The Outline Signature of a Convex Body

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Abstract. The orthogonal projection of a closed, bounded, strictly convex body in three-dimensional space onto a two-dimensional plane defines a closed strictly convex curve known as an outline of the body. The collection of all such outlines traces out a three-dimensional submanifold, that we call the total outline, of a certain vector bundle over two-dimensional projective space. We analyze the induced action of the three-dimensional Euclidean group of rigid motions (translations and rotations) on the total outline and use the method of equivariant moving frames to explicitly determine the fundamental first order differential invariant and the three invariant differential operators. We further prove that the entire differential invariant algebra is generated by the fundamental invariant through invariant differentiation. This enables us to construct a Euclidean-invariant outline signature that uniquely characterizes the total outline and hence the body up to Euclidean motion.

1. Introduction.

Constructing a three-dimensional model of a solid body through scanning — including computed tomography (CT), photogrammetry, laser scanning, structured light scanning, etc. — is time-consuming, expensive, and only feasible for certain types of objects, e.g., there are evident size and composition constraints. On the other hand, taking two-dimensional photos is, especially with the advent of universally accessible digital cameras and mobile phones, easy, almost cost-free, and achievable for well nigh any visible

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three-dimensional object. Moreover, since human and animal visual systems are extremely good at reconstructing (internal) three-dimensional models from their retinal image data, one can envision designing automatic computer-based algorithms that mimic constructing such models from a sufficient number of camera views. Indeed, there is an extensive literature on the reconstruction of solid bodies from two-dimensional images; see, for example, [**8**, **9**, **12**, **13**, **14**, **19**, **21**, **22**, **23**, **26**, **37**, **38**, **39**, **40**, **41**]; methods employed include photometric stereo (shape from shading), structure from motion, epipolar geometry, and stereoscopy.

In general, one must take into account the camera calibration. Here, we restrict our attention to the simplest scenario: a camera with infinite focal length. In this case, a "photo" of the object is obtained by orthogonal projection onto the camera image plane. The projection of a solid three-dimensional body produces a planar region, whose boundary is known as its *outline*, *apparent contour*, *profile*, or *silhouette*, [**8**, **9**, **13**, **22**, **23**]. Thus, the most basic reconstruction problem is to determine the bounding surface of the body from the complete collection of such outlines. The complications are evident. First, the shape of the outline varies with the viewing plane, although parallel planes produce identical projections (or, more accurately, parallel projections) and so we need only consider the planes passing through the origin, i.e., the Grassmannian of two-dimensional subspaces of $\mathbb{R}^3$, [**42**], which, by identifying each plane with its normal line, is isomorphic to the projective space $\mathbb{RP}^2$. Secondly, if the body is not convex, then there will be occlusions and interesting singularities of the outlines. Here, though, we will further simplify the problem by assuming the body is strictly convex and compact, although, since our results are local, they do apply, with appropriate restrictions, to the visible parts of non-convex bodies. Convexity and smoothness of the boundary of the body implies that each outline is a simple closed curve that is smooth and strictly convex, and can be identified as the projection of a simple closed smooth plane curve contained in the body's surface, known as the *rim*, [**23**], or *contour generator*, [**9**]. See [**13**, **14**, **37**] for methods that enable one, in favorable situations, to reconstruct a body from its outlines.

Subjecting a body to a rigid motion — a combination of rotations and translations[†] — does not alter its overall shape, and hence the reconstruction should take such transformations into account. Let SE(3) denote the Euclidean group consisting of all rigid motions of three-dimensional space. The equivalence problem for surfaces, that is characterizing when one can be mapped to the other by a rigid motion, can be solved, using the Cartan equivalence method, [**6**, **28**, **31**, **35**], by constructing a suitable differential invariant signature, which, in this case, is parametrized by the basic differential invariants: mean and Gauss curvature and their invariant derivatives. The question to be addressed here is how rigid motions affect the outlines of bodies, to construct the corresponding differential invariants, and thereby prescribe an outline signature that will characterize the outlines of (locally) rigidly equivalent bodies. Practical implementations and applications of these results to object recognition and symmetry detection will be the subject of future research. We remark that there is now a wide range of applications of other types of differential in-

[†] In this paper, we will ignore the action of reflections, although our analysis can be readily adapted to include them.

variant signatures in image processing, including cancer detection, [**16**, **17**], the reassembly of broken objects, such as jigsaw puzzles, [**20**], egg shells, [**15**], bones and lithics, [**1**], as well as in classical invariant theory, [**3**, **29**], and algebraic geometry, [**10**, **25**].

Remark: In [**4**, **5**, **24**], the behavior of Euclidean invariants of space curves under projection was analyzed. However, these results are not directly applicable to the present problem because the outlines of a solid body are not projections of a unique space curve.

We begin our analysis, in Section 2, by presenting the basic geometry of convex bodies in three-dimensional space and their projected outlines. The collection of all such outlines forms a three-dimensional submanifold of a certain vector bundle over projective space, that we call the *total outline*, and we are interested in how the Euclidean group acts thereon. The method of normal forms will enable us to derive and generalize a striking formula due to Koenderink, [**22**], relating the curvature of the outline curve to the curvature of the normal curve in the camera direction and the Gaussian curvature of the surface. In Section 3 we apply the method of equivariant moving frames to determine the differential invariants and invariant differential operators for the total outline of the body. Finally, in Section 4, we prove that the outline differential invariant algebra is generated by a single first order differential invariant, which enables us to prescribe an outline signature for a generic convex body, parametrized by its invariant derivatives of order at most 2. Appendix A contains a brief introduction to the method of equivariant moving frames, while Appendix B summarizes the normal form theory of plane and space curves, and of surfaces in three-dimensional space under the Euclidean group.

2. Outlines and Rims.

We begin by describing the geometry underlying the orthogonal projection of a solid body onto planes, which serves to model taking photographs at various angles with a camera that has infinite focal length. We use the standard Euclidean metric and associated dot product throughout. The relevant transformation group is the six-dimensional Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$ consisting of orientation-preserving rigid motions — rotations and translations:

$$\mathbf{p} \longmapsto R\,(\mathbf{p} + \mathbf{a}), \qquad R \in \mathrm{SO}(3), \qquad \mathbf{a} \in \mathbb{R}^3, \qquad \mathbf{p} \in \mathbb{R}^3, \tag{2.1}$$

where, in order to simplify later calculations, we act by first translating and then rotating.

Let $\mathbb{RP}^2$ be the real projective plane consisting of all lines passing through the origin in $\mathbb{R}^3$. Each line $\ell \in \mathbb{RP}^2$ defines a plane $\ell^\perp = \{\, \mathbf{w} \in \mathbb{R}^3 \mid \mathbf{w} \cdot \ell = 0 \,\}$ through the origin, namely its orthogonal complement, and vice versa. The *tautological bundle* $\mathcal{K} \to \mathbb{RP}^2$ is defined so that the fiber of $\mathcal{K}$ over a line $\ell \in \mathbb{RP}^2$ is the line itself: $\mathcal{K}|_\ell = \ell$. It is not hard to prove that $\mathcal{K}$ is an analytic line bundle, [**42**]. We let $\mathcal{K}^\perp \to \mathbb{RP}^2$ be its dual, the *normal plane bundle*, whose fiber over a line $\ell \in \mathbb{RP}^2$ is its orthogonal complement: $\mathcal{K}^\perp|_\ell = \ell^\perp$. Observe that the vector bundle $\mathcal{K}^\perp$ has two-dimensional fibers and hence forms a four-dimensional analytic manifold, which we will refer to as the *outline bundle*, for reasons that will shortly become clear. We can alternatively identify $\mathcal{K}^\perp$ as the tautological bundle over the Grassmannian of two-dimensional subspaces of $\mathbb{R}^3$.

