XX}, \ \ldots\ ) \cdot (x, X, X_x, X_{xx}, \ \ldots\ )$$
$$= (x, \mathbf{X}, \mathbf{X}_X\, X_x, \mathbf{X}_X\, X_{xx} + \mathbf{X}_{XX}\, X_x^2, \ \ldots\ )$$

$\Longrightarrow$ Composition and truncation of Taylor series

The higher order terms are expressed in terms of Bell polynomials according to the general Fàa–di–Bruno formula.

# Lie Pseudogroups

**Definition.** A *Lie pseudo-group* $\mathcal{G} \subset \mathcal{D}$ is defined by an involutive system of partial differential equations.

Determining equations:

$$F^{(n)}(z, Z^{(n)}) = 0.$$

$\mathcal{G}^{(n)} \subset \mathcal{D}^{(n)}$ — jet sub-groupoid (subbundle)

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Regularity:

(a) $\pi_n^{n+1} : \mathcal{G}^{(n+1)} \to \mathcal{G}^{(n)}$ is a bundle map,

(b) $\mathcal{G}_0^{(n)} \subset \mathcal{F}^{(n)}$ forms a subbundle of the $n^{\text{th}}$ order frame bundle,

(c) every smooth local solution $Z = \varphi(z)$ to the determining system $\mathcal{G}^{(n)}$ belongs to $\mathcal{G}$,

(d) $\mathcal{G}^{(n)} = \operatorname{pr}^{(n-n^\star)} \mathcal{G}^{(n^\star)}$ is obtained by prolongation.

for all $n \geq n^\star$ — the *order* of the pseudo-group

# Infinitesimal Generators

$\mathfrak{g}$ — space of infinitesimal generators of $\mathcal{G}$

Locally defined vector fields:

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a}$$

*Infinitesimal determining equations*

$$L^{(n)}(z, \zeta^{(n)}) = 0$$

$\Longrightarrow$ obtained by linearization

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*Remark*: If $\mathcal{G}$ is the symmetry group of a system of differential equations, then the linearized system is the usual determining equations for the symmetry group.

# First Example

$$\boxed{X = f(x), \qquad Y = y, \qquad U = \frac{u}{f'(x)}}$$

$\Longrightarrow$ Lie, Vessiot, Kumpera

Involutive system:

$$\begin{array}{lll} & Y = y, & \\ X_x = \dfrac{u}{U}, & Y_x = Y_u = 0, & U_y = 0, \\ X_y = X_u = 0, & Y_y = 1, & U_u = \dfrac{U}{u}, \end{array}$$

$$X_{xx} = -\frac{u\,U_x}{U^2}, \qquad X_{xu} = X_{uu} = 0, \qquad U_{xu} = \frac{U_x}{u}, \qquad U_{uu} = 0.$$

Infinitesimal generator:

$$\mathbf{v} \;=\; \xi\,\frac{\partial}{\partial x} + \varphi\,\frac{\partial}{\partial u} \;=\; a(x)\,\frac{\partial}{\partial x} - a_x(x)\,u\,\frac{\partial}{\partial u}$$

Infinitesimal determining system:

$$\xi_x = -\frac{\varphi}{u}, \qquad \xi_u = 0, \qquad \varphi_u = \frac{\varphi}{u},$$

Taylor coordinates:

$$x, y, u, X, U, U_x, U_{xx}, \ \dots \ , U_n \, .$$

Parametric coordinates:

$$x, y, u, f, f_x, f_{xx}, \ \dots \ , f_{n+1}$$

Parametrization of $\mathcal{G}^{(1)}$:

$$\begin{aligned} & X_x = f_x, && X_y = X_u = 0, \\ & Y_x = Y_u = 0, && Y_y = 1, \\ & U_x = -\frac{u\, f_{xx}}{f_x^2}\,, && U_y = 0, \qquad U_u = \frac{1}{f_x}\,. \end{aligned}$$

## Second Example

$$X = f(x) \qquad\qquad Y = f'(x)\, y + g(x)$$
$$U = u + \frac{f''(x)\, y + g'(x)}{f'(x)}$$

Involutive system

$$X_y = X_u = 0, \qquad Y_u = 0,$$
$$Y_x = (U - u) X_x, \qquad U_u = 1.$$
$$Y_y = X_x \neq 0,$$

Infinitesimal generator:

$$\begin{aligned}\mathbf{v} &= \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \varphi \frac{\partial}{\partial u} \\ &= a(x) \frac{\partial}{\partial x} + [\, a_x(x)\, y + b(x)\,] \frac{\partial}{\partial y} \\ &\qquad + [\, a_{xx}(x)\, y + b_x(x)\,] \frac{\partial}{\partial u}\end{aligned}$$

Infinitesimal determining system:

$$\xi_x = \eta_y \qquad \xi_y = \xi_u = \eta_u = \varphi_u = 0 \qquad \eta_x = \varphi$$

## Maurer–Cartan Forms for the Diffeomorphism Groupoid

Variational bicomplex over $\mathcal{D}^{(\infty)} \subset \mathrm{J}^\infty(M, M)$:

Horizontal forms:

$$dz^1, \ \ldots \ , dz^m$$

Contact forms:

$$\Upsilon^a_J = d_G\, Z^a_J = dZ^a_J - \sum_{i=1}^{m} Z^a_{J,i}\, dz^i$$

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*The Maurer–Cartan forms are right-invariant contact forms on* $\mathcal{D}^{(\infty)}$*!*

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## Right-invariant coframe on $\mathcal{D}^{(\infty)}$

*Key observation*: The target coordinate functions $Z^a$ are right-invariant.

