

INVARIANT GEOMETRIC EVOLUTIONS OF SURFACES AND VOLUMETRIC SMOOTHING*

PETER J. OLVER[†], GUILLERMO SAPIRO[‡], AND ALLEN TANNENBAUM[§]

Abstract. The study of geometric flows for smoothing, multiscale representation, and analysis of two- and three-dimensional objects has received much attention in the past few years. In this paper, we first survey the geometric smoothing of curves and surfaces via geometric heat-type flows, which are invariant under the groups of Euclidean and affine motions. Second, using the general theory of differential invariants, we determine the general formula for a geometric hypersurface evolution which is invariant under a prescribed symmetry group. As an application, we present the simplest affine invariant flow for (convex) surfaces in three-dimensional space, which, like the affine-invariant curve shortening flow, will be of fundamental importance in the processing of three-dimensional images.

Key words. invariant surface evolutions, partial differential equations, geometric smoothing, symmetry groups

AMS subject classifications. 35K22, 53A15, 53A55, 53A20, 35B99

PII. S0036139994266311

1. Introduction. Geometric smoothing, multiscale representation, and analysis of two-dimensional (2D) and three-dimensional (3D) objects are of extreme importance in different applications of computer graphics, computer-aided geometric design (CAGD), and image analysis. These can be used for smoothing out noise or for the representation of objects at different levels of detail. When one is interested in the geometry of the given object, it is important to perform these operations in an intrinsic geometric manner. Thus image processing via geometric driven diffusion-type flows has become a major topic of research in the last few years [58]. In our work, the object is deformed via a partial differential equation which is invariant with respect to a given symmetry group.

The smoothing and multiscale representation of planar objects was originally performed by filtering their boundary with a Gaussian filter [9, 38, 72]. This process is equivalent to deforming the curve via the classical heat flow which is an extrinsic process unrelated to the geometry of the given image. As we will see in section 2, this and other problems of the classical heat flow can be effectively solved by replacing it with geometric heat flows that were developed during the last few years [27, 28, 56, 59, 61, 63, 64].

The first question that we want to address in this paper is the problem of finding analogous flows for smoothing and multiscale representation of 3D objects. The main

*Received by the editors April 20, 1994; accepted for publication (in revised form) November 7, 1995. This research was supported in part by National Science Foundation grants DMS-9116672, DMS-9204192, DMS-8811084, and ECS-9122106; Air Force Office of Scientific Research grants AFOSR-90-0024 and F49620-94-1-0058DEF; Army Research Office grants DAAL03-92-G-0115, DAAL03-91-G-0019, DAAH04-93-G-0332, and DAAH04-94-G-0054; the Rothschild Foundation–Yad Hanadiv; and Image Evolutions, Ltd.

<http://www.siam.org/journals/siap/57-1/26631.html>

[†]Department of Mathematics, University of Minnesota, Minneapolis, MN 55455 (olver@ima.umn.edu).

[‡]Center for Intelligent Control Systems and Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA 02139. Present address: Hewlett-Packard Laboratories, Palo Alto, CA 94304 (guille@hpl.hp.com).

[§]Department of Electrical Engineering, University of Minnesota, Minneapolis, MN 55455 (tannenba@ee.umn.edu).

goal of this part is to review the literature on surface evolution relevant to volumetric smoothing. We first describe the available results on geometric smoothing of graphs (images) via geometric smoothing of their level sets. We then discuss the smoothing of surfaces via properly 3D flows, where the surface deforms with velocity given by functions of its principal curvatures. In order to make the paper accessible to the largest possible audience, many of the background results are presented in an informal way, i.e., without the mathematical details which may be found in the relevant references.

In the second part of the paper, we extend the results, first reported in [55, 56] for planar curves, to any dimension and any Lie group. We present the most general form of an invariant geometric flow for hypersurfaces. We show that the invariant flows can be formulated as functions of the invariant metric and invariant curvature, which are the basic differential invariant descriptors, together with the variational (Euler–Lagrange) derivative corresponding to this metric. We also show that if the transformation group is volume preserving, the variational derivative is invariant as well. This result extends for geometric flows the classical classification of differential invariant signatures. Then, as an example, we derive the simplest affine invariant geometric flow for (convex) 3D surfaces.

This paper is organized as follows: in section 2, we describe some of the key results related to planar curve geometric smoothing, which will be helpful to motivate and understand the surface theory. Section 3 reviews the literature in the geometric flows of surfaces, first provided by smoothing via level sets, and then fully 3D geometric smoothing. Section 4 describes the main contribution of the paper. Then in section 5, we discuss affine invariant flows of surfaces, and discussion and concluding remarks are given in section 6.

2. Planar curve smoothing. In this section, we review some results on geometric smoothing of planar curves that we wish to extend to surfaces and, more generally, hypersurfaces in Euclidean space of arbitrary dimension. Unfortunately, as we shall see, some of the desirable results for curves do not hold for surfaces. A family of plane curves (or hypersurfaces) will deform in time according to some evolution equation, where “time” represents “scale” in our multiscale resolution.

We begin with the case of curves in the plane. Let $\mathcal{C}(p, t)$ denote a family of embedded closed curves in the plane \mathbf{R}^2 . Here t parametrizes the family, and p , independent of t , parametrizes each curve. For each fixed t , the curve parametrized by $\mathcal{C}(p, t)$ will be the image (trace), denoted by $\text{Img}[\mathcal{C}(p, t)]$. This frees us from dependence on the explicit parametrization. In other words, if we reparametrize the curve by using $q = q(p, t)$, where $\partial q / \partial p > 0$, as our new parameter, then the curve $\mathcal{C}(p, t) = \widehat{\mathcal{C}}(q(p, t), t)$ has the same image: $\text{Img}[\mathcal{C}(p, t)] = \text{Img}[\widehat{\mathcal{C}}(q, t)]$.

Originally, the classical heat flow

$$(1) \quad \frac{\partial \mathcal{C}}{\partial t} = \frac{\partial^2 \mathcal{C}}{\partial p^2}, \quad \mathcal{C}(p, 0) = \mathcal{C}_0(p),$$

was proposed for smoothing curves, [9, 38, 39, 40, 41, 44, 47, 58, 72, 73, 74]. The solution to the heat equation (1) is obtained by convolution of the initial data with the Gaussian kernel

$$(2) \quad \mathcal{G}(p, t) := \frac{1}{\sqrt{4\pi t}} \exp \left\{ \frac{-p^2}{4t} \right\}.$$

The problem is that different parametrizations of the curve will give different results in (1), i.e., different Gaussian multiscale representations. This is an undesirable property, since parametrizations are in general arbitrary and may not be connected with the geometry of the curve. We attempt to solve this problem by choosing a parametrization which is intrinsic to the curve, i.e., that can be computed when only $\text{Img}[\mathcal{C}]$ is given. The most natural intrinsic parametrization is the *Euclidean arc-length* (traditionally denoted by s)

$$(3) \quad v(p, t) := \int_0^p \left\| \frac{\partial \mathcal{C}(\xi, t)}{\partial \xi} \right\| d\xi,$$

which means that the curve is traversed with constant velocity: $\| \mathcal{C}_v \| \equiv 1$. The Gaussian filter $\mathcal{G}(v, t)$, or the corresponding heat flow, can then be applied, but the problem is that the arc-length parameter is time dependent. Furthermore, this kind of evolution violates some of the basic properties of scale space. For example, the order is not preserved; i.e., if one initial curve is contained within another, it is not guaranteed that their images at later times will necessarily have this property. Also, the semigroup property, which means that $\mathcal{C}(v, t_1)$ can be obtained from $\mathcal{C}(v, t_2)$ for any $0 \leq t_2 < t_1$, can be violated. The theory described below rectifies these difficulties.