We introduce the following coordinate systems on the outline bundle $\mathcal{K}^\perp$. On $\mathbb{RP}^2$, we can employ homogeneous coordinates $[\alpha, \beta, \gamma]$, where $(\alpha, \beta, \gamma) \in \mathbb{R}^3 \setminus \{0\}$ is a nonzero vector in the direction of a line ℓ, and we identify $[\alpha, \beta, \gamma] \simeq \lambda[\alpha, \beta, \gamma]$ whenever $\lambda \neq 0$. Alternatively, let $P_0 = \{\gamma = 0\}$ denote the $\alpha\beta$ plane. On the dense open subset $U_0 = \{\ell \notin P_0\} \simeq \mathbb{R}^2 \subset \mathbb{RP}^2$, we employ inhomogeneous coordinates $(p, q) = (\alpha/\gamma, \beta/\gamma)$ to represent the line $[p, q, 1] \simeq [\alpha, \beta, \gamma]$.

On the Cartesian product $\mathbb{RP}^2 \times \mathbb{R}^3$, given a line ℓ with homogeneous coordinates $[\alpha, \beta, \gamma]$, a point $\mathbf{w} = (\xi, \eta, \zeta) \in \ell^\perp$ if and only if $\alpha\xi + \beta\eta + \gamma\zeta = 0$. The normal plane bundle can thus be identified with the algebraic codimension 1 submanifold

$$\mathcal{K}^\perp = \{\, [\alpha, \beta, \gamma; \xi, \eta, \zeta] \mid \ \alpha\xi + \beta\eta + \gamma\zeta = 0 \,\} \subset \mathbb{RP}^2 \times \mathbb{R}^3. \tag{2.2}$$

Over the open dense subset $U_0 \subset \mathbb{RP}^2$ coordinatized by (p, q) as above, we can identify its restriction as the graph of a simple bilinear function:

$$\mathcal{K}^\perp \mid U_0 = \{\, (p, q; \xi, \eta, \zeta) \mid \ \zeta = -p\xi - q\eta \,\} \subset \mathbb{R}^5. \tag{2.3}$$

We will employ the local coordinates $(p, q; \xi, \eta)$ on $\mathcal{K}^\perp$ to perform our calculations.

Let $\Omega \subset \mathbb{R}^3$ be a convex three-dimensional *body*, meaning that it is compact, with non-empty interior and smooth (C^2) boundary $S = \partial\Omega$. We will assume that Ω is *strictly convex*, by which we mean that the principal curvatures (relative to the outwards normal) of S are everywhere strictly positive. Given $\ell \in \mathbb{RP}^2$, let $\pi_\ell\colon \mathbb{R}^3 \to \ell^\perp$ denote the orthogonal projection along the line ℓ onto its orthogonal complement $\ell^\perp$. Let $D_\ell = \pi_\ell(\Omega) \subset \ell^\perp$ denote the planar domain obtained by projecting Ω onto $\ell^\perp$. Our convexity assumption implies that D_ℓ is a compact strictly convex planar domain, whose boundary $C_\ell = \partial D_\ell$ is called the *outline* of Ω in the direction ℓ, [**23**]. Convexity and smoothness of $S = \partial\Omega$ implies that C_ℓ is a strictly convex closed plane curve of class at least C^3, [**23**]. (When Ω is non-convex, the outline curves can exhibit interesting singularities, [**9**, **22**, **38**].)

The *rim* $R_\ell \subset S$ associated with a line $\ell \in \mathbb{RP}^2$ is defined as the set of all points in the surface that project to the outline: $\pi_\ell(R_\ell) = C_\ell$, [**23**]. Strict convexity of S implies that $\pi_\ell\colon R_\ell \xrightarrow{\ \sim\ } C_\ell$ is one-to-one. A point $\mathbf{p} \in S$ belongs to R_ℓ if and only if $\ell \in TS|_p$, or, equivalently, $\mathbf{n} \cdot \ell = 0$, where $\mathbf{n}$ is the unit outward normal to S at $\mathbf{p}$.

In this study, we are particularly interested in how the differential invariants of S, its rim curves R_ℓ, and its outlines C_ℓ are interrelated. To this end, we will apply normal form techniques, as described in Appendix B; see also [**36**]. Let us assume, by applying a suitable rotation, that the projection direction is parallel to the y axis, i.e., the line $\ell_y = [0, 1, 0]$. Thus, at each point $\mathbf{p} \in R_y = R_{\ell_y}$ in the corresponding rim curve, the tangent plane $TS|_{\mathbf{p}}$ contains ℓ_y. We can further rotate around ℓ_y so that the tangent plane $TS|_{\mathbf{p}}$ also contains $\ell_x = [1, 0, 0]$, i.e., that it coincides with the xy plane. By translating $\mathbf{p}$ to the origin, the effect is to place the surface at the point $\mathbf{p}$ in *rotated normal form*, obtained by rotating the standard Euclidean normal form expansion for a surface — see (B.5) — through an angle θ equal to the angle between the y axis and the second principal direction. In other words, the surface can be identified as the graph of a function passing through the origin with Taylor expansion

$$u(x, y) = \tfrac{1}{2}\kappa_1 \widehat{x}^2 + \tfrac{1}{2}\kappa_2 \widehat{y}^2 + \tfrac{1}{6}\kappa_{1,1} \widehat{x}^3 + \tfrac{1}{2}\kappa_{1,2} \widehat{x}^2\widehat{y} + \tfrac{1}{2}\kappa_{2,1} \widehat{x}\widehat{y}^2 + \tfrac{1}{6}\kappa_{2,2} \widehat{y}^3 + \ \cdots\ , \tag{2.4}$$

where

$$\widehat{x} = x\cos\theta - y\sin\theta, \qquad \widehat{y} = x\sin\theta - y\cos\theta. \tag{2.5}$$

Here κ_1, κ_2 are the principal curvatures of the surface at the point $\mathbf{p}$, and the additional numerical subscripts on the coefficients of the third order terms indicate the third order differential invariants obtained by invariant differentiation of the principal curvatures; see the discussion following (B.5). Higher order terms in the Taylor expansion can be systematically determined using the equivariant moving frame recurrence formulae (A.2).

For example, if a piece† of the surface under consideration is identified with the graph of a function $z = u(x, y)$, then the piece of the rim curve R_y associated with projection along the y axis is given implicitly by the pair of equations

$$z = u(x, y), \qquad u_y(x, y) = 0, \tag{2.6}$$

where we use subscripts to indicate derivatives. Assuming $u_{yy} \neq 0$, the second equation can be locally uniquely solved for $y = y(x)$, from which we obtain $z(x) = u(x, y(x))$. The resulting piece of the rim curve R_y is parametrized by $(x, y(x), z(x))$ while the corresponding piece of the outline curve $C_y = \pi_y(R_y)$ is obtained by projecting the rim onto the $x\,z$ plane and hence parametrized by $(x, z(x))$. Their derivatives are obtained by implicit differentiation:

$$\begin{aligned} y_x &= -\frac{u_{xy}}{u_{yy}}, \\ y_{xx} &= -\frac{u_{xxy} + 2\,u_{xyy}\,y_x + u_{yyy}\,y_x^2}{u_{yy}} = -\frac{u_{yy}^2 u_{xxy} - 2\,u_{xy}\,u_{yy}u_{xyy} + u_{xy}^2 u_{yyy}}{u_{yy}^3}, \end{aligned} \tag{2.7}$$

where we evaluate all derivatives at $x = y = 0$, and, similarly,

$$\begin{aligned} z_x &= u_x + u_y y_x = 0, \\ z_{xx} &= u_{xx} + 2\,u_{xy}\,y_x + u_{yy}\,y_x^2 + u_y y_{xx} = \frac{u_{xx}u_{yy} - u_{xy}^2}{u_{yy}}, \end{aligned} \tag{2.8}$$

and so on, where the second expression follows because the first order derivatives of u in rotated normal form all vanish at the origin: $u_x = u_y = 0$.