Decompose

$$dZ^a = \sigma^a + \mu^a$$

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Right-invariant horizontal forms:

$$\sigma^a = d_M\, Z^a = \sum_{i=1}^{m} Z_i^a\, dz^i$$

Right-invariant contact forms:

$$\mu^a = d_G\, Z^a = \Upsilon^a = dZ^a - \sum_{i=1}^{m} Z_i^a\, dz^i$$

$$\mu_J^a = \mathbb{D}_Z^J \mu^a = \mathbb{D}_Z^J \Upsilon^a$$

$\Longrightarrow$ *Maurer–Cartan forms*

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Invariant total differential operators:

$$\mathbb{D}_{Z^a} = \sum_{i=1}^{m} w_a^i\, \mathbb{D}_{z^i} \qquad w_a^i(z, Z^{(1)}) = (\, Z_i^a\,)^{-1}$$

## One-dimensional case: $M = \mathbb{R}$

Contact forms:

$$\Upsilon = d_G X = dX - X_x \, dx$$

$$\Upsilon_x = \mathbb{D}_x \Upsilon = dX_x - X_{xx} \, dx$$

$$\Upsilon_{xx} = \mathbb{D}_x^2 \Upsilon = dX_{xx} - X_{xxx} \, dx$$

Right-invariant horizontal form:

$$\sigma = d_M X = X_x \, dx$$

Invariant differentiation:

$$\mathbb{D}_X = \frac{1}{X_x} \mathbb{D}_x$$

Right-invariant Maurer–Cartan forms:

$$\mu = \Upsilon$$

$$\mu_X = \mathbb{D}_X \mu = \frac{\Upsilon_x}{X_x}$$

$$\mu_{XX} = \mathbb{D}_X^2 \mu = \frac{X_x \, \Upsilon_{xx} - X_{xx} \, \Upsilon_x}{X_x^3}$$

## **Two-dimensional case:** $M = \mathbb{R}^2$

Coordinates on $\mathcal{D}^{(\infty)}(\mathbb{R}^2)$:

$$(x, u, X, U, X_x, X_u, U_x, U_u, X_{xx}, X_{xu}, \ \ldots\ )$$

Contact forms on $\mathcal{D}^{(2)}(\mathbb{R}^2)$:

$$\Upsilon = dX - X_x\, dx - X_u\, du \qquad \Phi = dU - U_x\, dx - U_u\, du$$

$$\Upsilon_x = dX_x - X_{xx}\, dx - X_{xu}\, du \qquad \Phi_x = dU_x - U_{xx}\, dx - U_{xu}\, du$$

$$\Upsilon_u = dX_u - X_{xu}\, dx - X_{uu}\, du \qquad \Phi_u = dU_u - U_{xu}\, dx - U_{uu}\, du$$

Maurer–Cartan forms:

$$\sigma = d_M X = X_x\, dx + X_u\, du, \qquad \widetilde{\sigma} = d_M U = U_x\, dx + U_u\, du,$$

$$\mu = \Upsilon = d_G X \qquad \widetilde{\mu} = \Phi = d_G U$$

$$\mu_X = \frac{U_u\, \Upsilon_x - U_x\, \Upsilon_u}{X_x U_u - X_u U_x} \qquad \widetilde{\mu}_X = \frac{U_u\, \Phi_x - U_x\, \Phi_u}{X_x U_u - X_u U_x}$$

$$\mu_U = \frac{X_x\, \Upsilon_u - X_u\, \Upsilon_x}{X_x U_u - X_u U_x} \qquad \widetilde{\mu}_U = \frac{X_x\, \Phi_u - X_u\, \Phi_x}{X_x U_u - X_u U_x}$$

Right-invariant differentiations:

$$\mathbb{D}_X = \frac{U_u\, \mathbb{D}_x - U_x\, \mathbb{D}_u}{X_x U_u - X_u U_x}\,, \qquad \mathbb{D}_U = \frac{-\, X_u\, \mathbb{D}_x + X_x\, \mathbb{D}_u}{X_x U_u - X_u U_x}\,.$$

## The Diffeomorphism Structure Equations

$$d\mu[\![\, H \,]\!] = \nabla_H \mu[\![\, H \,]\!] \wedge (\, \mu[\![\, H \,]\!] - dZ \,)$$

$$d\sigma = -\, d\mu[\![\, 0 \,]\!] = \nabla_H \mu[\![\, 0 \,]\!] \wedge \sigma$$

Invariant contact forms:

$$\mu[\![\, H \,]\!] = \Upsilon[\![\, h \,]\!] \qquad \text{when} \qquad H = Z[\![\, h \,]\!] - Z[\![\, 0 \,]\!]$$

Invariant horizontal forms:

$$\sigma = dZ - \mu[\![\, 0 \,]\!]$$

$$\Upsilon^a[\![\, h \,]\!] = \sum_J \frac{1}{J!}\, \Upsilon^a_J\, h^J \qquad \mu^a[\![\, H \,]\!] = \sum_J \frac{1}{J!}\, \mu^a_J\, H^J$$

Contact forms Maurer–Cartan forms

## One-dimensional case: $M = \mathbb{R}$

$$\sigma = X_x\, dx \qquad \begin{aligned} \Upsilon[\![\, h \,]\!] &= \Upsilon + \Upsilon_x\, h + \tfrac{1}{2}\,\Upsilon_{xx}\, h^2 + \cdots \\ \mu[\![\, H \,]\!] &= \mu + \mu_X\, H + \tfrac{1}{2}\,\mu_{XX}\, H^2 + \cdots \end{aligned}$$

Maurer–Cartan form series:

$$\mu[\![\, H \,]\!] = \Upsilon[\![\, h \,]\!]$$

when

$$H = X[\![\, h \,]\!] - X = X_x\, h + \tfrac{1}{2}\, X_{xx} h^2 + \ \cdots$$

Structure equations:

$$d\sigma = \mu_X \wedge \sigma, \qquad d\mu[\![\, H \,]\!] = \mu_H[\![\, H \,]\!] \wedge (\mu[\![\, H \,]\!] - dZ),$$

$$\mu_H[\![\, H \,]\!] = \mu_X + \mu_{XX}\, H + \tfrac{1}{2}\,\mu_{XXX}\, H^2 + \ \cdots$$

$$\mu[\![\, H \,]\!] - dZ = -\,\sigma + \mu_X\, H + \tfrac{1}{2}\,\mu_{XX}\, H^2 + \ \cdots$$

In components:

$$d\sigma = \mu_1 \wedge \sigma$$

$$\begin{aligned} d\mu_n &= -\,\mu_{n+1} \wedge \sigma \ + \sum_{i=0}^{n-1} \binom{n}{i}\, \mu_{i+1} \wedge \mu_{n-i} \\ &= \sigma \wedge \mu_{n+1} - \sum_{j=1}^{\left[\frac{n+1}{2}\right]} \frac{n-2j+1}{n+1} \binom{n+1}{j}\, \mu_j \wedge \mu_{n+1-j}. \end{aligned}$$

$\Longrightarrow$ Cartan

## Two-dimensional case: $M = \mathbb{R}^2$

Maurer–Cartan form series:

$$dX = \sigma + \mu \qquad\qquad dY = \widetilde{\sigma} + \widetilde{\mu}$$

$$\mu[\![ H, K ]\!] = \mu + \mu_H H + \mu_K K + + \tfrac{1}{2}\mu_{HH} H^2 + \mu_{HK} H K + \tfrac{1}{2}\mu_{KK} K^2 + \cdots ,$$

$$\widetilde{\mu}[\![ H, K ]\!] = \widetilde{\mu} + \widetilde{\mu}_H H + \widetilde{\mu}_K K + + \tfrac{1}{2}\widetilde{\mu}_{HH} H^2 + \widetilde{\mu}_{HK} H K + \tfrac{1}{2}\widetilde{\mu}_{KK} K^2 + \cdots ,$$