Assume now that the family $\mathcal{C}(p, t)$ evolves according to the following general evolution equation:

$$(4) \quad \frac{\partial \mathcal{C}}{\partial t} = \alpha \vec{T} + \beta \vec{N}, \quad \mathcal{C}(p, 0) = \mathcal{C}_0(p),$$

where \vec{N} is the inward Euclidean unit normal and \vec{T} is the unit tangent [68]. The coefficients α and β are the tangential and normal components of the evolution velocity, respectively. The following lemma shows that under certain conditions the tangential component does not affect the curve images.

LEMMA 2.1. *Let β be a geometric quantity for a curve, i.e., a function whose definition is independent of the parametrization. Then a family of curves which evolves according to (4) can be converted into the solution of*

$$\hat{\mathcal{C}}_t = \bar{\alpha} \vec{T} + \bar{\beta} \vec{N}$$

for any continuous function $\bar{\alpha}$ by some reparametrization of the original solution. Moreover, since β is a geometric function, $\beta = \bar{\beta}$ as a function of the image curve.

For proofs of the lemma, see [21] and [61]. In other words, if the normal component β of the velocity is a geometric function of the curve, then the image curves $\text{Img}[\mathcal{C}(p, t)]$ are affected only by β . The tangential component α affects only the parametrization and not $\text{Img}[\mathcal{C}(p, t)]$, which is independent of the parametrization by definition. In particular, we can choose the tangential component to vanish, $\bar{\alpha} = 0$, and hence replace any geometric curve evolution by one purely in the normal direction:

$$(5) \quad \frac{\partial \mathcal{C}}{\partial t} = \beta \vec{N},$$

where $\beta = \vec{v} \cdot \vec{N}$ is the projection of the velocity vector on the normal direction.

One of the most important curve evolutions of the form (5) is

$$(6) \quad \frac{\partial \mathcal{C}}{\partial t} = \kappa \vec{N},$$

obtained for $\beta = \kappa$, the Euclidean curvature. Recalling the classical formula $\kappa \vec{N} = C_{vv}$ for the curvature in terms of the derivatives with respect to Euclidean arc-length (3) [68], we see that equation (6) can also be written as

$$(7) \quad C_t = C_{vv}.$$

Although equation (7) looks like the standard heat flow (1), it is genuinely nonlinear, since the arc-length parameter v is a function of time. Equation (6) or, equivalently, (7) has its origins in physical phenomena [6, 24, 29]. It is called the (*Euclidean*) *geometric heat flow*, or the *Euclidean curve shortening flow*, since the Euclidean perimeter shrinks as fast as possible when the curve evolves according to (6) [29]. Indeed, the fundamental results of Gage and Hamilton [27] and Grayson [28] show that any smooth, embedded, planar curve evolving according to (6) remains smoothly embedded, first becoming convex and then converging to a round point; i.e., for some finite T^* it becomes asymptotically circular while shrinking to a point as $t \rightarrow T^*$. Note that in spite of the local character of the evolution, global properties are obtained, which is a very interesting feature of this flow. For other results related to the Euclidean shortening flow, see [1, 6, 21, 27, 28, 29, 36].

Equation (7) (or (6)) has been proposed by different researchers [37, 44, 73] as an intrinsic, geometric, multiscale representation of closed curves that avoids the undesirable features of the classical heat flow. Proofs that it satisfies the basic properties required for a multiscale smoothing can be found in [44, 64]. These results are straightforward consequences of the results in [6, 27, 28].

Note that equation (6) is only invariant under the Euclidean group, consisting of translations, rotations, and reflections, since it is based on Euclidean differential geometry. In [59, 60, 61, 62, 63] we extended this geometric theory to the affine group, which consists of all area-preserving linear transformations together with translations. A general approach to the formulation of geometric curve flows for any Lie group appears in [55, 56, 63]. In general, let r denote the invariant arc-length of a given Lie group, i.e., its simplest invariant parameterization. The *geometric heat flow* of the group is obtained via

$$(8) \quad \frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial r^2}$$

and is invariant under the prescribed transformation group, since $\partial/\partial r$ is the unique invariant derivative of the group (see [56, 63]). More general invariant flows are obtained if arbitrary functions Ψ depending on the group-invariant curvature χ and its derivatives with respect to the group-invariant arc length (these are the fundamental differential invariants of the group) are incorporated into the flow:

$$(9) \quad C_t = \Psi(\chi, \chi_r, \chi_{rr}, \dots) C_{rr}.$$

In [56] we proved that (9) is indeed the most general geometric invariant flow for subgroups of the projective group, and the geometric heat flow is the simplest possible one for a number of important groups. One of the main purposes of the present paper is to extend these results to higher dimensions and to more general transformation groups.

The group normal C_{rr} is in general not perpendicular to the curve; i.e., it is not parallel to the Euclidean unit normal \vec{N} . Based on Lemma 2.1, we know that the effective velocity is obtained by the projection of the group normal onto the Euclidean

normal and expressing the group curvature in terms of the Euclidean curvature and its derivatives. For example, in the affine case, where r is replaced by the affine arc-length [12, 59]

$$s(p, t) := \int_0^p [C_\xi, C_{\xi\xi}]^{1/3} d\xi,$$

the affine-invariant geometric flow analogue of (8) is given by [59, 60, 61, 63]

$$(10) \quad \mathcal{C}_t = \kappa^{1/3} \vec{\mathcal{N}}.$$

This flow was also discovered by Alvarez et al. in their remarkable work [2] and used for image enhancement; see also [62, 65]. Using the theory of viscosity solutions and evolution of graphs, they also proved the uniqueness of the flow under a number of conditions which are natural for image processing. In [56] we proved that the evolution equation (10) can be uniquely characterized (up to constant multiple) as the “simplest flow having the affine group as symmetry group.” As in the Euclidean case, any smoothly embedded closed curve evolves under the affine flow in a smooth manner by first becoming convex and then shrinking to an “elliptical point,” meaning that as $t \rightarrow T^*$, the curve shrinks to a point while its shape becomes asymptotically an ellipse [8, 59, 60]. Moreover, all the properties of scale spaces hold [61]. For results on other interesting invariant flows incorporating invariance under the similarity and projective groups, see [56, 63, 64, 75]. It is important to note that, in contrast with the Euclidean and affine cases, in these cases the evolving curve may develop singularities.

Before concluding this section, let us point out another of the undesirable properties of Gaussian filtering that is also solved using geometric heat flows. A curve deforming according to the classical heat flow shrinks in a noncomputable form. This is due to the fact that the Gaussian filter also affects low frequencies of the curve coordinate functions [47]. Different authors proposed different solutions to this problem while always remaining in the area of Gaussian or linear filtering, i.e., nongeometric smoothers [32, 41, 47]. When a curve evolves according to a geometric heat flow, the shrinking factor can be computed, since the rate of change of area, length, or any other geometric quantity can be computed exactly. Based on this, in [64] we showed how to replace the geometric heat flow (8) by an analogous one, which keeps the area (length) constant. The approach is based on formulating a new geometric flow which deforms the curve according to the flow (8) while simultaneously expanding the plane in order to preserve area (length). This way, a geometric smoother without shrinking is obtained.