Observe that since $z(0) = z_x(0) = 0$, the outline curve C_y, parametrized by $(x, z(x))$, is in planar Euclidean normal form (B.1), and hence we can identify

$$z(x) = \tfrac{1}{2}\,\kappa^O x^2 + \tfrac{1}{6}\,\kappa_s^O x^3 + \cdots,$$

where κ^O, κ_s^O are, respectively, its curvature and the derivative of curvature with respect to arc length at the projected point. Thus, using (2.4–5) to evaluate (2.8) and its higher order counterparts, we deduce, after algebraic simplification, the following key result.

† By a *piece*, we mean a connected subset whose interior is non-empty and whose boundary is piecewise smooth, cf. [**34**].

Theorem 2.1. *Given a projection direction $\ell \in \mathbb{RP}^2$, the curvature of the outline curve C_ℓ at a point $\mathbf{q} = \pi_\ell(\mathbf{p})$ is given by*

$$\kappa^O = \frac{K}{\kappa(\theta)}, \tag{2.9}$$

where $K = \kappa_1 \kappa_2$ is the Gauss curvature of S at the point $\mathbf{p}$, while

$$\kappa(\theta) = \kappa_1 \sin^2\theta + \kappa_2 \cos^2\theta, \tag{2.10}$$

is Euler's formula *for the curvature of the curve $C = P \cap S$ obtained by intersecting the surface with the plane $P = \operatorname{span}\{\mathbf{n}, \ell\}$ spanned by the normal $\mathbf{n}$ to S at $\mathbf{p}$ and the projection line ℓ at an angle θ with the second principal direction. Furthermore,*

$$\kappa^O_s = \frac{\psi(\theta)}{\kappa(\theta)^3}\,, \tag{2.11}$$

where

$$\psi(\theta) = \kappa_2^3 \kappa_{1,1} \cos^3\theta + 3\kappa_1\kappa_2^2\kappa_{1,2}\sin\theta\cos^2\theta + 3\kappa_1^2\kappa_2\kappa_{2,1}\sin^2\theta\cos\theta + \kappa_1^3\kappa_{2,2}\sin^3\theta. \tag{2.12}$$

The first of these, (2.9), is Koenderink's beautiful formula for the curvature of the outline curve, [**22**]; see also [**9**; p. 62]. The second, (2.11), appears to be new. Higher order counterparts can be obtained by writing out the higher order terms in the normal form Taylor expansions, making use of the recurrence formulae (A.2).

On the other hand, we can use the standard differential geometric formulas, [**18**], to compute the curvature and torsion for the rim curve, which is parametrized by $\mathbf{p}(x) = (x, y(x), z(x))$. The key quantities are

$$\|\,\mathbf{p}_x\,\| = \frac{\sqrt{\alpha(\theta)}}{\kappa(\theta)}, \qquad \|\,\mathbf{p}_x \times \mathbf{p}_{xx}\,\| = \frac{\sqrt{\beta(\theta)}}{\kappa(\theta)^3}, \qquad \mathbf{p}_x \times \mathbf{p}_{xx} \cdot \mathbf{p}_{xxx} = -\,\frac{\gamma(\theta)\,\delta(\theta)}{\kappa(\theta)^6}, \tag{2.13}$$

where

$$\begin{aligned}
&\alpha(\theta) = \kappa_1^2 \sin^2\theta + \kappa_2^2\cos^2\theta, \qquad\qquad \beta(\theta) = \kappa_1^2\kappa_2^2\,\kappa(\theta)^2\,\alpha(\theta) + \gamma(\theta)^2,\\
&\gamma(\theta) = \kappa_2^2\kappa_{1,2}\cos^3\theta + \kappa_2(2\kappa_1\kappa_{2,1} - \kappa_2\kappa_{1,1})\sin\theta\cos^2\theta\\
&\qquad\qquad + \kappa_1(\kappa_1\kappa_{2,2} - 2\kappa_2\kappa_{1,2})\sin^2\theta\cos\theta - \kappa_1^2\kappa_{2,1}\sin^3\theta,\\
&\delta(\theta) = \kappa_2^2(3\kappa_1\kappa_{2,1} + \kappa_2\kappa_{1,1})\cos^3\theta + 3\kappa_1\kappa_2(\kappa_1\kappa_{2,2} - \kappa_2\kappa_{1,2})\sin\theta\cos^2\theta\\
&\qquad\qquad + 3\kappa_1\kappa_2(-\kappa_1\kappa_{2,1} + \kappa_2\kappa_{1,1})\sin^2\theta\cos\theta + \kappa_1^2(\kappa_1\kappa_{2,2} + 3\kappa_2\kappa_{1,2})\sin^3\theta.
\end{aligned} \tag{2.14}$$

Theorem 2.2. *The curvature and torsion of the rim curve R_ℓ are given by*

$$\kappa^R = \sqrt{\frac{\beta(\theta)}{\alpha(\theta)^3}}, \qquad \tau^R = -\,\frac{\gamma(\theta)\,\delta(\theta)}{\beta(\theta)}. \tag{2.15}$$

3. The Total Outline and its Differential Invariants.

In this section, we apply the method of equivariant moving frames, as outlined in Appendix A, to explicitly construct the basic first order differential invariant, as well as the operators of invariant differentiation, for the total outline of a solid body under Euclidean motion. As we subsequently prove, all higher order differential invariants can be generated from the basic differential invariant by repeated invariant differentiation.

Given a strictly convex compact body $\Omega \subset \mathbb{R}^3$ with smooth bounding surface $S = \partial\Omega$, the subset of the outline bundle traced out by all of its outline curves forms a three-dimensional submanifold, which we call the *total outline* of the body and denote by

$$\mathcal{O} = \mathcal{O}(\Omega) = \left\{ \, (\ell, \mathbf{w}) \in \mathcal{K}^{\perp} \;\middle|\; \mathbf{w} \in C_\ell = \partial\pi_\ell(S) \subset \ell^{\perp} \, \right\} \subset \mathcal{K}^{\perp}. \tag{3.1}$$

The papers [**13**, **14**, **37**] explain how to reconstruct the surface S of the body from its total outline.

Remark: Not every three-dimensional submanifold of the outline bundle is a total outline. For this to be the case, its intersection with each fiber must be a closed convex curve. Thus, there are transversality, topological, and geometrical constraints on the submanifolds that form total outlines of convex bodies.

Two bodies are said to be *rigidly equivalent* if there is a Euclidean transformation mapping one to the other, so that $\widetilde{\Omega} = g \cdot \Omega$ for some $g = (R, \mathbf{a}) \in \mathrm{SE}(3)$, as in (2.1). We are interested in studying the induced action of the Euclidean group SE(3) on their total outlines, mapping $\mathcal{O}(\Omega)$ to $\mathcal{O}(\widetilde{\Omega})$. In other words, we wish to understand how translations and rotations act on the outline curves, and then determine their invariants.

