Moving Frames and Differential Invariants of Lie Pseudo–Groups

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Sur la théorie, si importante sans doute, mais pour nous si obscure, des ≪groupes de Lie infinis≫, nous ne savons rien que ce qui trouve dans les mémoires de Cartan, premiere exploration à travers une jungle presque impénétrable; mais cell-ci menace de refermer sur les sentiers déjà tracés, si l'on ne procède bientôt à un indispensable travail de défrichement.

— André Weil, 1947

Pseudo-groups in Action

- Lie — Medolaghi — Vessiot
- Cartan ... Guillemin, Sternberg
- Kuranishi, Spencer, Goldschmidt, Kumpera, ...
- Relativity
- Noether's Second Theorem
- Gauge theory and field theories
  Maxwell, Yang–Mills, conformal, string, ...
- Fluid Mechanics, Metereology
  Euler, Navier–Stokes,
  boundary layer, quasi-geostropic , ...
- Linear and linearizable PDEs
- Solitons (in $2 + 1$ dimensions)
  K–P, Davey-Stewartson, ...
- Kac–Moody
- *Lie groups*!

What's New?

Direct constructive algorithms for:

- Invariant Maurer–Cartan forms
- Structure equations
- Moving frames
- Differential invariants
- Invariant differential operators
- Basis Theorem
- Syzygies and recurrence formulae
- Further applications

$\Longrightarrow$ Symmetry groups of differential equations $\Longrightarrow$ Vessiot group splitting
$\Longrightarrow$ Gauge theories
$\Longrightarrow$ Calculus of variations

Symmetry Groups — Review

System of differential equations:

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, 2, \ldots, k$$

Prolonged vector field:

$$\mathbf{v}^{(n)} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\#J=0}^{n} \varphi_\alpha^J \frac{\partial}{\partial u_J^\alpha}$$

where

$$\varphi_\alpha^J = D_J \left(\varphi^\alpha - \sum_{i=1}^{p} u_i^\alpha \xi^i \right) + \sum_{i=1}^{p} u_{J,i}^\alpha \xi^i$$

$$\equiv \Phi_\alpha^J(x, u^{(n)}; \xi^{(n)}, \varphi^{(n)})$$

Infinitesimal invariance:

$$\mathbf{v}^{(n)}(\Delta_\nu) = 0 \qquad \text{whenever} \qquad \Delta = 0.$$

Infinitesimal determining equations:

$$\mathcal{L}(x, u; \xi^{(n)}, \varphi^{(n)}) = 0$$

$$\mathcal{L}(\ \ldots\ , x^i,\ \ldots\ , u^\alpha,\ \ldots\ , \xi_A^i,\ \ldots\ , \varphi_A^\alpha,\ \ldots\ ) = 0$$

$\Longrightarrow$ involutive completion

The Korteweg–deVries equation

$$\boxed{u_t + u_{xxx} + u u_x = 0}$$

Symmetry generator:

$$\mathbf{v} = \tau(t,x,u)\,\frac{\partial}{\partial t} + \xi(t,x,u)\,\frac{\partial}{\partial x} + \varphi(t,x,u)\,\frac{\partial}{\partial u}$$

Prolongation:

$$\mathbf{v}^{(3)} = \mathbf{v} + \varphi^t\,\frac{\partial}{\partial u_t} + \varphi^x\,\frac{\partial}{\partial u_x} + \;\cdots\; + \varphi^{xxx}\,\frac{\partial}{\partial u_{xxx}}$$

where

$$\begin{aligned}
\varphi^t &= \varphi_t + u_t\varphi_u - u_t\tau_t - u_t^2\tau_u - u_x\xi_t - u_t u_x \xi_u \\
\varphi^x &= \varphi_x + u_x\varphi_u - u_t\tau_x - u_t u_x\tau_u - u_x\xi_x - u_x^2\xi_u \\
\varphi^{xxx} &= \varphi_{xxx} + 3\,u_x\varphi_u + \;\cdots
\end{aligned}$$

Infinitesimal invariance:

$$\mathbf{v}^{(3)}(u_t + u_{xxx} + u u_x) = \varphi^t + \varphi^{xxx} + u\,\varphi^x + u_x\,\varphi = 0$$

on solutions

$$\boxed{u_t + u_{xxx} + u u_x = 0}$$

Infinitesimal determining equations:

$$\tau_x = \tau_u = \xi_u = \varphi_t = \varphi_x = 0$$

$$\varphi = \xi_t - \tfrac{2}{3}\, u\tau_t \qquad \varphi_u = -\tfrac{2}{3}\,\tau_t = -\,2\,\xi_x$$

$$\tau_{tt} = \tau_{tx} = \tau_{xx} = \ \cdots \ = \varphi_{uu} = 0$$

General solution:

$$\tau = c_1 + 3\,c_4 t, \qquad \xi = c_2 + c_3 t + c_4 x, \qquad \varphi = c_3 - 2 c_4 u.$$

Basis for symmetry algebra:

$$\partial_t, \qquad \partial_x, \qquad t\,\partial_x + \partial_u, \qquad 3\,t\,\partial_t + x\,\partial_x - 2\,u\,\partial_u.$$

The symmetry group $\mathcal{G}_{KdV}$ is four-dimensional

$$(x, t, u) \ \longmapsto \ (\lambda^3 t + a, \lambda x + c\,t + b, \lambda^{-2} u + c)$$

Differential Invariants

$\mathcal{G}$ — transformation group acting on p-dimensional submanifolds $N = \{ u = f(x) \} \subset M$

Differential invariant $\qquad I : \mathrm{J}^n \to \mathbb{R}$

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

$\Longrightarrow$ curvature, torsion, ...