Structure equations:

$$\begin{pmatrix} d\mu[\![ H, K ]\!] \\ d\widetilde{\mu}[\![ H, K ]\!] \end{pmatrix} = \begin{pmatrix} \mu_H[\![ H, K ]\!] & \mu_K[\![ H, K ]\!] \\ \widetilde{\mu}_H[\![ H, K ]\!] & \widetilde{\mu}_K[\![ H, K ]\!] \end{pmatrix} \wedge \begin{pmatrix} \mu[\![ H, K ]\!] - dX \\ \widetilde{\mu}[\![ H, K ]\!] - dU \end{pmatrix}$$

First order structure equations:

$$d\mu = -\,d\sigma = -\,\mu_X \wedge \sigma - \mu_U \wedge \widetilde{\sigma},$$

$$d\widetilde{\mu} = -\,d\widetilde{\sigma} = -\,\widetilde{\mu}_X \wedge \sigma - \widetilde{\mu}_U \wedge \widetilde{\sigma},$$

$$d\mu_X = -\,\mu_{XX} \wedge \sigma - \mu_{XU} \wedge \widetilde{\sigma} + \mu_U \wedge \widetilde{\mu}_X,$$

$$d\widetilde{\mu}_X = -\,\widetilde{\mu}_{XX} \wedge \sigma - \widetilde{\mu}_{XU} \wedge \widetilde{\sigma} + \widetilde{\mu}_X \wedge (\mu_X - \widetilde{\mu}_U),$$

$$d\mu_U = -\,\mu_{XU} \wedge \sigma - \mu_{UU} \wedge \widetilde{\sigma} + (\mu_X - \widetilde{\mu}_U) \wedge \mu_U,$$

$$d\widetilde{\mu}_U = -\,\widetilde{\mu}_{XU} \wedge \sigma - \widetilde{\mu}_{UU} \wedge \widetilde{\sigma} + \widetilde{\mu}_X \wedge \mu_U$$

## The Structure Equations for a Pseudo-group

The Maurer–Cartan forms $\mu^{(\infty)}$ are obtained by restricting the diffeomorphism Maurer–Cartan forms to the pseudo-group $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$.

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**Theorem.** The right-invariant Maurer–Cartan forms satisfy the right-invariant linearized determining equations

$$L^{(n)}(Z, \mu^{(n)}) = 0, \qquad (*)$$

obtained from the infinitesimal determining equations

$$L^{(n)}(z, \zeta^{(n)}) = 0$$

by replacing

- source variables $z$ by target variables $Z$
- derivatives of vector field coefficients $\zeta^a_J$ by right-invariant Maurer–Cartan forms $\mu^a_J$

**Theorem.** The structure equations for the pseudo-group are obtained by restriction of the diffeomorphism structure equations

$$d\mu[\![\, H \,]\!] = \nabla_H \mu[\![\, H \,]\!] \wedge (\, \mu[\![\, H \,]\!] - dZ \,)$$

to the kernel of the linearized involutive system

$$L^{(n)}(Z, \mu^{(n)}) = 0.$$

## Example

$$\boxed{X = f(x) \qquad U = \frac{u}{f'(x)}}$$

Linearized determining system

$$\xi_x = -\frac{\varphi}{u} \qquad \xi_u = 0 \qquad \varphi_u = \frac{\varphi}{u}$$

Maurer–Cartan forms:

$$\sigma = \frac{u}{U}\, dx = f_x\, dx, \qquad \widetilde{\sigma} = U_x\, dx + \frac{U}{u}\, du = \frac{-\, u\, f_{xx}\, dx + f_x\, du}{f_x^2}$$

$$\mu = dX - \frac{U}{u}\, dx = df - f_x\, dx, \qquad \widetilde{\mu} = dU - U_x\, dx - \frac{U}{u}\, du = -\frac{u}{f_x^2}\,(\, df_x - f_{xx}\, dx\,)$$

$$\mu_X = \frac{du}{u} - \frac{dU - U_x\, dx}{U} = \frac{df_x - f_{xx}\, dx}{f_x}, \qquad \mu_U = 0$$

$$\widetilde{\mu}_X = \frac{U}{u}\,(dU_x - U_{xx}\, dx) - \frac{U_x}{u}\,(dU - U_x\, dx)$$

$$= -\frac{u}{f_x^3}(\, df_{xx} - f_{xxx}\, dx\,) + \frac{u\, f_{xx}}{f_x^4}(\, df_x - f_{xx}\, dx\,)$$

$$\widetilde{\mu}_U = -\frac{du}{u} + \frac{dU - U_x\, dx}{U} = -\frac{df_x - f_{xx}\, dx}{f_x}$$

Right-invariant linearized system:

$$\mu_X = -\frac{\widetilde{\mu}}{U} \qquad \mu_U = 0 \qquad \widetilde{\mu}_U = \frac{\widetilde{\mu}}{U}$$

First order structure equations:

$$d\mu = -\, d\sigma = \frac{\widetilde{\mu} \wedge \sigma}{U}, \qquad d\widetilde{\mu} = -\, \widetilde{\mu}_X \wedge \sigma \; - \; \frac{\widetilde{\mu} \wedge \widetilde{\sigma}}{U}$$

$$d\widetilde{\mu}_X = -\, \widetilde{\mu}_{XX} \wedge \sigma \; - \; \frac{\widetilde{\mu}_X \wedge (\widetilde{\sigma} + 2\, \widetilde{\mu})}{U}$$

# Action on Submanifolds

$\mathrm{J}^n = \mathrm{J}^n(M,p)$

— $n^{\text{th}}$ order jet bundle for $p$-dimensional submanifolds

Local coordinates

$$z^{(n)} = (x, u^{(n)}) = (\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ )$$

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## Variational Bicomplex

Horizontal forms:

$$dx^1, \ldots, dx^p$$

Contact forms:

$$\theta^\alpha_J = du^\alpha_J - \sum_{i=1}^{p} u^\alpha_{J,i}\, dx^i$$

Decompose the differential:

$$d = d_H + d_V$$

$$d_H\, F = \sum_{j=1}^{p} \mathrm{D}_{x^j} F\, dx^j \qquad d_V\, F = \sum_{\alpha,J} \frac{\partial F}{\partial u^\alpha_J}\, \theta^\alpha_J$$

$\mathrm{D}_{x^j}$ — total derivative

# Prolongation

The action of the diffeomorphism pseudo-group $\mathcal{D}$ on submanifolds $N \subset M$ induces an action of the jet groupoid $\mathcal{D}^{(n)}$ on $\mathrm{J}^n$ by prolongation.