Given the Euclidean action in the form (2.1), we claim that the induced action of $(R, \mathbf{a}) \in \mathrm{SE}(3)$ on the outline bundle $\mathcal{K}^{\perp}$ is provided by the formula

$$(\ell, \mathbf{w}) \longmapsto (L, \mathbf{W}) = (R\ell, R\widetilde{\mathbf{w}}), \qquad \text{where} \qquad \widetilde{\mathbf{w}} = \mathbf{w} + \mathbf{a} - (\mathbf{a} \cdot \mathbf{n})\,\mathbf{n}. \tag{3.2}$$

Here, $\ell \in \mathbb{RP}^2$ is a line and $\mathbf{n} \in \ell$ is a unit vector[†] in the camera direction, so $\| \mathbf{n} \| = 1$, while $\mathbf{w} \in \ell^{\perp}$ belongs to the orthogonal complement camera image plane. Observe that the transformation (3.2) maintains the orthogonality relation, meaning that $\widetilde{\mathbf{w}} \in \ell^{\perp}$ and hence $\mathbf{W} \in L^{\perp}$. We will refer to (3.2) as the *outline action* of the Euclidean group SE(3).

In terms of the above local coordinates (p, q, ξ, η) on the outline bundle $\mathcal{K}^{\perp}$, as presented in (2.3),

$$\mathbf{n} = \frac{(p, q, 1)}{n}, \qquad \text{where} \qquad n = \sqrt{1 + p^2 + q^2}\,. \tag{3.3}$$

The translations have trivial action on $\mathbb{RP}^2$, while on the fiber coordinates the translation corresponding to $\mathbf{a} = (a, b, c) \in \mathbb{R}^3$ maps the point $\mathbf{w} = (\xi, \eta, \zeta) \in \ell^{\perp}$ to the point

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[†] Keep in mind that there is a sign ambiguity in the choice of unit vector $\mathbf{n} \in \ell$, but this does not affect the formula (3.2). Also, $\mathbf{n}$ no longer denotes the surface normal.

$\widetilde{\mathbf{w}} = (\widetilde{\xi}, \widetilde{\eta}, \widetilde{\zeta}) \in \ell^\perp$, where

$$\widetilde{\xi} = \xi + \frac{a m^2 - b p q - c p}{n^2}, \quad \widetilde{\eta} = \eta + \frac{-a p q + b l^2 - c q}{n^2}, \quad \widetilde{\zeta} = \zeta + \frac{-a p - b q + c(p^2 + q^2)}{n^2}, \tag{3.4}$$

and where, for later convenience, we set

$$l = \| \, (1, 0, -p) \, \| = \sqrt{1 + p^2}, \qquad m = \| \, (0, 1, -q) \, \| = \sqrt{1 + q^2}. \tag{3.5}$$

Note that, as a consequence of (2.3), only the first and second entries of (3.4) are needed to prescribe the outline action of the translations.

On the other hand, the rotation $R \in \mathrm{SO}(3)$ acts by its standard representation on $(x, y, z) \in \mathbb{R}^3$ and $[\alpha, \beta, \gamma] \in \mathbb{RP}^2$, which in turn induces the linear fractional action

$$R\colon \, (p, q) \;\longmapsto\; (P, Q) = \left(\frac{\mathbf{r}^1 \cdot \mathbf{n}}{\mathbf{r}^3 \cdot \mathbf{n}}, \, \frac{\mathbf{r}^2 \cdot \mathbf{n}}{\mathbf{r}^3 \cdot \mathbf{n}} \right) = \left(\frac{r^1_1 p + r^1_2 q + r^1_3}{r^3_1 p + r^3_2 q + r^3_3}, \, \frac{r^2_1 p + r^2_2 q + r^2_3}{r^3_1 p + r^3_2 q + r^3_3} \right) \tag{3.6}$$

on the inhomogeneous coordinates. Here $\mathbf{r}^1, \mathbf{r}^2, \mathbf{r}^3$ are the rows of R, while r^i_j are its individual entries. Similarly, the rotations act via the usual representation on the translated fiber coordinates: $\widetilde{\mathbf{w}} \mapsto R\,\widetilde{\mathbf{w}}$, or, in full detail, in view of (2.3),

$$R\colon \, \big(\, \widetilde{\xi}, \widetilde{\eta} \,\big) \;\longmapsto\; (\Xi, H) = \big((r^1_1 - r^1_3 p)\, \widetilde{\xi} + (r^1_2 - r^1_3 q)\, \widetilde{\eta}, \; (r^2_1 - r^2_3 p)\, \widetilde{\xi} + (r^2_2 - r^2_3 q)\, \widetilde{\eta} \big). \tag{3.7}$$

The action

$$(p, q; \xi, \eta) \;\longmapsto\; (P, Q; \Xi, H) \tag{3.8}$$

of $\mathrm{SE}(3)$ on the total outline is given in local coordinates by the combined rotation and translation formulas (3.4–7).

Let us now implement a normalization process that places the body and hence its outline in a suitable normal form. The first step is to rotate the camera direction into normal form, by applying $\widehat{R} \in \mathrm{SO}(3)$ so that the rotated line $\widehat{R}\ell$ is parallel to the z axis[†], or, equivalently, normalize $p = q = 0$. Thus, in view of (3.6), this requires that the first and second rows of $\widehat{R}$ be orthogonal to the camera direction ℓ, and hence $\widehat{\mathbf{r}}^1, \widehat{\mathbf{r}}^2$ form an orthonormal basis of its orthogonal complement $\ell^\perp$. Consequently (again ignoring sign ambiguities) the third row is $\widehat{\mathbf{r}}^3 = \mathbf{n}$, as in (3.3). The first two rows are then determined modulo a subsequent rotation around the z axis.

We shall employ the orthonormal basis obtained by applying Gram–Schmidt to the evident non-orthogonal basis $(1, 0, -p), (0, 1, -q)$ of $\ell^\perp$, producing

$$\widehat{\mathbf{r}}^1 = \frac{(1, 0, -p)}{l}\,, \qquad \widehat{\mathbf{r}}^2 = \frac{(-pq, l^2, -q)}{l\,n}\,, \qquad \widehat{\mathbf{r}}^3 = \mathbf{n} = \frac{(p, q, 1)}{n}\,. \tag{3.9}$$

[†] In the preceding section, it was more convenient to use the y axis for the camera direction. This could be maintained here, but would require a different system of local coordinates on $\mathcal{K}^\perp$.

As noted above, this normalization specifies the rotation matrix R up to a rotation around the z axis, and hence

$$R = R_\varphi \widehat{R}, \qquad R_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \widehat{R} = \begin{pmatrix} \widehat{\mathbf{r}}^1 \\ \widehat{\mathbf{r}}^2 \\ \mathbf{r}^3 \end{pmatrix}, \tag{3.10}$$

where φ is the remaining as yet unnormalized rotation parameter.

The next step is to apply a translation to move the point $\mathbf{w} \in \ell^\perp$ to the origin. Setting $\widetilde{\xi} = \widetilde{\eta} = 0$ in (3.4) serves to specify two of the translation parameters:

$$a = pc - l^2\xi - pq\eta, \qquad b = qc - pq\xi - m^2\eta. \tag{3.11}$$

Note that the rotated point $\mathbf{W} = R\widetilde{\mathbf{w}}$ remains at the origin.

To normalize the remaining group parameters, namely c and φ, we prolong the group action to the first jet space $\mathrm{J}^1(\mathcal{K}^\perp, 3)$ that is determined by three-dimensional submanifolds (total outlines) of the outline bundle. In terms of the above local coordinates, we assume that the total outline $\mathcal{O} \subset \mathcal{K}^\perp$ can be locally written as the graph of a smooth function of the form

$$\eta = f(p, q, \xi), \tag{3.12}$$

which is valid under an appropriate transversality assumption on $\mathcal{O}$. Keep in mind that not every graph (3.12) can be locally identified with a total outline; see the remarks following equation (3.1).