Invariant differential operators:

$$\mathcal{D}_1, \ldots, \mathcal{D}_p$$

$\Longrightarrow$ arc length derivative

$\star\star$ $\mathcal{I}(\mathcal{G})$ — the algebra of differential invariants $\star\star$

Tresse's Theorem. $\mathcal{I}(\mathcal{G})$ is generated by a finite number of differential invariants $I_1, \ldots, I_\ell$, meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

$$\mathcal{D}_J I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa.$$

$\Longrightarrow$ *Kumpera*

Main Goals

Given a system of partial differential equations:

- Find the structure of its symmetry (pseudo-) group $\mathcal{G}$
  directly from the determining equations.
- Find and classify its differential invariants.
- Determine the structure of the
  differential invariant algebra $\mathcal{I}(\mathcal{G})$:

  ♢ generating invariants: $I_1, \ldots, I_\ell$

  ♢ invariant differential operators: $\mathcal{D}_1, \ldots, \mathcal{D}_p$ $\Longrightarrow$
  commutation relations

  ♢ syzygies: $H(\ \ldots \ \mathcal{D}_J I_\kappa \ \ldots \ ) \equiv 0$

  $\Longrightarrow$ Gauss–Codazzi relations

Pseudo-groups

Definition. A *pseudo-group* is a collection of local diffeomorphisms $\varphi\colon M \to M$ such that

- *Identity*: $\mathbf{1}_M \in \mathcal{G}$,
- *Inverses*: $\varphi^{-1} \in \mathcal{G}$,
- *Restriction*: $U \subset \operatorname{dom} \varphi \implies \varphi \mid U \in \mathcal{G}$,
- *Composition*: $\operatorname{im} \varphi \subset \operatorname{dom} \psi \implies \psi \circ \varphi \in \mathcal{G}$.

Definition. A *Lie pseudo-group* $\mathcal{G}$ is a pseudo-group whose transformations are the solutions to an involutive system of partial differential equations:

$$F(z, \varphi^{(n)}) = 0.$$

$\star$ Nonlinear determining equations

$\implies$ *analytic* (*Cartan–Kähler*)

$\star\star$ Key complication: $\nexists$ Abstract object $\mathcal{G}$??? $\star\star$

Infinitesimal Generators

$\mathfrak{g}$ — space of infinitesimal generators of
the pseudo-group $\mathcal{G}$

Locally defined vector fields:

$$\mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi_\alpha \frac{\partial}{\partial u^\alpha}$$

subject to:

Infinitesimal Determining Equations

$$\mathcal{L}(z, \zeta^{(n)}) = 0 \qquad (*)$$

$\Longrightarrow$ obtained by linearization

Remark: If $\mathcal{G}$ is the symmetry group of a system of differential equations $\Delta(x, u^{(n)}) = 0$, then $(*)$ is the (involutive completion of) the usual Lie determining equations for the symmetry group.

The Diffeomorphism Pseudogroup

M — smooth m-dimensional manifold

$\mathcal{D} = \mathcal{D}(M)$

— pseudo-group of all local diffeomorphisms

$$Z = \varphi(z) \qquad \begin{cases} z = (x,u) \text{ — source coordinates} \\ Z = (X,U) \text{ — target coordinates} \end{cases}$$

$\mathcal{D}^{(n)} = \mathcal{D}^{(n)}(M) \subset \mathrm{J}^n(M,M)$ — n^{th} order jets

$\Longrightarrow$ groupoid

Local coordinates on $\mathcal{D}^{(n)}$:

$$g^{(n)} = (z, Z^{(n)}) = (\ \ldots\ z^a\ \ldots\ Z^b_A\ \ldots\ )$$

$$= (\ \ldots\ x^i\ \ldots\ u^\alpha\ \ldots\ X^i_A\ \ldots\ U^\alpha_A\ \ldots\ )$$

The multi-indices A indicate partial derivatives with respect to $z = (x,u)$

$\Longrightarrow$ *The Maurer–Cartan forms for the diffeomorphism pseudo-group are the right-invariant contact forms on* $\mathcal{D}^{(\infty)}$.

Diffeomorphism Jets and the Variational Bicomplex

$$\mathcal{D}^{(\infty)} \subset \mathrm{J}^\infty(M, M)$$

Local coordinates:

$$\underbrace{z^1, \ \ldots \ , z^m}_{\text{source}}, \quad \underbrace{Z^1, \ \ldots \ , Z^m}_{\text{target}}, \quad \underbrace{\ldots \ Z^b_A, \ \ldots}_{\text{jet}}$$

Horizontal forms:

$$dz^1, \ \ldots \ , dz^m$$

Contact forms:

$$\Theta^b_A = \, d_G \, Z^b_A = dZ^b_A - \sum_{a=1}^{m} \, Z^a_{A,a} \, dz^a$$

Maurer–Cartan forms:

$$\mu^b_A = \mathbb{D}^A_Z \Theta^b = \mathbb{D}_{Z^{a_1}} \ \cdots \ \mathbb{D}_{Z^{a_n}} \Theta^b$$

$$b = 1, \ldots, m, \ \#A \geq 0$$

Maurer–Cartan forms for $\mathcal{D}^{(\infty)}$

Key observation: The target coordinate functions Z^a are right-invariant.

Decompose

$$dZ^a = \underset{\text{horizontal}}{\sigma^a} + \underset{\text{contact}}{\mu^a}$$

Invariant horizontal forms:

$$\sigma^a = d_M Z^a = \sum_{b=1}^{m} Z^a_b \, dz^b$$

Invariant total differentiation (dual operators):

$$\mathbb{D}_{Z^a} = \sum_{b=1}^{m} (Z^a_b)^{-1} \mathbb{D}_{z^b}$$

Invariant contact forms:

$$\mu^b = d_G Z^b = \Theta^b = dZ^b - \sum_{a=1}^{m} Z^b_a \, dz^a$$

$$\mu^b_A = \mathbb{D}^A_Z \Theta^b = \mathbb{D}_{Z^{a_1}} \cdots \mathbb{D}_{Z^{a_n}} \Theta^b$$

$$b = 1, \ldots, m, \ \#A \geq 0$$

$\Longrightarrow$ *Maurer–Cartan forms*

One-dimensional case: $M = \mathbb{R}$

Contact forms:

$$\Theta = d_G X = dX - X_x\, dx$$

$$\Theta_x = \mathbb{D}_x \Theta = dX_x - X_{xx}\, dx$$

$$\Theta_{xx} = \mathbb{D}_x^2 \Theta = dX_{xx} - X_{xxx}\, dx$$

Right-invariant horizontal form:

$$\sigma = d_M X = X_x\, dx$$

Invariant differentiation:

$$\mathbb{D}_X = \frac{1}{X_x}\, \mathbb{D}_x$$

Maurer–Cartan forms:

$$\mu = \Theta$$

$$\mu_X = \mathbb{D}_X \mu = \frac{\Theta_x}{X_x}$$

$$\mu_{XX} = \mathbb{D}_X^2 \mu = \frac{X_x\, \Theta_{xx} - X_{xx}\, \Theta_x}{X_x^3}$$