- The prolonged group transformation formulae are obtained by implicit differentiation.

Pull-back diffeomorphism bundle:

$$\begin{array}{ccc} \mathcal{D}^{(n)} & \longleftarrow & \mathcal{E}^{(n)} \\ \downarrow & & \downarrow \\ M & \longleftarrow & \mathrm{J}^n. \end{array}$$

Local coordinates $\mathbf{g}^{(n)} = (z^{(n)}, g^{(n)})$ on $\mathcal{E}^{(n)}$:

$$z^{(n)} = (x, u^{(n)})$$

$$g^{(n)} = (z, Z^{(n)}) = (x, u, X^{(n)}, U^{(n)})$$

Groupoid — double fibration

$$\begin{array}{ccccc} & & \mathcal{E}^{(n)} & & \\ & \boldsymbol{\sigma}^{(n)} \swarrow & & \searrow \boldsymbol{\tau}^{(n)} & \\ \mathrm{J}^n & & & & \mathrm{J}^n. \end{array}$$

## Example $M = \mathbb{R}^2$

Local coordinates on $\mathrm{J}^\infty = \mathrm{J}^\infty(\mathbb{R}^2, 1)$:

$$z^{(\infty)} = (x, u^{(\infty)}) = (x, u, u_x, u_{xx}, \ \ldots \ )$$

Induced coordinates on $\mathcal{E}^{(\infty)} \to \mathrm{J}^\infty$:

$$(z^{(\infty)}, Z^{(\infty)}) =$$

$$(x, u, u_x, u_{xx}, \ldots, X, U, X_x, X_u, U_x, U_u, X_{xx}, X_{xu}, X_{uu}, U_{xx}, \ldots)$$

Total derivative operator on $\mathcal{E}^{(\infty)}$

$$\mathrm{D}_x = \mathbb{D}_x + u_x \, \mathbb{D}_u + u_{xx} \frac{\partial}{\partial u_x} + u_{xxx} \frac{\partial}{\partial u_{xx}} + \ \cdots \ ,$$

$$\mathbb{D}_x = \frac{\partial}{\partial x} + X_x \frac{\partial}{\partial X} + U_x \frac{\partial}{\partial U} + X_{xx} \frac{\partial}{\partial X_x} + U_{xx} \frac{\partial}{\partial U_x} + X_{xu} \frac{\partial}{\partial X_u} + U_{xu} \frac{\partial}{\partial U_u} + \cdots ,$$

$$\mathbb{D}_u = \frac{\partial}{\partial u} + X_u \frac{\partial}{\partial X} + U_u \frac{\partial}{\partial U} + X_{xu} \frac{\partial}{\partial X_x} + U_{xu} \frac{\partial}{\partial U_x} + X_{uu} \frac{\partial}{\partial X_u} + U_{uu} \frac{\partial}{\partial U_u} + \cdots .$$

Horizontal one-form:

$$d_H X = \mathrm{D}_x X \ dx = (X_x + u_x X_u) \, dx$$

Implicit differentiation operator:

$$\mathrm{D}_X = \frac{1}{X_x + u_x X_u} \, \mathrm{D}_x$$

Prolonged action of $\mathcal{D}^{(\infty)}$ on $\mathrm{J}^\infty$:

$$U_X = \mathrm{D}_X U = \frac{\mathrm{D}_x U}{\mathrm{D}_x X} = \frac{U_x + u_x U_u}{X_x + u_x X_u}$$

$$U_{XX} = \mathrm{D}_X^2 U = \frac{\mathrm{D}_x^2 U \, \mathrm{D}_x X - \mathrm{D}_x U \, \mathrm{D}_x^2 X}{(\mathrm{D}_x X)^3}$$

# Moving Frames for Pseudo–Groups

Pull-back pseudo-group jet bundle:

$$\begin{array}{ccc} \mathcal{G}^{(n)} & \longleftarrow & \mathcal{H}^{(n)} \\ \downarrow & & \downarrow \\ M & \longleftarrow & \mathrm{J}^n. \end{array}$$

$\Longrightarrow$ In the finite-dimensional Lie group situation,

$$\mathcal{H}^{(n)} \sim \mathrm{J}^n \times G$$

**Definition.** A (right) *moving frame* of *order* $n$ is a right-equivariant section $\rho^{(n)} : \mathcal{V}^n \to \mathcal{H}^{(n)}$ defined on an open subset $\mathcal{V}^n \subset \mathrm{J}^n$.

**Proposition.** A moving frame of order $n$ exists if and only if $\mathcal{G}^{(n)}$ acts *freely* and regularly on an open subset of $\mathrm{J}^n$.

# Freeness

Isotropy subgroup

$$\mathcal{S}^{(n)}_{z^{(n)}} = \left\{ g^{(n)} \in \mathcal{G}^{(n)}_z \;\middle|\; g^{(n)} \cdot z^{(n)} = z^{(n)} \right\}$$

**Definition.** The pseudo-group $\mathcal{G}$ acts

- *freely* at $z^{(n)} \in \mathrm{J}^n$ if $\mathcal{S}^{(n)}_{z^{(n)}} = \{ \mathbf{1}^{(n)}_z \}$
- *locally freely* if $\mathcal{S}^{(n)}_{z^{(n)}}$ is a discrete subgroup of $\mathcal{G}^{(n)}_z$

**Theorem.** If $\mathcal{G}^{(n)}$ acts locally freely on an open subset of $\mathrm{J}^n$ for $n \gg 0$ then it acts locally freely on an open subset of $\mathrm{J}^k$ for all $k > n$.

$\Longrightarrow$ Gröbner basis for the symbol module

**Theorem.** Suppose $\mathcal{G}^{(n)}$ acts freely and regularly on $\mathcal{V}^n \subset \mathrm{J}^n$. Let $\mathcal{K}^n \subset \mathcal{V}^n$ be a (local) cross-section to the pseudo-group orbits. Given $z^{(n)} \in \mathcal{V}^n$, if $\rho^{(n)}(z^{(n)}) \in \mathcal{E}^{(n)}$ denotes the unique groupoid jet such that

$$I^{(n)}(z^{(n)}) = \boldsymbol{\tau}^{(n)}(\rho^{(n)}(z^{(n)})) \in \mathcal{K}^n.$$

Then $\rho^{(n)} : \mathcal{V}^n \to \mathcal{E}^{(n)}$ is a moving frame for $\mathcal{G}$. The local cross-section coordinates of $I^{(n)}(z^{(n)})$ provide a complete system of functionally independent $n^{\text{th}}$ order differential invariants for the pseudo-group action.