In other words, in our jet bundle computations, we treat p, q, ξ as independent variables and η as a dependent variable. The corresponding first order jet coordinates are η_p, η_q, η_ξ, representing partial derivatives of (3.12), and similarly for their higher order counterparts. In non-transversal situations, one can employ a different system of local coordinates, or, alternatively, derive the formulae for the differential invariants for a general parametrized total outline submanifold, incorporating the infinite-dimensional reparametrization pseudo-group in the (more challenging) calculation; this will not be attempted here. Moreover, it is not hard to produce the parametric formulas directly from their non-parametric versions by replacing the derivatives of the dependent variable with respect to the independent variables by their more complicated parametric counterparts, although the resulting expressions are too unwieldy to write down in this paper.

The required formulas for the prolonged actions of the Euclidean group on jet bundles are obtained by applying the associated implicit differentiation operators, [**11**], which are

$$\begin{pmatrix} \mathrm{D}_P \\ \mathrm{D}_Q \\ \mathrm{D}_\Xi \end{pmatrix} = \mathbf{J}^{-T} \begin{pmatrix} \mathrm{D}_p \\ \mathrm{D}_q \\ \mathrm{D}_\xi \end{pmatrix} \qquad \text{where} \qquad \mathbf{J} = \begin{pmatrix} \mathrm{D}_p P & \mathrm{D}_q P & \mathrm{D}_\xi P \\ \mathrm{D}_p Q & \mathrm{D}_q Q & \mathrm{D}_\xi Q \\ \mathrm{D}_p \Xi & \mathrm{D}_q \Xi & \mathrm{D}_\xi \Xi \end{pmatrix}. \tag{3.13}$$

Here $\mathbf{J}$ denotes the total Jacobian matrix for the transformed independent variables, as given in (3.8), whose invertibility is required to maintain transversality of the transformed outline hypersurface $\overline{S} = g \cdot S$ for $g \in \mathrm{SE}(3)$:

$$S = \{\, \eta = f(p, q, \xi) \,\} \quad \longmapsto \quad \overline{S} = g \cdot S = \{\, H = F(P, Q, \Xi) \,\}. \tag{3.14}$$

The prolonged Euclidean action on the jet coordinates is obtained by repeatedly applying $\mathrm{D}_P, \mathrm{D}_Q, \mathrm{D}_\Xi$ to the transformed dependent variable H. The resulting expressions are too lengthy to write out, but are readily handled by symbolic manipulation software, e.g., Mathematica. *Note*: It is important that the formulas for the prolonged action be computed *before* any moving frame normalizations are implemented. An alternative approach would be to employ the recursive moving frame algorithm in [**32**], but the additional intricacies are not necessary in this relatively simple situation.

Substituting the preceding normalizations (3.10, 11) into the prolonged action formulas, after simplification we find the transformed jet coordinate η_ξ is given by

$$H_\Xi = \mathrm{D}_\Xi H = \frac{\rho \sin\varphi + n\,\eta_\xi \cos\varphi}{\rho\cos\varphi - n\,\eta_\xi \sin\varphi}, \qquad \text{where} \qquad \begin{aligned} \rho &= l^2 + p\,q\,\eta_\xi, \\ \sigma &= p\,q + m^2 \eta_\xi, \end{aligned} \tag{3.15}$$

the latter quantity appearing in subsequent calculations. We thus normalize $H_\Xi = 0$ by setting $\tan\varphi = -\,n\,\eta_\xi/\rho$, and so (ignoring sign ambiguities as usual),

$$\cos\varphi = \frac{\rho}{l\,\tau}, \qquad \sin\varphi = -\,\frac{n\,\eta_\xi}{l\,\tau}, \tag{3.16}$$

where

$$\tau = \sqrt{\rho + \sigma\,\eta_\xi} = \sqrt{l^2 + 2\,p\,q\,\eta_\xi + m^2\eta_\xi^2} = \sqrt{1 + \eta_\xi^2 + (p + q\,\eta_\xi)^2} \tag{3.17}$$

is real. Substituting all the preceding normalizations (3.10, 11, 16) into the two remaining formulas for the prolonged action on η_p, η_q, we find that the first, namely $H_P = \mathrm{D}_P H$, is independent of the as yet unormalized translation parameter c, and thus provides the fundamental first order differential invariant

$$I = \frac{n^2(\rho\,\eta_p + \sigma\,\eta_q) + (q - p\,\eta_\xi)\,(\rho\,\xi + \sigma\,\eta)}{\rho + \sigma\,\eta_\xi}, \tag{3.18}$$

where n, ρ, σ are given in (3.3, 15). We remark that the absolute rational differential invariant I is a ratio of two relative polynomial differential invariants having a common weight, [**28**]. In particular, when the total outline (3.12) goes through the origin $p = q = \xi = \eta = 0$, then the formula (3.18) for the fundamental differential invariant simplifies to

$$I\,\big|_0 = \frac{\eta_p + \eta_q \eta_\xi}{1 + \eta_\xi^2}\,. \tag{3.19}$$

On the other hand, the normalized formula for the prolonged action on η_q does contain the remaining translation parameter c, which can be normalized by solving $H_Q = \mathrm{D}_Q H = 0$, producing

$$\begin{aligned} c = \tau^{-2}\Big\{ l^2(\eta_q - \eta_p\eta_\xi) &+ \big[\,p\,l^2 + (2\,p^2 - 1)\,q\,\eta_\xi - p\,(q^2 + 2)\,\eta_\xi^2\,\big]\,\xi \\ &+ \big[\,(p^2 + 2)\,q + p\,(2\,q^2 - 1)\,\eta_\xi + q\,m^2\eta_\xi^2\,\big]\,\eta \Big\}. \end{aligned} \tag{3.20}$$

The combined formulas (3.9, 10, 16, 11, 20) serve to prescribe the equivariant moving frame $\rho\colon \mathrm{J}^1(\mathcal{K}^\perp, 3) \longrightarrow \mathrm{SE}(3)$.

Finally, the explicit formulas for the invariant differential operators $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ are obtained by substituting the moving frame formulas for the group parameters into the implicit differentiations $\mathrm{D}_P, \mathrm{D}_Q, \mathrm{D}_\Xi$ as given in (3.13), respectively. After simplification with the aid of MATHEMATICA, the resulting expressions are

$$\begin{aligned} \mathcal{D}_1 &= \frac{n}{\tau^3}\left\{ \tau^2(\rho\,\mathrm{D}_p + \sigma\,\mathrm{D}_q) + (\alpha\,\eta_p + \beta\,\eta_q + \gamma\,\xi + \delta\,\eta)\,\mathrm{D}_\xi \right\}, \qquad \mathcal{D}_3 = \frac{1}{\tau}\,\mathrm{D}_\xi, \\ \mathcal{D}_2 &= \frac{n^2}{\tau^3}\left\{ \tau^2(-\eta_\xi \mathrm{D}_p + \mathrm{D}_q) + \left[\,\sigma\,(\eta_p\eta_\xi - \eta_q) + (p + q\,\eta_\xi)(\xi\,\eta_\xi - \eta)\,\right]\mathrm{D}_\xi \right\}, \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} \alpha &= l^2\sigma + m^2\eta_\xi\rho = p\,q\,l^2 + 2\,l^2m^2\eta_\xi + p\,q\,m^2\eta_\xi^2, \qquad & \beta &= m^2\tau^2 - 2\,n^2, \\ \gamma &= l^2\,(p + 2\,q\,\eta_\xi) + p\,(q^2-1)\,\eta_\xi^2, & \delta &= (p^2-1)\,q + m^2(2\,p + q\,\eta_\xi)\,\eta_\xi. \end{aligned}$$

The invariant differential operators (3.21) map the differential invariants to higher order differential invariants. Moreover, as we will prove below in Theorem 4.1, *all* higher order differential invariants can be obtained by repeatedly applying the invariant differential operators to the basic first order differential invariant (3.18), which therefore generates the entire outline differential invariant algebra.