3. Geometric surface evolution. We now turn to the generalizations of multiscale smoothing of plane curves to the smoothing of surfaces. The first class of surfaces we consider are those described by the graph $z = \Phi(x, y)$ of a (smooth) function. Graphs are of particular importance in image processing since gray scale images are usually defined by (a discrete version of) a function $\Phi: U \rightarrow \mathbf{R}^+$, so that $\Phi(x, y)$ represents the gray value at the point $(x, y) \in U \subset \mathbf{R}^2$. The object now is to find a multiscale smoothing of the graph of the function Φ . One approach to smoothing a surface given as a graph is to smooth its level sets according to one of the geometric heat flows described in section 2. This topic has been studied in different works [2, 3, 15, 22, 23, 57, 62]; here we review some of the basic results.

As before, the time variable t represents the scale, and we consider a parametrized family of graphs (images) $\Phi(x, y, t)$. Let

$$(11) \quad \mathcal{X}_c(t) := \{(x, y) : \Phi(x, y, t) = c\}$$

denote the level set for a fixed gray value c at time t which, at regular values, is a (union of) plane curves. Assume that this level set evolves according to a geometric flow, which we take in the normal direction as in (5). Note that for the level sets, at a regular point, the unit normal is given by

$$(12) \quad \vec{\mathcal{N}} = \frac{\nabla \Phi}{\|\nabla \Phi\|}.$$

Differentiating (11) with respect to t and using (12) to substitute for the normal, we obtain the general level set evolution:

$$(13) \quad \Phi_t = -\beta \|\nabla \Phi\|.$$

Evaluating the geometric normal component β in terms of Φ and its derivative produces the evolution equation of the graph when its level sets evolve according to (5).

For example, if $\beta = \kappa$ so that (5) defines the Euclidean curve shortening flow, then the corresponding graphical evolution is

$$(14) \quad \Phi_t = \|\nabla \Phi\| \operatorname{div} \left(\frac{\nabla \Phi}{\|\nabla \Phi\|} \right) = \frac{\Phi_y^2 \Phi_{xx} - 2\Phi_x \Phi_y \Phi_{xy} + \Phi_x^2 \Phi_{yy}}{\Phi_x^2 + \Phi_y^2}.$$

In other words, if $\Phi(x, y, t)$ is a solution to (14), then its level sets move according to the Euclidean heat flow [3, 57]. See Alvarez, Lions, and Morel [3] for modifications of (14) for image selective smoothing and edge detection. For general results concerning the evolution of level sets, see [15, 22, 57, 66].

In the affine case, the invariant curve flow is given by (10). Therefore, the corresponding graphical version is

$$(15) \quad \Phi_t = (\Phi_y^2 \Phi_{xx} - 2\Phi_x \Phi_y \Phi_{xy} + \Phi_x^2 \Phi_{yy})^{1/3}.$$

If $\Phi(x, y, t)$ is a solution to (15), then its level sets move according to the affine-invariant heat flow [2, 62]. Interestingly, because there is no denominator, the affine version is better from both an analytical and a numerical point of view. As pointed out in [62], the numerical implementation of the affine image smoothing is more stable than the Euclidean one. In [46] the affine, Euclidean, and classical heat flows were compared for the processing of MRI images; the affine version produces much better results, as expected. In [65] we studied the affine flow for MRI smoothing as well.

In real applications, like image smoothing, the original surface and its level sets are nonsmooth. In [2, 15, 22, 23], the evolution of surfaces via level-sets type flows was extended to nonsmooth curves based on the theory of viscosity solutions; see [18]. The existence of a unique solution for Lipschitz initial curves was studied for the affine heat flow in [8]. The theory of level-sets flows is well developed for nonsmooth initial curves as well, allowing the practical implementation of this kind of smoothing process in real applications like image smoothing.

Turning to the multiscale smoothing of more general surfaces, we consider the geometric evolution of a (closed) surface in its normal direction. In contrast to the graphical case where the surface flow was driven by 2D evolutions of level sets, the flows now will be governed by properly 3D equations. We consider the surface analogue of the curve flow (5) in which β is a geometric function of the surface, i.e., is independent of the particular parametrization. (As with curves, the tangential components of a geometric evolution are not important, as they only influence the particular

parametrization.) In contrast to the planar case, certain geometric constraints must be imposed on the initial surface in order that the evolving surface remain smooth. However, for convex initial surfaces, many of the planar results remain valid.

The most popular choice for β is so that it depends on the principal curvatures of the surface. (We refer the reader to [13, 31, 68] for the fundamentals in the differential geometry of surfaces in 3D space.) The most important special case is the *mean curvature flow*, when β is the mean curvature or average of the principal curvatures, which was first investigated by Brakke [11]. Another important choice is when β is a function of the Gaussian curvature or product of the principal curvatures, although this case is not as well understood.

For inward mean curvature flow, Huisken [33] proved that a convex surface evolves into a round point, meaning that it becomes asymptotically spherical before collapsing to a point in finite time. Chow [17] proved the same result when the (inward) velocity is given by the square root of the Gaussian curvature. Urbas investigated the expanding evolution of convex surfaces in [70, 71], again, under certain conditions, proving they become asymptotically spherical. The situation for nonconvex surfaces is much more complicated and still the subject of much research; see, for example, [7, 30, 67]. In general, a nonconvex surface evolving according to the mean curvature will not remain smooth, or even connected, as illustrated by the famous dumbbell example. Gerhardt [26] proves that for certain expanding evolutions depending on the principal curvatures (which includes the mean curvature flow) an initially star-shaped surface remains smooth and star shaped and becomes asymptotically spherical; see also [69].

Initial boundary value problems in which the initial surface is bounded and must obey certain boundary constraints have been looked at by various authors. For mean curvature flows, Huisken [34] showed that a surface with a vertical contact angle at the boundary smooths and, provided the boundary has nonnegative mean curvature, converges to the solution of the minimal surface equation. See Chopp [16] for the computation of minimal surfaces using this geometric flow. On the other hand, Oliker and Uraltseva [50] showed that mean curvature evolution with fixed boundary may produce singularities at the boundary at some finite time. The authors also provide sufficient conditions on the domain and the initial surface for this problem to have classical solutions for all time. A normalized solution of the mean curvature flow with fixed boundary was shown to asymptotically approach the first eigenfunction of the Laplace operator with Dirichlet data in Ω ; the evolution also “picks up” the symmetries of the domain Ω . For example, if Ω is a sphere then asymptotically the solution becomes radially symmetric. See [51] for extensions and [48, 50] for a similar analysis of the Gaussian curvature flow. Further results on the evolution of graphs defined over all of \mathbf{R}^2 were obtained by Ecker and Huisken in [19]; under certain growth conditions, the mean curvature flow has a solution for all time. Moreover, the surface converges to a self-similar solution if and only if the initial graph is asymptotically linear at large distances; see also [49]. A nice review of mean curvature flows can be found in [35].