Two-dimensional case: $M = \mathbb{R}^2$

Coordinates on $\mathcal{D}^{(\infty)}(\mathbb{R}^2)$:

$$(x, u, X, U, X_x, X_u, U_x, U_u, X_{xx}, X_{xu}, \ \ldots \ )$$

Contact forms on $\mathcal{D}^{(2)}(\mathbb{R}^2)$:

$$\Upsilon = dX - X_x\, dx - X_u\, du \qquad \Phi = dU - U_x\, dx - U_u\, du$$

$$\Upsilon_x = dX_x - X_{xx}\, dx - X_{xu}\, du \qquad \Phi_x = dU_x - U_{xx}\, dx - U_{xu}\, du$$

$$\Upsilon_u = dX_u - X_{xu}\, dx - X_{uu}\, du \qquad \Phi_u = dU_u - U_{xu}\, dx - U_{uu}\, du$$

Maurer–Cartan forms:

$$\sigma = d_M X = X_x\, dx + X_u\, du, \qquad \tau = d_M U = U_x\, dx + U_u\, du,$$

$$\mu = \Upsilon = d_G X \qquad \nu = \Phi = d_G U$$

$$\mu_X = \mathbb{D}_X \mu = \frac{U_u\, \Upsilon_x - U_x\, \Upsilon_u}{X_x U_u - X_u U_x} \qquad \nu_X = \mathbb{D}_X \nu = \frac{U_u\, \Phi_x - U_x\, \Phi_u}{X_x U_u - X_u U_x}$$

$$\mu_U = \mathbb{D}_U \mu = \frac{X_x\, \Upsilon_u - X_u\, \Upsilon_x}{X_x U_u - X_u U_x} \qquad \nu_U = \mathbb{D}_U \nu = \frac{X_x\, \Phi_u - X_u\, \Phi_x}{X_x U_u - X_u U_x}$$

Right-invariant differentiations:

$$\mathbb{D}_X = \frac{U_u\, \mathbb{D}_x - U_x\, \mathbb{D}_u}{X_x U_u - X_u U_x}, \qquad \mathbb{D}_U = \frac{-\, X_u\, \mathbb{D}_x + X_x\, \mathbb{D}_u}{X_x U_u - X_u U_x}.$$

The Universal Diffeomorphism

Structure Equations

Maurer–Cartan formal series:

$$\mu^b[\![\, H \,]\!] = \sum_A \frac{1}{A!}\, \mu^b_A \, H^A$$

$$\Longrightarrow \quad H = (H^1, \ldots, H^m) \; \text{— parameters}$$

Universal Structure Equations

$$d\mu[\![\, H \,]\!] = \nabla_H \mu[\![\, H \,]\!] \wedge (\, \mu[\![\, H \,]\!] - dZ \,)$$

$$d\sigma = -\, d\mu[\![\, 0 \,]\!] = \nabla_H \mu[\![\, 0 \,]\!] \wedge \sigma$$

$\Longrightarrow$ equate powers of H

One-dimensional case: $M = \mathbb{R}$

Structure equations:

$$d\sigma = \mu_X \wedge \sigma \qquad d\mu[\![H]\!] = \mu_H[\![H]\!] \wedge (\mu[\![H]\!] - dZ)$$

where

$$\sigma = X_x\, dx = dX - \mu$$

$$\mu[\![H]\!] = \mu + \mu_X\, H + \tfrac{1}{2}\,\mu_{XX}\, H^2 + \cdots$$

$$\mu[\![H]\!] - dZ = -\,\sigma + \mu_X\, H + \tfrac{1}{2}\,\mu_{XX}\, H^2 + \ \cdots$$

$$\mu_H[\![H]\!] = \mu_X + \mu_{XX}\, H + \tfrac{1}{2}\,\mu_{XXX}\, H^2 + \ \cdots$$

In components:

$$d\sigma = \mu_1 \wedge \sigma$$

$$d\mu_n = -\,\mu_{n+1} \wedge \sigma \ + \ \sum_{i=0}^{n-1} \binom{n}{i}\ \mu_{i+1} \wedge \mu_{n-i}$$

$$= \sigma \wedge \mu_{n+1} - \sum_{j=1}^{\left[\frac{n+1}{2}\right]} \frac{n-2j+1}{n+1} \binom{n+1}{j}\ \mu_j \wedge \mu_{n+1-j}.$$

$\Longrightarrow$ Cartan

Two-dimensional case: $M = \mathbb{R}^2$

Maurer–Cartan form series:

$$dX = \sigma + \mu \qquad\qquad dY = \tau + \nu$$

$$\begin{aligned}\mu[\![\, H, K \,]\!] &= \mu + \mu_H\, H + \mu_K\, K + \\ &\qquad + \tfrac{1}{2}\,\mu_{HH}\, H^2 + \mu_{HK}\, H\, K + \tfrac{1}{2}\,\mu_{KK}\, K^2 + \ \cdots\ , \\ \nu[\![\, H, K \,]\!] &= \nu + \nu_H\, H + \nu_K\, K + \\ &\qquad + \tfrac{1}{2}\,\nu_{HH}\, H^2 + \nu_{HK}\, H\, K + \tfrac{1}{2}\,\nu_{KK}\, K^2 + \ \cdots\ ,\end{aligned}$$

Structure equations:

$$\begin{pmatrix} d\mu[\![\, H, K \,]\!] \\ d\nu[\![\, H, K \,]\!] \end{pmatrix} = \begin{pmatrix} \mu_H[\![\, H, K \,]\!] & \mu_K[\![\, H, K \,]\!] \\ \nu_H[\![\, H, K \,]\!] & \nu_K[\![\, H, K \,]\!] \end{pmatrix} \ \wedge\ \begin{pmatrix} \mu[\![\, H, K \,]\!] - dX \\ \nu[\![\, H, K \,]\!] - dU \end{pmatrix}$$