# First Example

$$\boxed{X = f(x), \qquad Y = y, \qquad U = \frac{u}{f'(x)}}$$

Horizontal coframe

$$d_H X = f_x \, dx, \qquad d_H Y = dy,$$

Implicit differentiations

$$\mathrm{D}_X = \frac{1}{f_x} \, \mathrm{D}_x, \qquad \mathrm{D}_Y = \mathrm{D}_y.$$

Prolonged pseudo-group transformations on surfaces $S \subset \mathbb{R}^3$

$$X = f \qquad Y = y \qquad U = \frac{u}{f_x}$$

$$U_X = \frac{u_x}{f_x^2} - \frac{u\, f_{xx}}{f_x^3} \qquad U_Y = \frac{u_y}{f_x}$$

$$U_{XX} = \frac{u_{xx}}{f_x^3} - \frac{3 u_x\, f_{xx}}{f_x^4} - \frac{u\, f_{xxx}}{f_x^4} + \frac{3u\, f_{xx}^2}{f_x^5}$$

$$U_{XY} = \frac{u_{xy}}{f_x^2} - \frac{u_y\, f_{xx}}{f_x^3} \qquad U_{YY} = \frac{u_{yy}}{f_x}$$

$\Longrightarrow$ action is free at every order.

Coordinate cross-section

$$X = 0, \qquad U = 1, \qquad U_X = 0, \qquad U_{XX} = 0.$$

Moving frame

$$f = 0, \qquad f_x = u, \qquad f_{xx} = u_x, \qquad f_{xxx} = u_{xx}.$$

Differential invariants

$$U_Y \longmapsto J = \frac{u_y}{u}$$

$$U_{XY} \longmapsto J_1 = \frac{u u_{xy} - u_x u_y}{u^3} \qquad\qquad U_{YY} \longmapsto J_2 = \frac{u_{yy}}{u}$$

Horizontal invariant coframe

$$d_H X \longmapsto u\,dx, \qquad\qquad d_H Y \longmapsto dy,$$

Invariant differentiations

$$\mathcal{D}_1 = \frac{1}{u}\,\mathrm{D}_x \qquad\qquad \mathcal{D}_2 = \mathrm{D}_y$$

Higher order differential invariants: $\mathcal{D}_1^m\,\mathcal{D}_2^n\,J$

$$J_{,1} = \mathcal{D}_1 J = \frac{u u_{xy} - u_x u_y}{u^3} = J_1,$$

$$J_{,2} = \mathcal{D}_2 J = \frac{u u_{yy} - u_y^2}{u^2} = J_2 - J^2.$$

# Taylor Series Method

$$X = f(x), \qquad Y = y, \qquad U = \frac{u}{f'(x)}$$

Taylor parameters $\qquad f_n = f_{nx}$

$$H = f[\![\, h \,]\!] = \sum_{n=0}^{\infty} \frac{1}{n!}\, f_n\, h^n, \qquad K = k.$$

Taylor expansions for prolonged pseudo-group action

$$X = f[\![\, h \,]\!] \qquad Y = y$$

$$U[\![\, H, K \,]\!] = \frac{u[\![\, h, k \,]\!]}{f_h[\![\, h \,]\!]} = \frac{u[\![\, f^{-1}[\![\, H \,]\!], K \,]\!]}{f_x[\![\, f^{-1}[\![\, H \,]\!] \,]\!]} = \sum_{m,n} \frac{1}{m!\, n!}\, U_{m,n}\, H^m\, K^n$$

Normalization equations

$$U[\![\, H, 0 \,]\!] = 1 \qquad \text{or} \qquad u[\![\, h, 0 \,]\!] = f_h[\![\, h \,]\!]$$

Solution:

$$H = f[\![\, h \,]\!] = \int u[\![\, h, 0 \,]\!]\, dh \quad \Longrightarrow \quad f_m = u_{m-1,0}$$

Differential invariants

$$U[\![\, H, K \,]\!] \quad \longmapsto \quad I[\![\, H, K \,]\!] = 1 + K\, J[\![\, H, K \,]\!],$$

where

$$J[\![\, H, K \,]\!] = \sum_{m,n \geq 0} J_{m,n} \, \frac{H^m\, K^n}{m!\,(n+1)!} = \frac{u[\![\, h, k \,]\!] - u[\![\, h, 0 \,]\!]}{k\, u[\![\, h, 0 \,]\!]}$$

when

$$H = \int u[\![\, h, 0 \,]\!]\, dh = u\, h + \tfrac{1}{2}\, u_x\, h^2 + \tfrac{1}{6}\, u_{xx}\, h^3 + \cdots$$

$\Longrightarrow$ The Taylor coefficients $J_{m,n} = I_{m,n+1}$ are the fundamental differential invariants.

Explicitly,

$$J[\![\, H, K \,]\!] = \frac{u_y}{u} + \frac{u u_{xy} - u_x u_y}{u^3}\, H + \frac{u_{yy}}{2u}\, K$$
$$+ \frac{u^2 u_{xxy} - u u_y u_{xx} - 3 u u_x u_{xy} + 3 u_x^2 u_y}{2u^5}\, H^2$$
$$+ \frac{u u_{xyy} - u_x u_{yy}}{2u^3}\, H\, K + \frac{u_{yyy}}{6u}\, K^2 + \cdots \; .$$

# Second Example

$$X = f, \qquad Y = f_x\, y + g, \qquad U = u + \frac{f_{xx}\, y + g_x}{f_x}\,.$$

Horizontal coframe

$$d_H X = f_x\, dx, \qquad d_H Y = (f_{xx}\, y + g_x)\, dx + f_x\, dy.$$

Implicit differentiation

$$\mathrm{D}_X = \frac{1}{f_x}\,\mathrm{D}_x - \frac{f_{xx}\, y + g_x}{f_x^2}\,\mathrm{D}_y, \qquad \mathrm{D}_Y = \frac{1}{f_x}\,\mathrm{D}_y,$$

Prolonged pseudo-group transformations on surfaces $S \subset \mathbb{R}^3$

$$\begin{aligned}
U_X &= \frac{u_x}{f_x} + \frac{f_{xxx}\, y + g_{xx} - (f_{xx}\, y + g_x)u_y}{f_x^2} - 2\,\frac{f_{xx}(f_{xx}\, y + g_x)}{f_x^3} \\
U_Y &= \frac{u_y}{f_x} + \frac{f_{xx}}{f_x^2} \\
U_{XX} &= \cdots \\
U_{XY} &= \frac{u_{xy}}{f_x^2} - \frac{f_{xx}u_y + (f_{xx}\, y + g_x)u_{yy}}{f_x^3} + \frac{f_{xxx}}{f_x^3} - 2\,\frac{f_{xx}^2}{f_x^4} \\
U_{YY} &= \frac{u_{yy}}{f_x^2}
\end{aligned}$$

$\Longrightarrow$ free at every order $\geq 2$.