We conclude this section with some remarks on weak solutions of the aforementioned geometric flows. As pointed out in section 2 in [15, 22] the geometric evolution of level sets was studied in the framework of viscosity solutions. In [22] the mean curvature flow is analyzed, while in [15] more general evolution equations are studied. In both papers the authors showed the existence of a unique weak solution for partial differential equations in which the level sets evolve in time according to the mean

curvature. Short-term existence of a classical (smooth) solution is proved as well; see also [20, 23]. Therefore, even if the initial surface does not satisfy the properties which are required for long-term existence of classical solutions, for example convexity, nevertheless, a unique weak solution can be constructed based on the theory of viscosity solutions. These results allows one to generalize the definition of mean curvature flows also for nonsmooth surfaces. Of course, the generalized definition coincides with the classical one when the surface is smooth and the flow can be defined in the framework of classical differential geometry. These generalized flows also satisfy some of the analogous properties to the planar case, e.g., they preserve order, so that initial surfaces contained in each other remain that way.

Finally, the evolution of surfaces as level sets of functions was proposed and also studied experimentally by Osher and Sethian in [57] and Sethian in [66].

4. Invariant hypersurface flows. We now move on to present our general classification results for invariant evolution equations admitting prescribed symmetry groups. Of course our main interest is in evolution equations which describe some geometrically based diffusion of curves or surfaces of interest to image processing. Thus, the evolution equation will be most interesting in dimensions two and three. Moreover, the prescribed symmetry group will usually manifest itself as a subgroup of the full projective group. Nevertheless, the treatment of the general situation is not any more difficult, and so we will proceed in a completely general fashion.

We will be considering the evolution of hypersurfaces, which, for simplicity, we assume to be represented by the graph of a function. (In this section, since our considerations are local, we are not losing any generality. Moreover, the methods can be readily extended to parametrized hypersurfaces.) Thus, consider the $p+1$ -dimensional Euclidean space $E \simeq \mathbf{R}^p \times \mathbf{R}$, with coordinates $x = (x^1, \dots, x^p)$ representing the independent variables and $u \in \mathbf{R}$ the dependent variable. (Generalizations to several dependent variables are certainly possible, but again for simplicity we stick to the scalar case here.) Our hypersurface $\mathcal{S} \subset E$ will be identified with the graph of a function $u(x)$, defined on a domain $x \in D \subset \mathbf{R}^p$. The symmetry group G will be a finite-dimensional, connected transformation group acting on E . Each group transformation $g \in G$ will map hypersurfaces to hypersurfaces by pointwise transformation. For example, if G is the group of rotations on \mathbf{R}^{p+1} , then each hypersurface is rotated by the group transformations. Of course, if g is not sufficiently close to the identity, the transformed hypersurface may no longer be given by a graph; however, this does not cause any difficulties in the infinitesimal approach to be used in the analysis.

In Lie's theory of symmetry groups [53], one replaces the actual group transformations by their infinitesimal generators, which are vector fields on the domain E , taking the general form

$$(16) \quad \mathbf{v} = \xi(x, u) \frac{\partial}{\partial x} + \varphi(x, u) \frac{\partial}{\partial u} = \xi^1(x, u) \frac{\partial}{\partial x^1} + \dots + \xi^p(x, u) \frac{\partial}{\partial x^p} + \varphi(x, u) \frac{\partial}{\partial u}.$$

Each vector field generates a local one-parameter group of transformations (or flow) on E , obtained by integrating the associated system of ordinary differential equations

$$(17) \quad \frac{dx}{d\varepsilon} = \xi(x, u), \quad \frac{du}{d\varepsilon} = \varphi(x, u),$$

where ε represents the group parameter. In other words, the group transformations have the Taylor expansion

$$(18) \quad x(\varepsilon) = x + \varepsilon \xi(x, u) + \dots, \quad u(\varepsilon) = u + \varepsilon \varphi(x, u) + \dots.$$

The order ε terms in (18) are known as the infinitesimal group transformations and can be identified with the generating vector field (16). The different one-parameter groups combine to generate the entire connected group action of G .

Fixing the vector field (16), we let $u(x, \varepsilon)$ denote the one-parameter family of the family of hypersurfaces (functions) obtained from a given hypersurface $u(x, 0) = u(x)$ by applying the group transformation with parameter ε . (We assume ε is sufficiently small so that the transformed hypersurface remains a graph.) The infinitesimal change in the hypersurface is found by expanding in powers of ε using Taylor's theorem and the chain rule. Thus, the value of the transformed function u at the new point $x(\varepsilon)$ is given by

$$(19) \quad u(x(\varepsilon), \varepsilon) = u(x) + \varepsilon\varphi(x, u(x)) + \cdots .$$

On the other hand, if we are interested in the value of the transformed function at the original point $x = x(0)$, we substitute (18) into (19) to deduce the alternative expansion

$$(20) \quad u(x, \varepsilon) = u(x) + \varepsilon Q[u(x)] + \cdots .$$

The function

$$(21) \quad Q[u] = \varphi(x, u) - \sum_{i=1}^p \xi^i(x, u) \frac{\partial u}{\partial x^i}$$

is known as the *characteristic* of the vector field (16). The characteristic Q depends on first-order derivatives $u_i = \partial u / \partial x^i$ because the group transformations are acting on the independent variable x as well as the dependent variable u . In particular, a G -invariant hypersurface is independent of the group parameter ε and hence satisfies the first-order partial differential equation $Q(x, u^{(1)}) = 0$, indicating its "infinitesimal invariance" under the vector field \mathbf{v} . Vice versa, any infinitesimally invariant function, i.e., any solution to the characteristic equation $Q = 0$, is in fact invariant under the entire connected transformation group.

Consider a function $F[u] = F(x, u^{(n)})$ depending on x , u , and the derivatives of u , denoted by $u_J = D_J u$. Here $D_J = D_{j_1} D_{j_2} \cdots D_{j_k}$ are the total derivative operators which differentiate treating u as a function of x . For example, $D_i u = u_i$, $D_{ij} u = D_i(D_j u) = u_{ij} = \partial^2 u / \partial x^i \partial x^j$, etc. The infinitesimal variation in the function $F[u]$ under the group generated by the vector field \mathbf{v} is then given by

$$(22) \quad \left. \frac{d}{d\varepsilon} F[u(x, \varepsilon)] \right|_{\varepsilon=0} = \sum_J \frac{\partial F}{\partial u_J} D_J Q.$$

In (22) we evaluate F and u at the original point x . If we are interested in the value at the transformed point $x(\varepsilon)$, we must include an additional term arising from the change of independent variable, as in the passage from (20) to (19). We deduce the expansion

$$(23) \quad F[x(\varepsilon), u^{(n)}(x, \varepsilon)] = F(x, u^{(n)}) + \varepsilon \text{pr } \mathbf{v}(F) + \cdots ,$$

where

$$(24) \quad \text{pr } \mathbf{v}(F) = \sum_J \frac{\partial F}{\partial u_J} D_J Q + \sum_i \xi^i D_i F$$

defines the “prolongation” of the vector field \mathbf{v} , denoted $\text{pr } \mathbf{v}$, which forms the infinitesimal generator of the prolonged group action on the space of derivatives. See [53] for details.

A function $F(x, u^{(n)})$ is called a *differential invariant* if its value is not affected by the group transformations. Thus we require that the left-hand side of (23) be independent of ε . The infinitesimal invariance condition is obtained by differentiating with respect to ε . This produces

$$(25) \quad 0 = \text{pr } \mathbf{v}(F) = \sum_J \frac{\partial F}{\partial u_J} D_J Q + \sum_i \xi^i D_i F.$$

Condition (25), for \mathbf{v} an arbitrary infinitesimal generator of G , is necessary and sufficient for F to be a differential invariant. The problem of classifying differential invariants can be solved by methods dating back to Lie [53, 54]; see also [31] for methods based on Cartan’s theory of moving frames. In the case of curves, every differential invariant is a function of the group-invariant curvature and its derivatives with respect to the group invariant arc length. For surfaces, the complete classification of differential invariants is known in a few examples, but the general computations remain to be completed. We refer the reader to [53, 54] and references therein for more details on the theory of differential invariants and their applications in computer vision.