First order structure equations:

$$\begin{aligned} d\mu &= -\, d\sigma = -\, \mu_X \wedge \sigma - \mu_U \wedge \tau, \\ d\nu &= -\, d\tau = -\, \nu_X \wedge \sigma - \nu_U \wedge \tau, \\ d\mu_X &= -\, \mu_{XX} \wedge \sigma - \mu_{XU} \wedge \tau + \mu_U \wedge \nu_X, \\ d\nu_X &= -\, \nu_{XX} \wedge \sigma - \nu_{XU} \wedge \tau + \nu_X \wedge (\mu_X - \nu_U), \\ d\mu_U &= -\, \mu_{XU} \wedge \sigma - \mu_{UU} \wedge \tau + (\mu_X - \nu_U) \wedge \mu_U, \\ d\nu_U &= -\, \nu_{XU} \wedge \sigma - \nu_{UU} \wedge \tau + \nu_X \wedge \mu_U \end{aligned}$$

The Structure Equations for a Lie Pseudo-group

Lie pseudo-group jets: $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$.
Infinitesimal determining equations:

$$\mathcal{L}(\; \ldots \; z^a \; \ldots \; \zeta^b_A \; \ldots \;) = 0 \qquad (\star)$$

The Maurer–Cartan forms for $\mathcal{G}$ are obtained by restricting the diffeomorphism Maurer–Cartan forms μ^b_A to $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$.

The resulting forms μ^b_A are no longer linearly independent:

Theorem. The Maurer–Cartan forms on $\mathcal{G}^{(\infty)}$ satisfy the invariant infinitesimal determining equations

$$\mathcal{L}(\; \ldots \; Z^a \; \ldots \; \mu^b_A \; \ldots \;) = 0 \qquad (\star\star)$$

obtained from the infinitesimal determining equations $(\star)$ by replacing

- source variables z^a by target variables Z^a
- derivatives of vector field coefficients ζ^b_A by right-invariant Maurer–Cartan forms μ^b_A

The Fundamental Structure Theorem

Theorem. The structure equations for the pseudo-group $\mathcal{G}$ are obtained by restricting the universal diffeomorphism structure equations

$$d\mu[\![\, H \,]\!] = \nabla_H \mu[\![\, H \,]\!] \wedge (\, \mu[\![\, H \,]\!] - dZ \,)$$

to the solution space of the linearized involutive system

$$\mathcal{L}(\; \ldots \; Z^a, \; \ldots \; \mu_A^b, \; \ldots \;) = 0.$$

♠ The structure equations are on the principal bundle $\mathcal{G}^{(\infty)}$; if G is a finite-dimensional Lie group, then $\mathcal{G}^{(\infty)} \simeq M \times G$, and the usual Lie group structure equations are found by neestriction to the target fibers $\{ \, Z = c \} \simeq G$.

Korteweg–deVries Equation

Diffeomorphism Maurer–Cartan forms:

$$\mu^t,\ \mu^x,\ \mu^u,\ \mu^t_T,\ \mu^t_X,\ \mu^t_U,\ \mu^x_T,\ \ldots,\ \mu^u_U,\ \mu^t_{TT},\ \mu^T_{TX},\ \cdots$$

Maurer–Cartan determining equations:

$$\mu^t_X = \mu^t_U = \mu^x_U = \mu^u_T = \mu^u_X = 0,$$

$$\mu^u = \mu^x_T - \tfrac{2}{3}\,U\mu^t_T, \qquad \mu^u_U = -\tfrac{2}{3}\,\mu^t_T = -\,2\,\mu^x_X,$$

$$\mu^t_{TT} = \mu^t_{TX} = \mu^t_{XX} = \ \cdots\ = \mu^u_{UU} = \ \ldots\ = 0.$$

Basis ($\dim \mathcal{G}_{KdV} = 4$):

$$\mu^1 = \mu^t, \qquad \mu^2 = \mu^x, \qquad \mu^3 = \mu^u, \qquad \mu^4 = \mu^t_T.$$

Structure equations:

$$d\mu^1 = -\,\mu^1 \wedge \mu^4,$$

$$d\mu^2 = -\mu^1 \wedge \mu^3 - \tfrac{2}{3}\,U\,\mu^1 \wedge \mu^4 - \tfrac{1}{3}\,\mu^2 \wedge \mu^4,$$

$$d\mu^3 = \tfrac{2}{3}\,\mu^3 \wedge \mu^4,$$

$$d\mu^4 = 0.$$

$$\boxed{d\mu^i = C^i_{jk}\,\mu^j \wedge \mu^k}$$

Lie–Kumpera Example

$$X = f(x) \qquad U = \frac{u}{f'(x)}$$

Linearized determining system

$$\xi_x = -\frac{\varphi}{u} \qquad \xi_u = 0 \qquad \varphi_u = \frac{\varphi}{u}$$

Maurer–Cartan forms:

$$\sigma = \frac{u}{U}\,dx = f_x\,dx, \qquad \tau = U_x\,dx + \frac{U}{u}\,du = \frac{-\,u\,f_{xx}\,dx + f_x\,du}{f_x^2}$$

$$\mu = dX - \frac{U}{u}\,dx = df - f_x\,dx, \qquad \nu = dU - U_x\,dx - \frac{U}{u}\,du = -\frac{u}{f_x^2}\,(\,df_x - f_{xx}\,dx\,)$$

$$\mu_X = \frac{du}{u} - \frac{dU - U_x\,dx}{U} = \frac{df_x - f_{xx}\,dx}{f_x}\,, \qquad \mu_U = 0$$

$$\nu_X = \frac{U}{u}\,(dU_x - U_{xx}\,dx) - \frac{U_x}{u}\,(dU - U_x\,dx)$$

$$= -\frac{u}{f_x^3}(\,df_{xx} - f_{xxx}\,dx\,) + \frac{u\,f_{xx}}{f_x^4}(\,df_x - f_{xx}\,dx\,)$$

$$\nu_U = -\frac{du}{u} + \frac{dU - U_x\,dx}{U} = -\frac{df_x - f_{xx}\,dx}{f_x}$$