4. The Outline Signature.

In the preceding section, we constructed the right-equivariant moving frame on the first order jet space corresponding to the cross-section

$$p = q = \xi = \eta = \eta_q = \eta_\xi = 0. \tag{4.1}$$

This amounts to placing the body in the *normal form* so that its total outline (3.12) passes through the origin, as defined by our choice of local coordinates on the outline bundle, and has the following Taylor expansion there:

$$\eta = I\,p + \tfrac{1}{2}\,J_1\,p^2 + J_2\,p\,q + \tfrac{1}{2}\,J_3\,q^2 + J_4\,p\,\xi + J_5\,q\,\xi + \tfrac{1}{2}\,J_6\,\xi^2 + \ \cdots\ . \tag{4.2}$$

As usual, [**36**], once the submanifold is placed in normal form, its non-constant Taylor coefficients, when expressed in terms of of the original (unnormalized) jet coordinates, form a complete system of functionally independent differential invariants. Thus, the total outline has a single first order differential invariant† $I = \iota(\eta_\xi)$, given by (3.18), followed by 6 independent second order differential invariants, 10 independent third order differential invariants, and so on.

There are three invariant differential operators, (3.21), and hence one expects to be able to produce 3 combinations of the second order differential invariants by differentiating the first order differential invariant I. The general theory, [**11**], says that, because the moving frame is of order 1, the differential invariants of order ≤ 2 generate all higher order differential invariants. In fact, as we shall prove below, one can produce all 6 second order

† Here ι denotes the *invariantization map* associated with the moving frame; see Appendix A.

differential invariants, and hence all the differential invariants, by invariant differentiation of I alone.

For the outline action of the Euclidean group, the recurrence formulas, cf. (A.2), relating the differentiated invariants can be explicitly constructed using its infinitesimal generators of the group action. A straightforward calculation based on the local coordinate formulas (3.4–7) produces the vector fields

$$\begin{aligned}
\mathbf{v}_1 &= \frac{1+q^2}{1+p^2+q^2}\,\frac{\partial}{\partial \xi} - \frac{p\,q}{1+p^2+q^2}\,\frac{\partial}{\partial \eta}, & \mathbf{v}_4 &= -\,p\,q\,\frac{\partial}{\partial p} - (1+q^2)\,\frac{\partial}{\partial q} + (p\,\xi + q\,\eta)\,\frac{\partial}{\partial \eta}, \\
\mathbf{v}_2 &= -\frac{p\,q}{1+p^2+q^2}\,\frac{\partial}{\partial \xi} + \frac{1+p^2}{1+p^2+q^2}\,\frac{\partial}{\partial \eta}, & \mathbf{v}_5 &= -\,(1+p^2)\,\frac{\partial}{\partial p} - p\,q\,\frac{\partial}{\partial q} + (p\,\xi + q\,\eta)\,\frac{\partial}{\partial \xi}, \\
\mathbf{v}_3 &= -\frac{p}{1+p^2+q^2}\,\frac{\partial}{\partial \xi} - \frac{q}{1+p^2+q^2}\,\frac{\partial}{\partial \eta}, & \mathbf{v}_6 &= -\,q\,\frac{\partial}{\partial p} + p\,\frac{\partial}{\partial q} - \eta\,\frac{\partial}{\partial \xi} + \xi\,\frac{\partial}{\partial \eta}.
\end{aligned} \tag{4.3}$$

The first three, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, generate translations in the coordinate directions, while the latter three, $\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6$, generate rotations around, respectively, the x, y and z axes. As the reader can check, they satisfy the usual commutation relations for the Lie algebra $\mathfrak{se}(3)$.

Using the general symbolic moving frame calculus, the initial recurrence formulae are given by

$$\mathcal{D}_1 I = J_1, \qquad \mathcal{D}_2 I = J_2, \qquad \mathcal{D}_3 I = J_4. \tag{4.4}$$

Furthermore, the moving frame symbolic calculus, [**11**, **27**, **33**], can be used to produce the commutator relations for the three invariant differential operators:

$$\begin{aligned}
[\,\mathcal{D}_1, \mathcal{D}_2\,] &= -\,J_4\,\mathcal{D}_1 - \,J_5\,\mathcal{D}_2 + (\,2\,I\,J_5 - J_3\,)\,\mathcal{D}_3, \\
[\,\mathcal{D}_1, \mathcal{D}_3\,] &= -\,J_6\,\mathcal{D}_2 + (\,2\,I\,J_6 - J_5\,)\,\mathcal{D}_3,
\end{aligned} \qquad [\,\mathcal{D}_2, \mathcal{D}_3\,] = J_6\,\mathcal{D}_1. \tag{4.5}$$

We can appeal to a variant of the commutator trick that was introduced in [**31**] to generate all the commutator invariants — that is, the coefficients of the invariant differential operators on the right hand side of the commutator formula (4.5) — by differentiation of I for sufficiently generic total outlines. In more detail, applying the commutators to I produces the required formulas:

$$\begin{aligned}
J_6 &= \frac{[\,\mathcal{D}_2, \mathcal{D}_3\,]I}{\mathcal{D}_1 I} = \frac{\mathcal{D}_2 J_4 - \mathcal{D}_3 J_2}{J_1}, \\
J_5 &= 2\,I\,J_6 - \frac{J_6\,\mathcal{D}_2 I + [\,\mathcal{D}_1, \mathcal{D}_3\,]I}{\mathcal{D}_3 I} = \frac{\mathcal{D}_3 J_1 - \mathcal{D}_1 J_4 + 2\,I\,J_4 J_6 - J_2 J_6}{J_4}, \\
J_3 &= 2\,I\,J_5 - \frac{J_4\,\mathcal{D}_1 I + J_5\,\mathcal{D}_2 I + [\,\mathcal{D}_1, \mathcal{D}_2\,]I}{\mathcal{D}_3 I} = \frac{\mathcal{D}_2 J_1 - \mathcal{D}_1 J_2 + 2\,I\,J_4 J_5 - J_1 J_4 - J_2 J_5}{J_4},
\end{aligned} \tag{4.6}$$

which, combined with (4.4), produces all 6 second order invariants as rational functions of the invariant derivatives of I — provided the denominators $J_1 = \mathcal{D}_1 I$, $J_4 = \mathcal{D}_3 I$ do not vanish, which is true generically. (It would be of interest to characterize those special outlines which do not satisfy this condition; see [**36**] for the analogous issue in Euclidean surface geometry.) We have thus proved:

Theorem 4.1. *The algebra of differential invariants of a generic total outline is generated by the normalized first order differential invariant $I = \iota(\eta_\xi)$ through invariant differentiation.*

One possibility for non-genericity is when the differential invariant I is constant. As far as I can tell, this does not necessarily imply that the outline differential invariants J_3, J_5, J_6 are constant, since the formulas (4.6) are no longer valid. On the other hand, by a general theorem due to Cartan, [**28**], the differential invariants are all constant if and only if the total outline coincides with a piece of a three-dimensional orbit of a three-dimensional subgroup of the Euclidean group SE(3). According to [**2**; Table 1] there are precisely 4 inequivalent three-dimensional subgroups. Two of these act transitively on $\mathbb{R}^3$ and so are not relevant to the problem at hand. The third is equivalent to SE(2) whose orbits are planes, and hence do not occur as pieces of the boundary of a strictly convex body. The last class is represented by the rotation subgroup SO(3), whose orbits are spheres. Observe that, while SO(3) acts non-freely as a symmetry group of the sphere, it acts freely on its total outline which can be identified with a circle bundle over $\mathbb{RP}^2$. The orbits of the other three classes of 3-dimensional subgroups contained in the outline bundle are, thus, not total outlines, even locally. This observation serves to justify the following result:

Theorem 4.2. *The differential invariant I is constant on (a piece of) the total outline if and only if the corresponding part of the surface coincides with (a piece of) a sphere.*

In the case when the body is not spherical, but its boundary includes one or more spherical pieces, the local symmetry groupoid of each piece contains an open neighborhood of the identity in SO(3). We refer the reader to [**34**] for further details on symmetry groups and symmetry groupoids.