A transformation group G is called a symmetry group of a differential equation

$$(26) \quad F(x, u^{(n)}) = 0$$

if it maps solutions to solutions. The differential equation (26) admits G as a symmetry group if and only if the infinitesimal invariance condition

$$(27) \quad \text{pr } \mathbf{v}[F] = 0 \quad \text{whenever} \quad F = 0$$

holds for all infinitesimal generators of G . See [53] for a detailed discussion of how one uses the infinitesimal invariance conditions (27) to systematically compute the most general symmetry group of a differential equation.

Our goal is to determine the general form that a G -invariant evolution equation

$$(28) \quad u_t = K(x, u^{(n)})$$

must take. Here we have introduced an additional variable t —the time or scale parameter—which is not affected by our group transformations. Thus, for $p = 1$, we will determine all possible invariant curve evolutions in the plane under a given transformation group, while for $p = 2$ we find the invariant surface evolutions. According to (24), the infinitesimal change in the t -derivative of u at the transformed point is

$$(29) \quad \left. \frac{d}{d\varepsilon} u_t(x, t, \varepsilon) \right|_{\varepsilon=0} = D_t Q + \sum_{i=1}^p \xi^i D_i u_t = Q_u u_t,$$

where

$$(30) \quad Q_u = \frac{\partial Q}{\partial u} = \frac{\partial \varphi}{\partial u} - \sum_{i=1}^p \frac{\partial \xi^i}{\partial u} \frac{\partial u}{\partial x^i}.$$

Therefore, using the infinitesimal condition (27) and substituting for u_t according to the equation (28), we deduce the basic invariance condition that an evolution equation must satisfy in order to admit a prescribed symmetry group.

LEMMA 4.1. *A connected transformation group G is a symmetry group of the evolution equation $u_t = K[u]$ if and only if the infinitesimal condition*

$$(31) \quad \text{pr } \mathbf{v}(K) = Q_u K$$

holds for every infinitesimal generator \mathbf{v} of the group G with associated characteristic Q .

In the language of representation theory [54], the invariance condition (31) says that the function K must be a relative differential invariant of weight Q_u under the prolonged action of the transformation group G . In particular, if $u_t = K_0$ is a particular G -invariant evolution equation, then every other G -invariant evolution equation has the form $u_t = K$, where $K = IK_0$ and I is an arbitrary differential invariant for the group G . Thus, our analysis of invariant evolution equations requires us to determine a single particular case from which the general case can be deduced using the complete system of differential invariants.

For example, in the case of the Euclidean group acting on the plane, the simplest invariant evolution equation is the optical curve flow $u_t = \sqrt{1 + u_x^2}$, in which one moves by a fixed amount in the normal direction; e.g., we chose $\beta = 1$ in (5). Every other Euclidean invariant evolution equation has the form $u_t = I(\kappa, \kappa_v, \dots)\sqrt{1 + u_x^2}$, where I is an arbitrary function of the Euclidean curvature κ and its derivatives with respect to Euclidean arc-length v . In particular, choosing $I = \kappa$ produces the Euclidean curve shortening flow $u_t = u_{xx}/(1 + u_x^2)$. This example was generalized to arbitrary subgroups of the projective group in the plane in [55].

In order to discover a G -invariant evolution equation for an arbitrary group, we consider the G -invariant functionals. An n th order *variational problem* consists of finding the extremals (maxima or minima) of a *functional*

$$(32) \quad \mathcal{L}_D[u] = \int_D L(x, u^{(n)}) dx = \int_D L(x, u^{(n)}) dx^1 \wedge \dots \wedge dx^p,$$

subject to certain boundary conditions (whose precise form will not concern us here). We refer the reader to [25, 53] for an introduction to the required basics from the calculus of variations. The integrand $L[u] = L(x, u^{(n)})$, known as the *Lagrangian*, is a smooth function depending on x , u , and the derivatives of u . A transformation group G is a symmetry group of a variational problem provided it leaves the functional (32) invariant. More precisely, given a function $u(x)$ defined on a domain D and a one-parameter subgroup of G , we let $u(x, \varepsilon)$ denote the transformed function, which is defined on a transformed domain $D(\varepsilon)$. Invariance of the functional requires that $\mathcal{L}_{D(\varepsilon)}[u(x, \varepsilon)] = \mathcal{L}_D[u(x)]$. Using the standard Jacobian change of variables formula for multiple integrals, the infinitesimal invariance condition is then found by differentiating:

$$(33) \quad \begin{aligned} 0 &= \left. \frac{d}{d\varepsilon} \mathcal{L}_{D(\varepsilon)}[u(x, \varepsilon)] \right|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon} \int_D L[u(x(\varepsilon), \varepsilon)] \det \left[\frac{\partial x(\varepsilon)}{\partial x} \right] dx \Big|_{\varepsilon=0} \\ &= \int_D [\text{pr } \mathbf{v}(L) + L \text{ Div } \xi] dx. \end{aligned}$$

Here $\text{Div } \xi = \sum D_i \xi^i$ is the total divergence arising from the infinitesimal change in the independent variables.

LEMMA 4.2. *A connected transformation group G is a symmetry group of the variational problem $\int L dx$ if and only if every infinitesimal generator \mathbf{v} satisfies the infinitesimal condition*

$$(34) \quad \text{pr } \mathbf{v}(L) + L \text{Div } \xi = 0.$$

In other words, a Lagrangian L defines a G -invariant functional if and only if it is a relative differential invariant of weight $-\text{Div } \xi$. If $L_0[u]$ is a particular G -invariant Lagrangian, then the most general G -invariant variational problem has the form $\mathcal{L}[u] = \int I L_0 dx$, where I is an arbitrary differential invariant for the group G . (In our applications, the p -form $L_0 dx$ represents the G -invariant element of arc length or surface area.) This remark is fundamental in modern physical theories (e.g., string theory) in which one uses a group-invariant Lagrangian to construct the physical field equations based on the assumed symmetry group of the theory.

The smooth extremals (maxima and minima) of a variational problem are known to satisfy the classical Euler–Lagrange equation

$$(35) \quad E(L) \equiv \sum_{\#J=0}^n (-D)_J \frac{\partial L}{\partial u_J} = 0, \quad \alpha = 1, \dots, q,$$

where $(-D)_J = (-D_{j_1})(-D_{j_2}) \cdots (-D_{j_k})$ is the signed total derivative. This condition is the infinite-dimensional analog of the vanishing gradient condition for maxima and minima of ordinary functions. The Euler–Lagrange equation is obtained by taking the variational derivative of the functional [25, 53]. For example, if \mathcal{L} represents the G -invariant arc length or surface area functional, the Euler–Lagrange equation will describe the G -invariant minimal curves or surfaces.

In general, the invariance of a variational problem under a given transformation group implies the invariance of its Euler–Lagrange equation. (The converse, however, is not true.) We will be interested in precisely how the Euler–Lagrange equation varies, and this is the result of the following key lemma.