Right-invariant linearized system:

$$\mu_X = -\frac{\nu}{U} \qquad \mu_U = 0 \qquad \nu_U = \frac{\nu}{U}$$

First order structure equations:

$$d\mu = -\,d\sigma = \frac{\nu \wedge \sigma}{U}\,, \qquad d\nu = -\,\nu_X \wedge \sigma \; - \; \frac{\nu \wedge \tau}{U}$$

$$d\nu_X = -\,\nu_{XX} \wedge \sigma \; - \; \frac{\nu_X \wedge (\tau + 2\,\nu)}{U}$$

Action of Pseudo-groups on Submanifolds

$\mathcal{G}$ — Lie pseudo-group acting on p-dimensional submanifolds — solutions to differential equations:

$$N = \{\, u = f(x) \,\} \subset M$$

$\mathrm{J}^n = \mathrm{J}^n(M, p)$ — n^{th} order jet bundle

Local coordinates

$$z^{(n)} = (x, u^{(n)}) = (\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ )$$

Prolongation

Prolonged action of $\mathcal{G}^{(n)}$ on submanifolds:

$$(x, u^{(n)}) \quad \longmapsto \quad (X, \widehat{U}^{(n)})$$

Coordinate formulae:

$$\widehat{U}^\alpha_J = F^\alpha_J(x, u^{(n)}, g^{(n)})$$

$\Longrightarrow$ Implicit differentiation.

Moving Frames for Pseudo–Groups

In the finite-dimensional Lie group case, a moving frame is defined as an equivariant map

$$\rho^{(n)} : \mathrm{J}^n \longrightarrow G$$

However, we no longer have an abstract object to represent our pseudo-group $\mathcal{G}$, and so the moving frame will be an equivariant section of the pulled-back pseudo-group principal jet bundle:

$$\begin{array}{ccc} \mathcal{G}^{(n)} & \longleftarrow & \mathcal{H}^{(n)} \\ \downarrow & & \downarrow \\ M & \longleftarrow & \mathrm{J}^n. \end{array}$$

Definition. A (right) *moving frame* of *order* n is a right-equivariant section $\rho^{(n)} : V^n \to \mathcal{H}^{(n)}$ defined on an open subset $V^n \subset \mathrm{J}^n$.

Freeness

Proposition. A moving frame of order n exists if and only if $\mathcal{G}^{(n)}$ acts *freely* and regularly.

♣ In finite-dimensions, freeness means no isotropy. For infinite-dimensional pseudo-groups, one must restrict to the transformation jets of order n.

Isotropy subgroup

$$\mathcal{G}^{(n)}_{z^{(n)}} = \left\{ \, g^{(n)} \in \mathcal{G}^{(n)}_z \;\middle|\; g^{(n)} \cdot z^{(n)} = z^{(n)} \, \right\}$$

Definition. The pseudo-group $\mathcal{G}$ acts

- *freely* at $z^{(n)} \in \mathrm{J}^n$ if $\mathcal{G}^{(n)}_{z^{(n)}} = \{ \, \mathbf{1}^{(n)}_z \, \}$
- *locally freely* if
  - $\mathcal{G}^{(n)}_{z^{(n)}}$ is a discrete subgroup of $\mathcal{G}^{(n)}_z$
  - the orbits have dim $= r_n = \dim \mathcal{G}^{(n)}_z$

Freeness Theorem

Theorem. If $\mathcal{G}^{(n)}$ acts locally freely at $z^{(n)} \in \mathrm{J}^n$, then it acts locally freely at any $z^{(k)} \in \mathrm{J}^k$ with

$\widetilde{\pi}^k_n(z^{(k)}) = z^{(n)}$

for $k > n$.

Normalization

♠ To construct a moving frame :

I. Choose a cross-section to the pseudo-group orbits:

$$u^{\alpha_\kappa}_{J_\kappa} = c_\kappa, \qquad \kappa = 1, \ldots, r_n = \text{fiber dim } \mathcal{G}^{(n)}$$

II. Solve the normalization equations

$$F^{\alpha_\kappa}_{J_\kappa}(x, u^{(n)}, g^{(n)}) = c_\kappa$$

for the pseudo-group parameters

$$g^{(n)} = \rho^{(n)}(x, u^{(n)})$$

III. Invariantization maps differential functions to differential invariants:

$$\iota : F(x, u^{(n)}) \longmapsto I(x, u^{(n)}) = F(\, \rho^{(n)}(x, u^{(n)}) \cdot (x, u^{(n)})\,)$$

$\Longrightarrow$ an algebra morphism and a projection:

$$\iota \circ \iota = \iota$$

$$I(x, u^{(n)}) = \iota(\, I(x, u^{(n)})\,)$$

Invariantization

◇ Functions $\Longrightarrow$ differential invariants

$$\iota(x^i) = H^i \qquad \iota(u^\alpha_J) = I^\alpha_J$$

- Phantom differential invariants: $\quad I^{\alpha_\kappa}_{J_\kappa} = c_\kappa$
- The non-constant invariants form a functionally independent generating set for the differential invariant algebra $\mathcal{I}(\mathcal{G})$
- Replacement Theorem

$$J(\ \ldots\ x^i\ \ldots\ u^\alpha_J\ \ldots\ ) = J(x, u^{(n)}) = \iota(\ J(x, u^{(n)})\ )$$
$$= J(\ \ldots\ H^i\ \ldots\ I^\alpha_J\ \ldots\ )$$

◇ Differential forms $\Longrightarrow$ invariant differential forms

$$\iota(\,dx^i) = \omega^i \qquad i = 1, \ldots, p$$

◇ Differential operators $\Longrightarrow$

invariant differential operators

$$\iota\,(\,\mathrm{D}_{x^i}\,) = \mathcal{D}_i \qquad i = 1, \ldots, p$$

Recurrence Formulae

The *recurrence formulae* connect the differentiated invariants with their normalized counterparts.