As a consequence of Theorem 4.1, the differential invariant I can be used to generate a signature that provides necessary and sufficient conditions for two suitably generic† total outlines to be locally equivalent under a Euclidean motion.

Theorem 4.3. *Generically, a total outline is locally uniquely characterized up to rigid motion by a differential invariant signature prescribed by I and its invariant derivatives of order ≤ 2.*

Proof: Since the total outline has dimension 3, in the absence of any continuous local symmetry group, a total outline has exactly three functionally independent differential invariants. Generically, these could be $I, \mathcal{D}_1 I, \mathcal{D}_2 I$ (or $I, \mathcal{D}_1 I, \mathcal{D}_3 I$, or $I, \mathcal{D}_2 I, \mathcal{D}_3 I$), and hence we can locally write $\mathcal{D}_3 I$ and the second derived invariants as functions thereof:

$$\mathcal{D}_3 I = \Phi_3(I, \mathcal{D}_1 I, \mathcal{D}_2 I), \qquad \mathcal{D}_i \mathcal{D}_j I = \Phi_{ij}(I, \mathcal{D}_1 I, \mathcal{D}_2 I), \qquad 1 \leq i, j, \leq 3. \tag{4.7}$$

However, some of the latter syzygies are redundant, since we can differentiate the first to determine some of the second, namely those for which $j = 3$. Nevertheless, once we know

† The notion of genericty here is not the same as in Theorem 4.1.

all the second order syzygies, the higher order ones can all be constructed through further invariant differentiation. *Q.E.D.*

In general, as described in detail in [**34**], at regular points, the local codimension of the signature prescribes the dimension of the local symmetry groupoid of the boundary of the body, which includes any global symmetries. For example, a piece of an ellipsoid with two equal semi-axes has a local one-dimensional symmetry group consisting of rotations around the other semi-axis, and hence its outline signature has codimension 1. Theorem 4.1 governs the case of a three-dimensional symmetry group, for which the outline is a single point, i.e., has dimension 0. In the generic case when the boundary has a discrete local symmetry group, the *index* of the outline signature, meaning the cardinality of the inverse image of a point, determines the number of local symmetries at the point.

Remark: For a compact body to admit a one-parameter local symmetry group, its boundary would have to locally be a surface of translation, a surface of revolution, or a helicoid surface, obtained by, respectively, rotating, translating, or screwing a plane curve, [**36**]. Another non-generic case not covered by Theorem 4.3 is when there is only one differentiated invariant $\mathcal{D}_i I$ that is functionally independent of I, but a second functionally independent invariant shows up among the second derived invariants; in this case, as in [**28**], one must extend to the third order derived invariants to construct a signature.

5. Further Directions.

At this point, a number of issues arise, which serve to indicate potentially fertile lines of research for further developing and applying the results in this paper. In particular:

- While the results here have some mathematically appealing aspects, a critical question is whether there are practical applications to imaging, which served to motivate the study. In general, one would not expect to have access to the full extent of the total outline of a solid body, and hence is discrete sampling thereof sufficient to classify and (approximately) reconstruct Euclidean equivalent bodies and characterize their (local) symmetries in real world situations? For example, as pointed out in [**37**], as far as the reconstruction problem goes, the total outline contains redundant information and one can reconstruct (a piece of) the body from a codimension one submanifold such as that formed by a drone that flies past it. The invariants and analysis of these sub-outlines would be worth pursuing.
- Most bodies are not convex, and so the extension of the analysis here to the non-convex situation is a particularly important direction.
- Further, extensions to more realistic camera models with finite focal length, [**9**, **23**] are also of great interest.
- As for the outline differential invariants and signature, in practical situations one does not have analytic formulas, and so the question arises as to how to construct effective numerical approximation schemes. As advocated in [**7**], the most promising approach is to design numerical schemes that retain the underlying Euclidean invariance, which will involve suitable combinations of the "joint outline invariants". The latter are in need of classification, which can be accomplished by implementing the equivariant moving frame method on the Cartesian product action, cf. [**30**].

Appendix A. Moving Frames, Invariantization, and Recurrence Formulae.

Here we will quickly review the basics of moving frames, referring the reader to [**11**, **27**, **33**, **36**] for details. Let G be a Lie group acting (locally) freely on a manifold M; typically M is an open subset of a sufficiently high order jet bundle, and G acts via prolongation, cf. [**28**]. (Another possibility is a Cartesian product action, [**30**].) A moving frame is specified by a choice of *cross-section* $K \subset M$ to the group orbits, meaning a submanifold of complementary dimension that intersects them transversally. Usually, the cross-section is specified by setting a certain number of the local coordinates to appropriately chosen constants. The right-equivariant *moving frame* map $\rho\colon M \to G$ is locally uniquely specified by solving the *normalization equations* $g\cdot z \in K$ for the group parameters $g = \rho(z)$ in terms of the point $z \in M$.

With the moving frame in hand, the *invariantization* $I = \iota(F)$ of a function $F(z)$ on M is obtained by transforming it, to obtain $F(g \cdot z)$, and then substituting the moving frame formulas for the group parameters; the result is an invariant function $I(z) = F(\rho(z) \cdot z)$. In particular, invariantization does not affect an invariant, $\iota(I) = I$. Moreover, invariantization respects all algebraic operations. Let $I_j(z) = \iota(z_j)$ denote the *fundamental invariants* obtained by invariantizing the coordinate functions. In particular, those used to specify the cross-section are the corresponding normalization constants, known as the *phantom invariants*; the remainder form a complete system of functionally independent invariants. The invariantization of a more general function is then found by merely replacing its coordinate expression by the corresponding algebraic combination of the fundamental invariants:

$$\iota\big[\, F(z_1, \ldots, z_m)\,\big] = F(I_1, \ldots, I_m). \tag{A.1}$$

In the particular case when $F = I$ is an invariant, and hence unaffected by invariantization, (A.1) becomes the *Replacement Rule* that allows one to immediately rewrite any invariant in terms of the fundamental invariants.

Invariantization can be extended to other objects, including differential forms and differential operators. As above, one transforms the object by the group transformations and then replaces all the group parameters by their moving frame formulas. In particular, on a jet bundle, invariantization of the differential functions produces the fundamental differential invariants. The invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ that map differential invariants to higher order differential invariants are obtained by invariantization of the total derivatives $\mathrm{D}_1, \ldots, \mathrm{D}_p$, where p denotes the number of independent variables $x = (x_1, \ldots, x_p)$, which transform into the implicit differentiation operators used to compute the prolonged group action, while invariantization of the differential forms leads to invariant differential forms, such as a G-invariant arc length form.