LEMMA 4.3. *Let $\text{pr } \mathbf{v}$ be the prolonged vector field (24). Let $L(x, u^{(n)})$ be a Lagrangian. Then*

$$(36) \quad E(\text{pr } \mathbf{v}(L) + L \text{Div } \xi) = \text{pr } \mathbf{v}(E(L)) + (Q_u + \text{Div } \xi)E(L).$$

The proof of this result will appear later. Let us first look at some important consequences, including our desired construction of an invariant evolution equation. Suppose that L is a G -invariant Lagrangian, e.g., defining the group invariant arc length or area. Then L satisfies the infinitesimal invariance condition (34), and hence (36) implies the identity

$$(37) \quad \text{pr } \mathbf{v}[E(L)] + (\text{Div } \xi + Q_u)E(L) = 0.$$

Equation (37) means that $E(L)$ is a relative differential invariant of weight $-\text{Div } \xi - Q_u$. In particular, the Euler–Lagrange equation $E(L) = 0$ is invariant under G , as claimed. On the other hand L itself is a relative invariant of weight $-\text{Div } \xi$. Since the prolonged vector field $\text{pr } \mathbf{v}$ acts as a derivation, the ratio $E(L)/L$ is a relative differential invariant weight $-Q_u$; i.e., it satisfies

$$(38) \quad \text{pr } \mathbf{v} \left[\frac{E(L)}{L} \right] + Q_u \left[\frac{E(L)}{L} \right] = 0.$$

Consequently, its reciprocal $L/E(L)$ (assuming $E(L) \neq 0$) satisfies (31) and defines a G -invariant evolution equation. We have therefore deduced our fundamental theorem.

THEOREM 4.4. *Let G be a transformation group, and let $L dx$ be a G -invariant Lagrangian with nonzero Euler–Lagrange equation $E(L) = 0$. Then every G -invariant evolution equation has the form*

$$(39) \quad u_t = \frac{L}{E(L)} I,$$

where I is a arbitrary differential invariant of G .

Proof of Lemma 4.3. We begin, as in the standard derivation of the Euler–Lagrange equation, by considering a one-parameter family of variations,

$$(40) \quad u(x, \kappa) = u(x) + \kappa v(x) + \dots,$$

of a fixed function $u(x)$. In order to maintain the boundary conditions, the variations (40) are taken to have compact support, so that outside a compact subset $K \subset D$, the functions are all the same: $u(x, \kappa) = u(x)$, $x \in D \setminus K$. Given a vector field \mathbf{v} generating a one-parameter subgroup of G , we let

$$(41) \quad u(x, \kappa, \varepsilon) = u(x, \kappa) + \varepsilon Q[u(x, \kappa)] + \dots$$

$$(42) \quad = u(x) + \kappa v(x) + \varepsilon Q[u(x)] + \kappa \varepsilon [Q_u[u(x)]v(x) - \xi \cdot Dv] + \dots$$

denote the corresponding transformed variations. Note that $u(x, \kappa, \varepsilon)$, as a function of x , will be defined on a transformed domain $D(\varepsilon)$ which, because the variations are of compact support in D , only depends on the group parameter ε . We consider the function

$$\begin{aligned} F(\kappa, \varepsilon) &= \mathcal{L}_{D(\varepsilon)}[u(x, \kappa, \varepsilon)] = \int_{D(\varepsilon)} L[u(x, \kappa, \varepsilon)] dx \\ &= \int_D L[u(x(\kappa, \varepsilon), \kappa, \varepsilon)] \det \left[\frac{\partial x(\kappa, \varepsilon)}{\partial x} \right] dx. \end{aligned}$$

As we shall see, (36) is just a statement of the equality of mixed partials $\partial^2 F / \partial \kappa \partial \varepsilon$ at $\kappa = \varepsilon = 0$, which we compute in two different ways.

First, according to the basic integration by parts formulation of the calculus of variations, the derivative with respect to κ is given by

$$\begin{aligned} \left. \frac{\partial}{\partial \kappa} \mathcal{L}[u(x, \kappa, \varepsilon)] \right|_{\kappa=0} &= \int_{D(\varepsilon)} E(L)[u(x, \varepsilon)] \cdot v(x, \varepsilon) dx \\ &= \int_D E(L)[u(x(\kappa, \varepsilon), \kappa, \varepsilon)] \cdot v(x(\kappa, \varepsilon), \varepsilon) \det \left[\frac{\partial x(\kappa, \varepsilon)}{\partial x} \right] dx, \end{aligned}$$

where $u(x, \varepsilon) = u(x, 0, \varepsilon)$ and

$$v(x, \varepsilon) = \frac{\partial u}{\partial \kappa}(x, 0, \varepsilon) = v(x) + \varepsilon (Q_u[u(x)]v(x) - \xi \cdot Dv).$$

Therefore, using the same computation as in (33), we find

$$\left. \frac{\partial^2 F}{\partial \kappa \partial \varepsilon} \right|_{\kappa=\varepsilon=0} = \int_D \{ \text{pr } \mathbf{v}(E(L)) + E(L)(Q_u + \text{Div } \xi) \} v dx.$$

On the other hand, if we use (33) to first differentiate with respect to ε , we find

$$\frac{\partial^2 F}{\partial \kappa \partial \varepsilon} \Big|_{\kappa=\varepsilon=0} = \frac{\partial}{\partial \kappa} \int_D \{\text{pr } \mathbf{v}(L) + L \text{Div } \xi\} dx \Big|_{\kappa=0} = \int_D E(\text{pr } \mathbf{v}(L) + L \text{Div } \xi) v dx.$$

Since the latter two integrals must agree for *arbitrary* variations v , we conclude the validity of the identity (36). \square

Remark. Theorem 4.4 and Lemma 4.3 also extend to several dependent variables (suitably reinterpreted, since you can't divide by $E(g)$). Here you need as many independent volume forms as the number of dependent variables, and the $1/E(g)$ becomes the matrix inverse of the variational derivatives of the volume forms. (See [54].)

We should also remark that an alternative proof of Theorem 4.4, based on the “variational bicomplex,” was communicated to us by Ian Anderson and Juha Pohjanpelto.

Although (39) defines the most general class of invariant evolution equations, the case when the differential invariant I is constant is not necessarily the simplest one. For example, in the planar Euclidean case, $L = \sqrt{1 + u_x^2}$ is the Euclidean arc length Lagrangian, so that

$$E(L) = -D_x \frac{\partial L}{\partial u_x} = -\frac{u_{xx}}{(1 + u_x^2)^{3/2}} = -\kappa.$$

Thus the general Euclidean-invariant evolution equation has the form

$$u_t = -\sqrt{1 + u_x^2} \frac{I}{\kappa},$$

where I is an arbitrary function of κ and its arc-length derivatives. Choosing $I = \kappa$ produces the simplest one, the optical flow, whereas $I = \kappa^2$ produces the Euclidean curve shortening flow (6).

The Euclidean group is a special case of a *volume-preserving* transformation group G . This means that it leaves the $(p + 1)$ form $dx \wedge du = dx^1 \wedge \cdots \wedge dx^p \wedge du$ invariant. Equivalently, using (30) the infinitesimal condition reads

$$(43) \quad 0 = \sum_{i=1}^p \frac{\partial \xi^i}{\partial x^i} + \frac{\partial \varphi}{\partial u} = \text{Div } \xi + Q_u.$$

PROPOSITION 4.5. *Suppose G is a connected transformation group and $L dx$ a G -invariant p form such that $E(L) \neq 0$. Then $E(L)$ is a differential invariant if and only if G is volume preserving.*

Proof. This follows immediately from the infinitesimal condition (43) and the invariance of the Euler–Lagrange equation (37). Condition (37) implies that $E(L)$ is invariant, i.e., $\text{pr } \mathbf{v}[E(L)] = 0$, if and only if (43) holds. \square

COROLLARY 4.6. *Let G be a connected volume-preserving transformation group. Then, up to constant multiple, the G -invariant flow of lowest order has the form*

$$(44) \quad u_t = L,$$

where $\omega = L dx^1 \wedge \cdots \wedge dx^p$ is the invariant p form of minimal order such that $E(L) \neq 0$.