$\star\star$ They can be found using only the infinitesimal generators and linear differential algebra

Key Items:

- Determination of a minimal basis $I_1, \ldots, I_\ell$ of differential invariants: $\mathcal{D}_K I_\kappa$
- Commutation formulae for the invariant differential operators:

$$[\,\mathcal{D}_j, \mathcal{D}_k\,] = \sum_{i=1}^{p} K_{j,k}^i\, \mathcal{D}_i$$

- Syzygies (functional relations) among differentiated invariants:

$$\Phi(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ ) \equiv 0$$

- Equivalence and signatures of submanifolds and characterization of moduli spaces
- Computation of invariant variational problems:

$$\int L(\ \ldots\ \mathcal{D}_K I^\alpha\ \ \ldots\ )\, \boldsymbol{\omega}$$

- Group splitting of PDEs

Fundamental Recurrence Formula

Normalized differential invariants:

$$\mathcal{D}_i I_J^\alpha = I_{J,i}^\alpha + M_{J,i}^\alpha$$

$$\Longrightarrow \quad M_{J,i}^\alpha \text{ — correction terms}$$

Key formula:

$$\boxed{d_H\, I_J^\alpha \;=\; \sum_{i=1}^{p} (\, \mathcal{D}_i I_J^\alpha \,)\, \omega^i \;=\; \sum_{i=1}^{p} I_{J,i}^\alpha\, \omega^i + \psi_\alpha^J}$$

where

$$\psi_\alpha^J = \iota(\, \varphi_\alpha^J \,) = \Phi_\alpha^J(\;\ldots\; H^i \;\ldots\; I_J^\alpha \;\ldots\; ; \;\ldots\; \gamma_A^b \;\ldots\;)$$

are the invariantized prolonged vector field coefficients,

$$\iota(\zeta_A^b) = \gamma_A^b = \left(\rho^{(\infty)}\right)^* \mu_A^b$$

are the (horizontal) pulled-back Maurer–Cartan forms.

$$d_H I_J^\alpha = \sum_{i=1}^{p} I_{J,i}^\alpha \, \omega^i + \psi_\alpha^J(\ldots \gamma_A^b \ldots)$$

Invariantized infinitesimal determining equations

$$\mathcal{L}(H^1, \ldots, H^p, I^1, \ldots, I^q, \ \ldots \ , \gamma_A^b, \ \ldots \ ) = 0$$

♠ Solve the phantom recurrence formulas

$$0 = d_H I_J^\alpha = \sum_{i=1}^{p} I_{J,i}^\alpha \, \omega^i + \psi_\alpha^J(\ldots \gamma_A^b \ldots)$$

for the basis pulled-back Maurer–Cartan forms:

$$\gamma_A^b = \sum_{i=1}^{p} J_{A,i}^b \, \omega^i$$

Substitute the results into the non-phantom recurrence formulae to obtain the correction terms.

♡ Only uses linear differential algebra based on the choice of cross-section.

♡ Does not require any explicit formulas for the moving frame, the differential invariants, the invariant differential operators, or even the Maurer–Cartan forms!

Syzygies

Theorem. Complete system of basic syzygies:

$$\mathcal{D}_K I_J^\alpha = c_{JK}^\alpha + M_{J,K}^\alpha,$$

$$\mathcal{D}_J I_{LK}^\alpha - \mathcal{D}_K I_{LJ}^\alpha = M_{LK,J}^\alpha - M_{LJ,K}^\alpha,$$

where

- I_J^α is a generating differential invariant
- $I_{JK}^\alpha = c_{JK}^\alpha$ is a phantom differential invariant
- I_{LK}^α and I_{LJ}^α are generating differential invariants with $K \cap J = \varnothing$.

Korteweg–deVries equation

Cross section:

$$H^1 = \iota(t) = 0, \qquad I_{00} = \iota(u) = 0,$$

$$H^2 = \iota(x) = 0, \qquad I_{10} = \iota(u_t) = 1.$$

$\Longrightarrow$ phantoms

Normalized differential invariants:

$$I_{01} = \iota(u_x) = \frac{u_x}{(u_t + u u_x)^{3/5}}$$

$$I_{20} = \iota(u_{tt}) = \frac{u_{tt} + 2u u_{tx} + u^2 u_{xx}}{(u_t + u u_x)^{8/5}}$$

$$I_{11} = \iota(u_{tx}) = \frac{u_{tx} + u u_{xx}}{(u_t + u u_x)^{6/5}}$$

$$I_{02} = \iota(u_{xx}) = \frac{u_{xx}}{(u_t + u u_x)^{4/5}}$$

$$I_{03} = \iota(u_{xxx}) = \frac{u_{xxx}}{u_t + u u_x}$$

$$\vdots$$

Replacement Theorem:

$$0 = \iota(u_t + u u_x + u_{xxx}) = 1 + I_{03} = \frac{u_t + u u_x + u_{xxx}}{u_t + u u_x}.$$

Invariant horizontal one-forms:

$$\omega^1 = \iota(dt) = (u_t + u u_x)^{3/5}\, dt,$$

$$\omega^2 = \iota(dx) = -u(u_t + u u_x)^{1/5}\, dt + (u_t + u u_x)^{1/5}\, dx.$$

Invariant differential operators:

$$\mathcal{D}_1 = \iota(D_t) = (u_t + u u_x)^{-3/5} D_t + u(u_t + u u_x)^{-3/5} D_x,$$

$$\mathcal{D}_2 = \iota(D_x) = (u_t + u u_x)^{-1/5} D_x.$$

Recurrence formulae:

$$dI_{jk} = I_{j+1,k}\,\omega^1 + I_{j,k+1}\,\omega^2 + \iota(\varphi^{jk})$$

Invariantizations of prolonged vector field coefficients ($\gamma_i = \rho^* \mu_i$):

$$\iota(\tau) = \gamma_1, \quad \iota(\xi) = \gamma_2, \quad \iota(\varphi) = \gamma_3, \quad \iota(\varphi^t) = -I_{01}\gamma_3 - \tfrac{5}{3}\,\gamma_4,$$
$$\iota(\varphi^x) = -I_{01}\gamma_4, \quad \iota(\varphi^{tt}) = -2I_{11}\gamma_3 - \tfrac{8}{3}I_{20}\gamma_4, \quad \ldots$$