Normalizations

$$
\begin{aligned}
X &= 0, & f &= 0, \\
Y &= 0, & g &= -\,y\,f_x, \\
U &= 0, & g_x &= -\,u\,f_x - y\,f_{xx}, \\
U_Y &= 0, & f_{xx} &= -\,u_y\,f_x, \\
U_X &= 0, & g_{xx} &= -\,u_x\,f_x - y\,f_{xxx}, \\
U_{YY} &= 1, & f_x &= -\sqrt{u_{yy}} \\
U_{XY} &= 0, & f_{xxx} &= \sqrt{u_{yy}}\,(\,u_{xy} + u u_{yy} - u_y^2\,), \\
U_{XX} &= 0, & g_{xxx} &= \sqrt{u_{yy}}\,(\,u_{xx} - u u_{xy} - 2u^2 u_{yy} - 2u_x u_y + u u_y^2\,) \\
& & &\qquad - y\,f_{xxxx}, \\
U_{XXY} &= 0, & f_{xxxx} &= \cdots, \\
U_{XXX} &= 0, & g_{xxx} &= \cdots.
\end{aligned}
$$

Differential invariants

$$
\begin{aligned}
U_{XYY} &\longmapsto J_1 = -\,\frac{u u_{yyy} + u_{xyy} + 2u_y u_{yy}}{u_{yy}^{3/2}} \\
U_{YYY} &\longmapsto J_2 = -\,\frac{u_{yyy}}{u_{yy}^{3/2}}\,, \\
U_{XXYY} &\longmapsto J_3, \\
U_{XYYY} &\longmapsto J_4, \\
U_{YYYY} &\longmapsto J_5.
\end{aligned}
$$

Invariant coframe

$$d_H X \quad \longmapsto \quad \omega^1 = -\sqrt{u_{yy}}\, dx,$$

$$d_H Y \quad \longmapsto \quad \omega^2 = -\sqrt{u_{yy}}\,(dy - u\,dx).$$

Invariant differential operators

$$\mathcal{D}_1 = \frac{1}{\sqrt{u_{yy}}}(\mathrm{D}_x + u\,\mathrm{D}_y), \qquad \mathcal{D}_2 = \frac{1}{\sqrt{u_{yy}}}\mathrm{D}_y.$$

Higher order differential invariants

$$\mathcal{D}_1^m\,\mathcal{D}_2^n\,J_1, \qquad \mathcal{D}_1^m\,\mathcal{D}_2^n\,J_2.$$

$\Longrightarrow$ Syzygies come from the recurrence formulae.

# Taylor Series Method

$$\boxed{X = f, \qquad Y = f_x\, y + g, \qquad U = u + \frac{f_{xx}\, y + g_x}{f_x},}$$

Taylor expansions

$$X = f[\![\, h \,]\!], \qquad\qquad Y = y + f_h[\![\, h \,]\!](\, k - a[\![\, h \,]\!]\,),$$

$$U[\![\, H, K \,]\!] = u[\![\, h, k \,]\!] + \frac{f_{hh}[\![\, h \,]\!]}{f_h[\![\, h \,]\!]}\,(\, k - a[\![\, h \,]\!]\,) - a_h[\![\, h \,]\!],$$

when

$$H = f[\![\, h \,]\!], \qquad K = y + f_h[\![\, h \,]\!]\,(\, k - a[\![\, h \,]\!]\,).$$

Also

$$a(x+h) = -\,y - \frac{g(x+h)}{f_x(x+h)}$$

Normalizations

$$U[\![\, H, 0 \,]\!] = 0, \qquad U_H[\![\, H, 0 \,]\!] = 0, \qquad U_{HH}[\![\, 0, 0 \,]\!] = 1.$$

Solution to first normalization equations

$$U[\![\, H, 0 \,]\!] = 0 \qquad \Longrightarrow \qquad u[\![\, h, a[\![\, h \,]\!] \,]\!] = a_h[\![\, h \,]\!].$$

First order nonlinear ODE

$$a_0 = 0, \quad a_1 = u, \quad a_2 = u_x + u\, u_y, \quad \ldots \quad a_k = (\mathrm{D}_x + u\,\mathrm{D}_y)^{j-1} u.$$

Solution to second normalization equations

$$U_H[\![\, H, 0 \,]\!] = 0 \quad \Longrightarrow \quad f_{hh}[\![\, h \,]\!] = -\, u_y[\![\, h, a[\![\, h \,]\!] \,]\!] \, f_h[\![\, h \,]\!].$$

First order linear ODE

$$f_0 = 0, \quad f_2 = -u_y\, f_1, \quad f_3 = (\, u_{xy} + u u_{yy} - u_y^2 \,) f_1, \quad \ldots$$

$$f_k = f_1\, (\mathrm{D}_x + u\,\mathrm{D}_y - u_y)^{k-1}(1) \qquad\qquad k \geq 2.$$

Solution to third normalization equation

$$U_{HH}[\![\, 0, 0 \,]\!] = 1 \quad \Longrightarrow \quad u_{yy} = f_1^2$$

so that $$f_1 = \sqrt{u_{yy}}$$

$$f_m = \sqrt{u_{yy}}\, (\mathrm{D}_x + u\,\mathrm{D}_y - u_y)^{m-1}(1), \qquad a_m = (\mathrm{D}_x + u\,\mathrm{D}_y)^{m-1} u,$$

Fundamental differential invariants

$$U[\![\, H, K \,]\!] \quad \longmapsto \quad K^2\, (\, 1 + J[\![\, H, K \,]\!] \,)$$

$$J[\![\, H, K \,]\!] = -\, \frac{u u_{yyy} + u_{xyy} + 2 u_y u_{yy}}{2 u_{yy}^{3/2}}\, H \; - \; \frac{u_{yyy}}{6 u_{yy}^{3/2}}\, K + \; \cdots$$

# Recurrence Formulae

The *recurrence formulae* serve to connect invariantly differentiated differential invariants and differential forms with their higher order normalized counterparts.