Remark. The p form of minimal order will be unique unless G has a differential invariant of equal or lower order than L .

5. Affine-invariant surface flows. In this section, we apply the preceding results to describe the simplest possible affine-invariant surface evolution. This gives for convex surfaces the surface version of the affine-shortening flow for curves. The group G is the (*special*) *affine group* $\mathrm{SL}(3, \mathbf{R})$, consisting of all 3×3 matrices with determinant 1, combined with the translations. Let \mathcal{S} be a smooth strictly convex surface in \mathbf{R}^3 , which we write locally as a graph $u = u(x, y)$. According to [10, 12], the simplest affine-invariant area form is constructed from the affine-invariant metric, which is given by

$$L dx \wedge dy = \kappa^{1/4} \sqrt{1 + u_x^2 + u_y^2} dx \wedge dy,$$

where

$$\kappa = \frac{u_{xx}u_{yy} - u_{xy}^2}{(1 + u_x^2 + u_y^2)^2}$$

denotes the usual Gaussian curvature of \mathcal{S} . Corollary 4.6 allows us to conclude the following.

COROLLARY 5.1. *Up to constant multiple, the simplest affine-invariant evolution equation has the form*

$$(45) \quad u_t = \kappa^{1/4} \sqrt{1 + u_x^2 + u_y^2}.$$

We conclude that the simplest affine-invariant surface flow is the global evolution

$$(46) \quad \mathcal{S}_t = \kappa^{1/4} \vec{\mathcal{N}},$$

where $\vec{\mathcal{N}}$ denotes the Euclidean inward normal to \mathcal{S} , called the *affine surface flow*. This equation was also derived using completely different methods by [2, 4]. Note that besides affine invariance, a number of properties were required in [2, 4] to obtain the flow we present below. (Some of these properties are related to the importance of the flow being an “evolution equation.”) In our approach, after the starting point of formulation of an evolution equation, the only requirement is that it be “the simplest flow which admits the affine group as its symmetry group.”

Remarks.

1. Recently, it has been announced that a convex (C^2) surface will converge to an ellipsoidal point under the affine surface flow (46); see [5, 52]. Indeed, one must verify that the affine curvature [31] becomes constant for the corresponding normalized dilated surface flow. (Another possibility would be to show that the affine isoperimetric inequality converges to the right value [42].) Of course, this result generalizes in a straightforward way to convex hypersurfaces in any dimension, where one uses the $(n + 2)$ nd root of the Gaussian curvature for n the dimension of the hypersurface.
2. In general, Chow [17] has shown that a convex hypersurface converges smoothly to a point under the flow defined by any power $\beta > 0$ of the Gaussian curvature. Moreover, it is shown that for $\beta = \frac{1}{n}$ where n is the dimension of the hypersurface, the point is round. Other than $\beta = \frac{1}{n}, \frac{1}{(n+2)}$, the shape of the point is not known. In [43] the authors gave a geometric interpretation of the flow obtained with $\beta = \frac{1}{2}$ (that is, motion by $\kappa_+^{1/2}$), derived for surface smoothing in [2, 4, 14] using a different approach.

3. The affine flow (46) is well defined only for convex surfaces. In order to extend this flow to nonconvex ones, the positive part of the Gaussian curvature must be taken [2]. Alvarez et al. proved that the resulting flow is indeed well posed in the viscosity framework. Note that this flow is of course included in the general form given by Theorem 4.4.
4. Finally, Caselles and Sbert [14] have recently shown that, in contrast to flow via mean curvature, a dumbbell does not become singular under the non-convex extension of the flow (46) using $\kappa_+^{1/4}$ as velocity. On the other hand, they also present examples where the flow disconnects an initially connected nonconvex surface.

6. Discussion and concluding remarks. In this work, we first reviewed basic results concerning geometric smoothing of surfaces. We considered both $2\frac{1}{2}$ D smoothing processes, based on smoothing graphs via level set smoothing, and pure 3D processes, based on the evolution via functions of the principal curvatures, such as the mean and Gaussian curvatures. Unfortunately, many of the results expected from the planar theory do not hold in the 3D case. An arbitrary regular surface can develop singularities when evolving according to the Gaussian or mean curvature, or even other more general functions as we described in this paper, so these kinds of flows cannot be used for smoothing general surfaces. However, they can be used for specific graphs or surfaces, e.g., star-shaped surfaces. We are currently investigating the evolution of surfaces by other functions of their principal curvature. Our goal with these functions is to achieve surface flows with analogous behavior to those of planar geometric flows and then to be able to perform geometric smoothing of more general surfaces. Another topic under investigation is the possibility of smoothing 3D surfaces via geometric 2D flows applied to curves on the surface, different from the level sets. One possibility is to smooth lines of curvature, or lines of maximal slope. The main advantage of smoothing 3D objects via 2D geometric flows is the existence of a well-developed theory for these kind of flows, as we saw in section 2.

In the second part of the paper we presented a general formulation for invariant geometric flows of hypersurfaces. This result completes the theory started in [55, 56] for planar curves. We showed that the invariant flows can be formulated as functions of the invariant metric and invariant curvatures, which are the basic differential invariant descriptors, together with the variational derivative of this metric. As an example, we derived the simplest affine-invariant geometric flow for (convex) 3D surfaces. We also showed that if the transformation group is volume preserving, this variational derivative is invariant as well. Note that the invariant geometric flows for planar curves are smoothing processes for both the Euclidean and the special affine groups but not for the similarity, full affine, and projective ones [56]. One of the key differences among these groups is that the first two are area preserving while the others are not. We are currently investigating whether there is any connection between the lack of smoothing and the lack of invariance of the variational derivative for non-area-preserving groups. For such groups, we are also investigating the use of different invariant metrics to define geometric smoothing processes. These metrics can be used either to define different “heat flows,” obtained via derivatives with respect to the corresponding arc-length, or to derive geometric variational problems which can define smoothing processes.

In this work, we have presented a general formulation and classification of invariant flows without performing an analysis of the resulting equations. Our general framework for new invariant flows leaves open the problem of short- and long-term

existence and regularity of the different invariant flows. The affine flow (46) gives an example of a flow derived directly from the general formula, for which the simplest flow is well posed only for convex surfaces and should be modified as in [2, 14] to extend it to general surfaces. Having the classification presented here, the door is open to find the specific flows for given applications. This was already done for a number of groups in two and three dimensions, but much more research should be done, specially for hypersurfaces, where satisfactory invariant smoothers are still unknown. With the classification in mind, we know how the basic flows should look, making the search much simpler but of course still not trivial.

Acknowledgments. G.S. thanks members of the European University Network for interesting conversations on parts of this work during the meeting in Las Palmas, Spain, in March 1994.