Phantom recurrence formulae:

$$0 = d_H\, H^1 = \omega^1 + \gamma_1,$$
$$0 = d_H\, H^2 = \omega^2 + \gamma_2,$$
$$0 = d_H\, I_{00} = I_{10}\omega^1 + I_{01}\omega^2 + \psi = \omega^1 + I_{01}\omega^2 + \gamma_3,$$
$$0 = d_H\, I_{10} = I_{20}\omega^1 + I_{11}\omega^2 + \psi^T = I_{20}\omega^1 + I_{11}\omega^2 - I_{01}\gamma_3 - \tfrac{5}{3}\,\gamma_4,$$

$\Longrightarrow$ Solve for $\gamma_1 = -\omega^1, \quad \gamma_2 = -\omega^2, \quad \gamma_3 = -\omega^1 - I_{01}\omega^2,$

$$\gamma_4 = \tfrac{3}{5}\,(I_{20} + I_{01})\omega^1 + \tfrac{3}{5}\,(I_{11} + I_{01}^2)\omega^2.$$

Recurrence formulae:

$$d_H\, I_{01} = I_{11}\omega^1 + I_{02}\omega^2 - I_{01}\gamma_4,$$
$$d_H\, I_{20} = I_{30}\omega^1 + I_{21}\omega^2 - 2I_{11}\gamma_3 - \tfrac{8}{3}I_{20}\gamma_4,$$
$$d_H\, I_{11} = I_{21}\omega^1 + I_{12}\omega^2 - I_{02}\gamma_3 - 2I_{11}\gamma_4,$$
$$d_H\, I_{02} = I_{12}\omega^1 + I_{03}\omega^2 - \tfrac{4}{3}I_{02}\gamma_4,$$
$$\vdots$$

Recurrence formulae:

$$\mathcal{D}_1 I_{01} = I_{11} - \tfrac{3}{5}I_{01}^2 - \tfrac{3}{5}I_{01}I_{20}, \qquad \mathcal{D}_2 I_{01} = I_{02} - \tfrac{3}{5}\,I_{01}^3 - \tfrac{3}{5}\,I_{01}I_{11},$$
$$\mathcal{D}_1 I_{20} = I_{30} + 2I_{11} - \tfrac{8}{5}I_{01}I_{20} - \tfrac{8}{5}I_{20}^2, \qquad \mathcal{D}_2 I_{20} = I_{21} + 2I_{01}I_{11} - \tfrac{8}{5}I_{01}^2I_{20} - \tfrac{8}{5}I_{11}I_{20},$$
$$\mathcal{D}_1 I_{11} = I_{21} + I_{02} - \tfrac{6}{5}I_{01}I_{11} - \tfrac{6}{5}I_{11}I_{20}, \qquad \mathcal{D}_2 I_{11} = I_{12} + I_{01}I_{02} - \tfrac{6}{5}I_{01}^2I_{11} - \tfrac{6}{5}I_{11}^2,$$
$$\mathcal{D}_1 I_{02} = I_{12} - \tfrac{4}{5}I_{01}I_{02} - \tfrac{4}{5}I_{02}I_{20}, \qquad \mathcal{D}_2 I_{02} = I_{03} - \tfrac{4}{5}I_{01}^2I_{02} - \tfrac{4}{5}I_{02}I_{11},$$
$$\vdots \qquad\qquad \vdots$$

Generating differential invariants:

$$I_{01} = \iota(u_x) = \frac{u_x}{(u_t + uu_x)^{3/5}},$$

$$I_{20} = \iota(u_{tt}) = \frac{u_{tt} + 2uu_{tx} + u^2 u_{xx}}{(u_t + uu_x)^{8/5}}.$$

Invariant differential operators:

$$\mathcal{D}_1 = \iota(D_t) = (u_t + uu_x)^{-3/5} D_t + u(u_t + uu_x)^{-3/5} D_x,$$

$$\mathcal{D}_2 = \iota(D_x) = (u_t + uu_x)^{-1/5} D_x.$$

Fundamental syzygy:

$$\mathcal{D}_1^2 I_{01} + \tfrac{3}{5}\, I_{01} \mathcal{D}_1 I_{20} - \mathcal{D}_2 I_{20} + \left(\tfrac{1}{5}\, I_{20} + \tfrac{19}{5}\, I_{01}\right) \mathcal{D}_1 I_{01}$$
$$- \mathcal{D}_2 I_{01} - \tfrac{6}{25}\, I_{01} I_{20}^2 - \tfrac{7}{25}\, I_{01}^2 I_{20} + \tfrac{24}{25}\, I_{01}^3 = 0.$$

Lie–Tresse–Kumpera Example

$$X = f(x), \qquad Y = y, \qquad U = \frac{u}{f'(x)}$$

Horizontal coframe

$$d_H X = f_x\, dx, \qquad d_H Y = dy,$$

Implicit differentiations

$$\mathrm{D}_X = \frac{1}{f_x}\,\mathrm{D}_x, \qquad \mathrm{D}_Y = \mathrm{D}_y.$$

Prolonged pseudo-group transformations on surfaces $S \subset \mathbb{R}^3$

$$X = f \qquad Y = y \qquad U = \frac{u}{f_x}$$

$$U_X = \frac{u_x}{f_x^2} - \frac{u\, f_{xx}}{f_x^3} \qquad U_Y = \frac{u_y}{f_x}$$

$$U_{XX} = \frac{u_{xx}}{f_x^3} - \frac{3u_x\, f_{xx}}{f_x^4} - \frac{u\, f_{xxx}}{f_x^4} + \frac{3u\, f_{xx}^2}{f_x^5}$$

$$U_{XY} = \frac{u_{xy}}{f_x^2} - \frac{u_y\, f_{xx}}{f_x^3} \qquad U_{YY} = \frac{u_{yy}}{f_x}$$

$\Longrightarrow$ action is free at every order.