$\Longrightarrow$ Using moving frames, they can be found using only the infinitesimal generators and linear differential algebra

**Applications:**

- Determination of a minimal basis $I^1, \ldots, I^\ell$ of differential invariants:

$$I^\alpha_{,K} = \mathcal{D}_K I^\alpha$$

- Commutation formulae for the invariant differential operators:

$$[\, \mathcal{D}_j, \mathcal{D}_k \,] = \sum_{i=1}^{p} C^i_{j,k}\, \mathcal{D}_i$$

- Syzygies (functional relations) among differentiated invariants:

$$\Phi( \ \ldots \ \mathcal{D}_K I^\alpha \ \ldots \ ) \equiv 0$$

- Equivalence and signatures of submanifolds
- Computation of invariant variational problems:

$$\int L( \ \ldots \ \mathcal{D}_K I^\alpha \ \ldots \ )\, \boldsymbol{\omega}$$

# Lifted Forms

**Definition.** The *lift* of a differential form $\omega$ on $\mathrm{J}^\infty$ is

$$\Omega = \boldsymbol{\lambda}(\omega) = \pi_J((\boldsymbol{\tau}^{(\infty)})^*\omega).$$

$\Longrightarrow$ Right-invariant form on $\mathcal{E}^{(\infty)} \to \mathrm{J}^\infty$.

Differentials:

$$d_J\,\Omega = \boldsymbol{\lambda}(d\omega) \qquad d_G\,\Omega = \mathbf{v}_\Psi^{(\infty)}(\Omega)$$

Lifted infinitesimal generator:

$$\mathbf{v}_\Psi^{(\infty)} = \sum_{i=1}^{p} \Xi^i \frac{\partial}{\partial X^i} + \sum_{\alpha=1}^{q} \sum_{\#K \geq 0} \Psi_K^\alpha \frac{\partial}{\partial U_K^\alpha},$$

$$\Xi^i = d_G\,X^i \qquad \Psi_K^\alpha = d_G\,U_K^\alpha$$

The coefficients of $\mathbf{v}_\Psi^{(\infty)}$ are obtained from the coefficients $\varphi_K^\alpha$ of a prolonged vector field by replacing:

- source jet variables $u_J^\alpha$ by their lifts $U_J^\alpha$
  (prolonged transformation formulae)
- derivatives of vector field coefficients $\zeta_J^a$ by
  right-invariant Maurer–Cartan forms $\mu_J^a$

# Invariantization

**Definition.** If $\Omega$ is any differential form on $\mathrm{J}^\infty$, then its *invariantization* is the invariant differential form

$$\iota(\Omega) = (\rho^{(\infty)})^* \, [\, \boldsymbol{\lambda}\,(\,\Omega\,)\,] = (\rho^{(\infty)})^* \, [\, \pi_J((\boldsymbol{\tau}^{(\infty)})^* \, \Omega)\,]$$

$\Longrightarrow$ Invariantization defines a projection from functions and forms to invariant functions and forms.

Invariantization of the jet coordinates and basis one-forms gives the fundamental differential invariants and invariant one-forms:

$$J^i = \iota(x^i) \qquad\qquad \varpi^i = \iota(dx^i) = \omega^i + \eta^i$$

$$I^\alpha_J = \iota(u^\alpha_J) \qquad\qquad \vartheta^\alpha_J = \iota(\theta^\alpha_J)$$

$\Longrightarrow$ invariant coframe

Pulled-back Maurer–Cartan forms:

$$\nu^{(\infty)} = (\rho^{(\infty)})^* \, \mu^{(\infty)}.$$

$\Longrightarrow$ These are determined from the recurrence formulae

## Fundamental Recurrence Formula

$$d\iota(\Omega) \;=\; \iota \left[\; d\Omega + \mathbf{v}_\psi^{(\infty)}(\Omega) \right]$$

$$\mathbf{v}_\psi^{(\infty)} = \sum_{i=1}^{p} \eta^i \frac{\partial}{\partial x^i} + \sum_{\alpha,K} \psi_K^\alpha \frac{\partial}{\partial u_J^\alpha}$$

$$\text{where} \qquad \eta^i = (\rho^{(\infty)})^* \, \Xi^i \qquad\qquad \psi_K^\alpha = (\rho^{(\infty)})^* \, \Psi_K^\alpha$$

The coefficients of $\mathbf{v}_\psi^{(\infty)}$ are obtained from the coefficients $\varphi_K^\alpha$ of a prolonged vector field by replacing:

- source jet variables $u_J^\alpha$ by their invariantizations $I_J^\alpha = \iota(u_J^\alpha)$
- derivatives of vector field coefficients $\zeta_J^a$ by the pull-backs $\nu_J^\alpha$ of the Maurer–Cartan forms

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- All recurrence formulae for differential invariants and invariant differential forms follow from specialization.
- In particular, the formulae for the pull-backs of the Maurer–Cartan forms $\nu_J^\alpha$ are found by solving the recurrence formula for the phantom differential invariants.

# First Example

$$X = f(x), \qquad Y = y, \qquad U = \frac{u}{f'(x)}$$

Prolonged infinitesimal generator:

$$\begin{aligned}\mathbf{v}^{(\infty)} = a\,\partial_x - u\,a_x\,\partial_u - (\,u\,a_{xx} + 2\,u_x\,a_x\,)\,\partial_{u_x} - u_y\,a_x\,\partial_{u_y} - \\ - (\,u\,a_{xxx} + 3\,u_x\,a_{xx} + 3\,u_{xx}\,a_x\,)\,\partial_{u_{xx}} - \\ - \left(u_y\,a_{xx} + 2\,u_{xy}\,a_x\right)\partial_{u_{xy}} - u_{yy}\,a_x\,\partial_{u_{yy}} - \cdots\end{aligned}$$

Lifted infinitesimal generator:

$$\begin{aligned}\mathbf{v}_\Psi^{(\infty)} = \mu\,\partial_X - U\,\mu_X\,\partial_U - \\ - (U\,\mu_{XX} + 2\,U_X\,\mu_X)\,\partial_{U_X} - U_Y\,\mu_X\,\partial_{U_Y} - \cdots\end{aligned}$$