In general, the *recurrence formulae* express the resulting differentiated invariants in terms of the fundamental differential invariants:

$$\mathcal{D}_i \iota(F) = \iota(\mathrm{D}_i F) + \sum_{\sigma=1}^{r} K_i^\sigma\, \iota\big[\, \mathbf{v}_\sigma(F)\,\big], \qquad i = 1, \ldots, p. \tag{A.2}$$

Here F denotes a differential function, $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are a basis for the prolonged infinitesimal generators of the Lie group action, while K_i^σ are certain differential invariants known as the *Maurer–Cartan invariants*. In particular, if we take F to be one of the cross-section coordinates, then its invariantization is a constant phantom invariant, and hence the left hand side of the recurrence formula (A.2) is zero. The phantom recurrence formulas form a system of linear equations which can be uniquely solved for the Maurer–Cartan invariants as rational functions of the normalized differential invariants. Substituting these expressions back into (A.2) produces the explicit recurrence relations that completely specifies the structure of the differential invariant algebra. Vice versa, the recurrence formulae can be used to re-express the fundamental differential invariants in terms of the differentiated invariants.

The recurrence formula (A.2) also applies as stated when F is a differential form, in which case the differential operators/infinitesimal generators act via Lie differentiation. In particular, setting $F \mapsto dx^i,\ \ i = 1, \ldots, p$, to be the differentials of one of the independent variables produces the *commutator formulas* for the invariant differential operators:

$$[\, \mathcal{D}_j, \mathcal{D}_k \,] = \mathcal{D}_j\, \mathcal{D}_k - \mathcal{D}_k\, \mathcal{D}_j = \sum_{i=1}^{p} Y^i_{jk} \mathcal{D}_i. \tag{A.3}$$

The coefficients

$$Y^i_{jk} = -\, Y^i_{kj} = \sum_{\sigma=1}^{r} \left[\, K^\sigma_k\, \iota(\mathrm{D}_j \xi^i_\sigma) - K^\sigma_j\, \iota(\mathrm{D}_k \xi^i_\sigma) \,\right] \tag{A.4}$$

are known as the *commutator invariants* and are explicitly determined in terms of the basic differential invariants by (A.4), in which ξ^i_σ denotes the coefficient of $\partial/\partial x^i$ in the infinitesimal generator $\mathbf{v}_\sigma$.

Appendix B. Normal Forms for Curves and Surfaces.

In the method of equivariant moving frames, the choice of cross-section in jet space can be viewed as placing a submanifold near a prescribed point into a distinguished *normal form* through judicious application of group transformations; see [**36**] for details. In the case of the planar Euclidean group acting on curves $C \subset \mathbb{R}^2$, we first apply a translation to move the base point $\mathbf{p} \in C$ to the origin. We then rotate the translated curve so that its tangent is horizontal. The only remaining ambiguity is a 180° rotation about the origin, but this can be fixed by specifying an orientation on the curve. The coefficients of the Taylor expansion of the resulting *Euclidean normal form*, when expressed in terms of the original jet coordinates, are thus the differential invariants obtained through the moving frame normalization process. A calculation based on the recurrence formulas (A.2) reveals

$$y = \tfrac{1}{2}\,\kappa x^2 + \tfrac{1}{6}\,\kappa_s\, x^3 + \tfrac{1}{4!}(\kappa_{ss} + 3\,\kappa^3)\, x^4 + \tfrac{1}{5!}(\kappa_{ssss} + 19\,\kappa^2 \kappa_s)\, x^5 + \ \cdots\ . \tag{B.1}$$

where κ is the curvature and the s subscripts on κ indicate its derivatives with respect to arc length.

In the case of space curves $C \subset \mathbb{R}^3$, under the action (2.1) of the three-dimensional Euclidean group SE(3), the standard normal form is obtained by translating and rotating the curve so that it goes through the origin, has tangent in the direction of the x axis, and, assuming we are not at an inflection point, has second order contact with the xy plane. The resulting Taylor expansions have the form

$$\begin{aligned} y &= \tfrac{1}{2}\kappa x^2 + \tfrac{1}{6}\kappa_s x^3 + \tfrac{1}{24}(\kappa_{ss} + 3\kappa^3 - \kappa\tau^2)x^4 + \ \cdots \ , \\ z &= \tfrac{1}{6}\kappa\tau x^3 + \tfrac{1}{24}(2\tau\kappa_s + \kappa\tau_s)x^4 + \ \cdots \ , \end{aligned} \tag{B.2}$$

where κ, τ are the curvature and torsion. As before, the formulas for the coefficients are found through application of the recurrence formulas (A.2). Observe that if $\tau \equiv 0$, so that the curve is planar, then the first equation in (B.2) reduces to the planar normal form (B.1).

Let us finally review the normal form of a surface at a point $\mathbf{p} \in S \subset \mathbb{R}^3$ under the action of the Euclidean group SE(3). We begin by translating so that $\mathbf{p}$ is mapped to the origin. One can then rotate the surface around the origin so that its tangent plane becomes horizontal, i.e., is the xy plane. Thus, the surface is locally prescribed by the graph of a function $z = u(x, y)$ whose Taylor expansion at the origin begins with quadratic terms. We can further rotate the surface around the z axis in order to eliminate the xy term in the expansion. Thus, the Euclidean normal form of the surface starts with the quadratic Taylor polynomial

$$u(x, y) = \tfrac{1}{2}\kappa_1 x^2 + \tfrac{1}{2}\kappa_2 y^2 + \ \cdots \ , \tag{B.3}$$

where the coefficients κ_1, κ_2 are differential invariants known as the *principal curvatures*. The *mean* and *Gauss curvatures* are given by the average and product thereof, respectively:

$$H = \tfrac{1}{2}(\kappa_1 + \kappa_2), \qquad K = \kappa_1 \kappa_2. \tag{B.4}$$

The point $\mathbf{p} \in S$ is *non-umbilic* if $\kappa_1 \neq \kappa_2$; in this case, the full Euclidean normal form expansion can be written down:

$$u(x, y) = \tfrac{1}{2}\kappa_1 x^2 + \tfrac{1}{2}\kappa_2 y^2 + \tfrac{1}{6}\kappa_{1,1} x^3 + \tfrac{1}{2}\kappa_{1,2} x^2 y + \tfrac{1}{2}\kappa_{2,1} x y^2 + \tfrac{1}{6}\kappa_{2,2} y^3 + \ \cdots \ . \tag{B.5}$$

Here the additional numerical subscripts on κ_1, κ_2 indicate invariant differentiation with respect to the underlying invariant differential operators $\mathcal{D}_1, \mathcal{D}_2$, so that $\kappa_{1,2} = \mathcal{D}_2\kappa_1$, $\kappa_{1,21} = \mathcal{D}_1\mathcal{D}_2\kappa_1$, and so on. See [**31, 35**] for higher order terms and explicit formulas, although these are not required here.

In the umbilic case, the remaining rotation around the normal direction cannot be fixed by normalization at order 2. If the order 3 Taylor coefficients are not all zero, then one can rotate in order to make one of them, say that of y^3, equal to zero. There is thus either a single distinguished direction or three distinguished directions depending upon whether the associated cubic polynomial has 1 or 3 real roots. The resulting normal form contains 5 third order basic differential invariants, namely the coefficients of the quadratic and cubic terms in the Taylor expansion. If the third order terms all vanish, then one can use the rotation to normalize one of the fourth order coefficients unless the fourth order terms are a multiple of $(x^2 + y^2)^2$. The chosen normalization constant need not be zero if the quartic polynomial has no real roots. And so on. The only case where there is a residual continuous rotational symmetry is when the surface is locally a surface of revolution, $z = h(x^2 + y^2)$, possessing (at least) a one-parameter isotropy group at the point in question.

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