Coordinate cross-section

$$X = 0, \qquad U = 1, \qquad U_X = 0, \qquad U_{XX} = 0.$$

Moving frame

$$f = 0, \qquad f_x = u, \qquad f_{xx} = u_x, \qquad f_{xxx} = u_{xx}.$$

Differential invariants

$$U_Y \longmapsto J = \frac{u_y}{u}$$

$$U_{XY} \longmapsto J_1 = \frac{u u_{xy} - u_x u_y}{u^3} \qquad U_{YY} \longmapsto J_2 = \frac{u_{yy}}{u}$$

Horizontal invariant coframe

$$d_H X \longmapsto u\, dx, \qquad d_H Y \longmapsto dy,$$

Invariant differentiations

$$\mathcal{D}_1 = \frac{1}{u}\, \mathrm{D}_x \qquad \mathcal{D}_2 = \mathrm{D}_y$$

Higher order differential invariants: $\mathcal{D}_1^m \mathcal{D}_2^n J$

$$J_{,1} = \mathcal{D}_1 J = \frac{u u_{xy} - u_x u_y}{u^3} = J_1,$$

$$J_{,2} = \mathcal{D}_2 J = \frac{u u_{yy} - u_y^2}{u^2} = J_2 - J^2.$$

$\Longrightarrow$ All higher order differential invariants are obtained from J by invariant differentiation

$$\mathbf{v}_\psi^{(\infty)} = \gamma\,\partial_x - u\,\gamma_1\,\partial_u - (\,u\,\gamma_2 + 2\,u_x\,\gamma_1\,)\,\partial_{u_x} - u_y\,\gamma_1\,\partial_{u_y} - \;\cdots$$

Phantom invariants:

$$0 = dH = \varpi^1 + \gamma, \qquad\qquad 0 = dI_{10} = J_1\,\varpi^2 + \vartheta_1 - \gamma_2,$$

$$0 = dI_{00} = J\,\varpi^2 + \vartheta - \gamma_1, \qquad\qquad 0 = dI_{20} = J_3\,\varpi^2 + \vartheta_3 - \gamma_3,$$

Solve for pulled-back Maurer–Cartan forms:

$$\gamma = -\,\varpi^1, \qquad\qquad \gamma_2 = J_1\,\varpi^2 + \vartheta_1,$$

$$\gamma_1 = J\,\varpi^2 + \vartheta, \qquad\qquad \gamma_3 = J_3\,\varpi^2 + \vartheta_3,$$

Recurrence formulae: $\qquad dy = \varpi^2$

$$dJ = J_1\,\varpi^1 + (J_2 - J^2)\,\varpi^2 + \vartheta_2 - J\,\vartheta,$$

$$dJ_1 = J_3\,\varpi^1 + (J_4 - 3\,J\,J_1)\,\varpi^2 + \vartheta_4 - J\,\vartheta_1 - J_1\,\vartheta,$$

$$dJ_2 = J_4\,\varpi^1 + (J_5 - J\,J_2)\,\varpi^2 + \vartheta_5 - J_2\,\vartheta,$$

$$\mathcal{D}_1 J = J_1, \qquad \mathcal{D}_2 J = J_2 - J^2, \qquad d_{\mathcal{V}}\,J = \vartheta_2 - J\,\vartheta,$$

$$\mathcal{D}_1 J_1 = J_3, \qquad \mathcal{D}_2 J_1 = J_4 - 3\,J\,J_1, \qquad d_{\mathcal{V}}\,J_1 = \vartheta_4 - J\,\vartheta_{10} - J_1\,\vartheta,$$

$$\mathcal{D}_1 J_2 = J_4, \qquad \mathcal{D}_2 J_2 = J_5 - J\,J_2, \qquad d_{\mathcal{V}}\,J_2 = \vartheta_5 - J_2\,\vartheta,$$

$\Longrightarrow$ All higher order differential invariants are obtained from J by invariant differentiation

Invariant horizontal forms

$$d\varpi^1 = -\,J\,\varpi^1 \wedge \varpi^2 + \vartheta \wedge \varpi^1, \qquad\qquad d\varpi^2 = 0.$$

Commmutation formula

$$[\,\mathcal{D}_1, \mathcal{D}_2\,] = J\,\mathcal{D}_1.$$

Gröbner Basis Approach

Identify the cross-section variables with the complementary monomials to a certain algebraic module $\mathcal{J}$, which is the pull-back of the symbol module of the pseudo-group under a certain explicit linear map.

$\Longrightarrow$ Compatible term ordering.

$\Longrightarrow$ Algebraic specification of compatible moving frames of all orders $n > n^\star$.

Theorem. Suppose $\mathcal{G}$ acts freely at order $n^\star$. Then a system of generating differential invariants is contained in the non-phantom normalized differential invariants of order $n^\star$ and those differential invariants corresponding to a Gröbner basis for the module $\mathcal{J}^{>n^\star}$.

The Symbol Module

Linearized determining equations

$$\mathcal{L}(z, \zeta^{(n)}) = 0$$

$$r = (r_1, \ldots, r_m), \qquad R = (R_1, \ldots, R_m)$$

$$\mathcal{R} = \left\{ P(r,R) = \sum_{a=1}^{m} P_a(r)\, R_a \right\} \simeq \mathbb{R}[r] \otimes \mathbb{R}^m \subset \mathbb{R}[r,R]$$

$\mathcal{S} \subset \mathcal{R}$ — symbol module

$$s = (s_1, \ldots, s_p), \qquad S = (S_1, \ldots, S_q),$$

$$\mathcal{Q} = \left\{ T(s,S) = \sum_{\alpha=1}^{q} T_\alpha(s)\, S_\alpha \right\} \simeq \mathbb{R}[s] \otimes \mathbb{R}^q \subset \mathbb{R}[s,S]$$

Define the linear map

$$s_i = \beta_i(r) = r_i + \sum_{\alpha=1}^{q} u_i^\alpha\, r_{p+\alpha}, \qquad i = 1, \ldots, p,$$

$$S_\alpha = B_\alpha(R) = R_{p+\alpha} - \sum_{i=1}^{p} u_i^\alpha\, R_i, \qquad \alpha = 1, \ldots, q.$$

Prolonged symbol module:

$$\boxed{\mathcal{J} = (\boldsymbol{\beta}^*)^{-1}(\mathcal{S})}$$

$\mathcal{N}$ — leading monomials $s^J S_\alpha$

$\Longrightarrow$ normalized differential invariants I_J^α

$\mathcal{K}$ — complementary monomials $s^K S_\beta$

$\Longrightarrow$ phantom differential invariants I_K^β