Normalization

$$X = 0, \quad U = 1, \quad U_X = 0, \quad U_{XX} = 0, \quad \ldots.$$

Fundamental differential invariants:

$$\iota(x) = F = 0, \quad \iota(y) = y, \quad \iota(u) = I_{00} = 1,$$

$$\iota(u_x) = I_{10} = 0, \quad \iota(u_y) = I_{10} = J,$$

$$\iota(u_{xx}) = I_{20} = 0, \quad \iota(u_{xy}) = I_{11} = J_1, \quad \iota(u_{yy}) = I_{02} = J_2,$$

Invariant horizontal forms:

$$\varpi^1 = \iota(dx) = u\,dx, \qquad \varpi^2 = \iota(dy) = dy,$$

Invariantized contact forms:

$$\vartheta = \iota(\theta), \quad \vartheta_1 = \iota(\theta_x), \quad \vartheta_2 = \iota(\theta_y), \quad \ldots$$

Recurrence formulae:

$$\mathbf{v}_\psi^{(\infty)} = \nu\,\partial_x - u\,\nu_1\,\partial_u - (\,u\,\nu_2 + 2\,u_x\,\nu_1\,)\,\partial_{u_x} - u_y\,\nu_1\,\partial_{u_y} - \cdots$$

Phantom invariants:

$$0 = dH = \varpi^1 + \nu, \qquad 0 = dI_{10} = J_1\, \varpi^2 + \vartheta_1 - \nu_2,$$

$$0 = dI_{00} = J\, \varpi^2 + \vartheta - \nu_1, \qquad 0 = dI_{20} = J_3\, \varpi^2 + \vartheta_3 - \nu_3,$$

Solve for pulled-back Maurer–Cartan forms:

$$\nu = -\,\varpi^1, \qquad \nu_2 = J_1\, \varpi^2 + \vartheta_1,$$

$$\nu_1 = J\, \varpi^2 + \vartheta, \qquad \nu_3 = J_3\, \varpi^2 + \vartheta_3,$$

Recurrence formulae: $dy = \varpi^2$

$$dJ = J_1\, \varpi^1 + (J_2 - J^2)\, \varpi^2 + \vartheta_2 - J\, \vartheta,$$

$$dJ_1 = J_3\, \varpi^1 + (J_4 - 3\, J\, J_1)\, \varpi^2 + \vartheta_4 - J\, \vartheta_1 - J_1\, \vartheta,$$

$$dJ_2 = J_4\, \varpi^1 + (J_5 - J\, J_2)\, \varpi^2 + \vartheta_5 - J_2\, \vartheta,$$

$$\mathcal{D}_1 J = J_1, \quad \mathcal{D}_2 J = J_2 - J^2, \qquad d_{\mathcal{V}}\, J = \vartheta_2 - J\, \vartheta,$$

$$\mathcal{D}_1 J_1 = J_3, \quad \mathcal{D}_2 J_1 = J_4 - 3\, J\, J_1, \qquad d_{\mathcal{V}}\, J_1 = \vartheta_4 - J\, \vartheta_{10} - J_1\, \vartheta,$$

$$\mathcal{D}_1 J_2 = J_4, \quad \mathcal{D}_2 J_2 = J_5 - J\, J_2, \qquad d_{\mathcal{V}}\, J_2 = \vartheta_5 - J_2\, \vartheta,$$

$\Longrightarrow$ All higher order differential invariants are obtained from $J$ by invariant differentiation

Invariant horizontal forms

$$d\varpi^1 = -\, J\, \varpi^1 \wedge \varpi^2 + \vartheta \wedge \varpi^1, \qquad d\varpi^2 = 0.$$

Commmutation formula

$$[\, \mathcal{D}_1, \mathcal{D}_2\, ] = J\, \mathcal{D}_1.$$

Invariant contact forms

$$d\vartheta = \varpi^1 \wedge \vartheta_1 + \varpi^2 \wedge (\, \vartheta_2 - J\, \vartheta\, ).$$

$$\mathcal{D}_1 \vartheta = \vartheta_1, \qquad \mathcal{D}_2 \vartheta = \vartheta_2 - J\, \vartheta,$$

and so on ...

# Taylor Series Method

Differential invariant series

$$\iota(u[\![ h, k ]\!]) = I[\![ H, K ]\!] = 1 + K\, J[\![ H, K ]\!].$$

Recurrence formulae

$$dI[\![ H, K ]\!] = \vartheta[\![ H, K ]\!] + I_H[\![ H, K ]\!]\,\varpi^1 + I_K[\![ H, K ]\!]\,\varpi^2 - \\ - \partial_H (I[\![ H, K ]\!]\,\widetilde{\nu}[\![ H ]\!]),$$

Pulled-back Maurer–Cartan forms:

$$\widetilde{\nu}[\![ H ]\!] = \nu[\![ H ]\!] - \nu[\![ 0 ]\!] = \nu_1\, H + \tfrac{1}{2}\nu_2\, H^2 + \ \cdots$$

Normalization:

$$I[\![ H, 0 ]\!] = 1 \quad \Longrightarrow \quad I_H[\![ H, 0 ]\!] = 0, \qquad dI[\![ H, 0 ]\!] = 0.$$

$$\widetilde{\nu}_H[\![ H ]\!] = I_K[\![ H, 0 ]\!]\,\varpi^2 + \vartheta[\![ H, 0 ]\!]$$

$$\widetilde{\nu}[\![ H ]\!] = \int (\, I_K[\![ H, 0 ]\!]\,\varpi^2 + \vartheta[\![ H, 0 ]\!]\,)\, dH$$

Complete system of recurrence formulae:

$$d_{\mathcal{H}}\, I[\![ H, K ]\!] = I_H[\![ H, K ]\!]\,\varpi^1 + \left[ I_K[\![ H, K ]\!] - \frac{\partial}{\partial H}\left( I[\![ H, K ]\!] \int I_K[\![ H, 0 ]\!]\, dH \right) \right] \varpi^2,$$

or, in components,

$$\mathcal{D}_1 I_{jk} = I_{j+1,k}, \qquad \mathcal{D}_2 I_{jk} = I_{j,k+1} - \sum_{i=0}^{j} \binom{j+1}{i} I_{ik}\, I_{j-i,1}.$$

Also

$$d_{\mathcal{V}}\, I[\![ H, K ]\!] = \vartheta[\![ H, K ]\!] - \frac{\partial}{\partial H}\left( I[\![ H, K ]\!] \int \vartheta[\![ H, 0 ]\!]\, dH \right),$$

or, in components,

$$d_{\mathcal{V}}\, I_{jk} = \vartheta_{j,k} - \sum_{i=0}^{j} \binom{j+1}{i} I_{ik}\, \vartheta_{j-i,0}.$$