REFERENCES

- [1] U. ABRESCH AND J. LANGER, *The normalized curve shortening flow and homothetic solutions*, J. Differential Geom., 23 (1986), pp. 175–196.
- [2] L. ALVAREZ, F. GUICHARD, P. L. LIONS, AND J. M. MOREL, *Axioms and fundamental equations of image processing*, Arch. Rational Mech. Anal., 123 (1993), p. 3.
- [3] L. ALVAREZ, P. L. LIONS, AND J. M. MOREL, *Image selective smoothing and edge detection by nonlinear diffusion*, SIAM J. Numer. Anal., 29 (1992), pp. 845–866.
- [4] L. ALVAREZ AND J. M. MOREL, *Formalization and Computational Aspects of Image Analysis*, Technical report, Dept. of Information and Systems, Universidad de Las Palmas de Gran Canaria, Spain, October 1993.
- [5] B. ANDREWS, *Contraction of Convex Hypersurfaces by Their Affine Normal*, Department of Mathematics, Stanford University, Stanford, CA, 1994, preprint.
- [6] S. ANGENENT, *Parabolic equations for curves on surfaces, II. Intersections, blow-up, and generalized solutions*, Ann. of Math., 133 (1991), pp. 171–215.
- [7] S. ANGENENT, *Some recent results on mean curvature flow*, in Recent Advances in Partial Differential Equations, M. A. Herrero and E. Zuazua, eds., John Wiley & Sons, New York, 1994.
- [8] S. ANGENENT, G. SAPIRO, AND A. TANNENBAUM, *On the Affine Heat Flow for Non-Convex Curves*, Technical report, Department of Engineering, University of Minnesota, Minneapolis, MN, March 1996.
- [9] J. BABAUD, A. P. WITKIN, M. BAUDIN, AND R. O. DUDA, *Uniqueness of the Gaussian kernel for scale-space filtering*, IEEE Trans. Pattern Anal. Machine Intell., 8 (1986), pp. 26–33.
- [10] W. BLASCHKE, *Vorlesungen über Differentialgeometrie II*, Verlag Von Julius Springer, Berlin, 1923.
- [11] K. A. BRAKKE, *The Motion of a Surface by Its Mean Curvature*, Princeton University Press, Princeton, NJ, 1978.
- [12] S. BUCHIN, *Affine Differential Geometry*, Gordon and Breach, New York, 1983.
- [13] M. P. DO CARMO, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, Englewood Cliffs, NJ, 1976.
- [14] V. CASELLES AND C. SBERT, *What is the best causal scale-space for 3D images?* SIAM J. Appl. Math., 56 (1996), pp. 1199–1246.
- [15] Y. G. CHEN, Y. GIGA, AND S. GOTO, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Differential Geom., 33 (1991), pp. 749–786.
- [16] D. L. CHOPP, *Computing Minimal Surfaces Via Level Set Curvature Flow*, Ph.D. dissertation, Mathematics Department, Lawrence Berkeley Laboratory, Berkeley, CA, 1991.
- [17] B. CHOW, *Deforming convex hypersurfaces by the n th root of the Gaussian curvature*, J. Differential Geom., 22 (1985), pp. 117–138.
- [18] M. G. CRANDALL, H. ISHII, AND P. L. LIONS, *User's guide to viscosity solutions of second order partial linear differential equations*, Bull. Amer. Math. Soc., 27 (1992), pp. 1–67.
- [19] K. ECKER AND G. HUISKEN, *Mean curvature evolution of entire graphs*, Ann. of Math., 130 (1989), pp. 453–471.
- [20] K. ECKER AND G. HUISKEN, *Interior estimates for hypersurfaces moving by mean curvature*, Invent. Math., 105 (1991), pp. 547–569.
- [21] C. L. EPSTEIN AND M. GAGE, *The curve shortening flow*, in Wave Motion: Theory, Modeling, and Computation, A. Chorin and A. Majda, eds., Springer-Verlag, New York, 1987.

- [22] L. C. EVANS AND J. SPRUCK, *Motion of level sets by mean curvature*, I, J. Differential Geom., 33 (1991), pp. 635–681.
- [23] L. C. EVANS AND J. SPRUCK, *Motion of level sets by mean curvature*, II, Trans. Amer. Math. Soc., 330 (1992), pp. 321–332.
- [24] M. GAGE, *On an area-preserving evolution equation for plane curves*, Contemp. Math., 51 (1986), pp. 51–62.
- [25] I. M. GEL'FAND AND S. V. FOMIN, *Calculus of Variations*, Prentice-Hall, Englewood Cliffs, NJ, 1963.
- [26] C. GERHARDT, *Flow of nonconvex hypersurfaces into spheres*, J. Differential Geom., 32 (1990), pp. 299–314.
- [27] M. GAGE AND R. S. HAMILTON, *The heat equation shrinking convex plane curves*, J. Differential Geom., 23 (1986), pp. 69–96.
- [28] M. GRAYSON, *The heat equation shrinks embedded plane curves to round points*, J. Differential Geom., 26 (1987), pp. 285–314.
- [29] M. GRAYSON, *Shortening embedded curves*, Ann. of Math., 129 (1989), pp. 71–111.
- [30] M. GRAYSON, *A short note on the evolution of a surface by its mean curvature*, Duke Math. J., 58 (1989), pp. 555–558.
- [31] H. W. GUGGENHEIMER, *Differential Geometry*, McGraw-Hill, New York, 1963.
- [32] B. K. P. HORN AND E. J. WELDON, JR., *Filtering closed curves*, IEEE Trans. Pattern Anal. Machine Intell., 8 (1986), pp. 665–668.
- [33] G. HUISKEN, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geom., 20 (1984), pp. 237–266.
- [34] G. HUISKEN, *Non-parametric mean curvature evolution with boundary conditions*, J. Differential Equations, 77 (1989), pp. 369–378.
- [35] G. HUISKEN, *Local and global behavior of hypersurfaces moving by mean curvature*, Proc. Symp. Pure Math., 54 (1993), pp. 175–191.
- [36] B. B. KIMIA, A. TANNENBAUM, AND S. W. ZUCKER, *On the evolution of curves via a function of curvature*, I: *The classical case*, J. Math. Anal. Appl., 163 (1992), pp. 438–458.
- [37] B. B. KIMIA, A. TANNENBAUM, AND S. W. ZUCKER, *Shapes, shocks, and deformations*, I, Internat. J. Comp. Vision, 15 (1995), pp. 189–224.
- [38] J. J. KOENDERINK, *Solid Shape*, MIT Press, Cambridge, MA, 1990.
- [39] T. LINDBERG, *Linear Scale-Space in Computer Vision*, Kluwer, the Netherlands, 1993.
- [40] T. LINDBERG AND J. O. EKLUNDH, *On the computation of a scale-space primal sketch*, J. Visual Comm. Image Rep., 2 (1991), pp. 55–78.
- [41] D. G. LOWE, *Organization of smooth image curves at multiple scales*, Internat. J. Comput. Vision, 3 (1989), pp. 119–130.
- [42] E. LUTWAK, *On some affine isoperimetric inequalities*, J. Differential Geom., 23 (1986), pp. 1–13.
- [43] P. NESKOVIC AND B. KIMIA, *Three-dimensional shape representation from curvature dependent surface evolution*, in Proc. IEEE-ICIP, Austin, TX, November 1994, pp. 6–10.
- [44] F. MOKHATARIAN AND A. MACKWORTH, *A theory of multiscale, curvature-based shape representation for planar curves*, IEEE Trans. Pattern Anal. Machine Intell., 14 (1992), pp. 789–805.
- [45] J. L. MUNDY AND A. ZISSERMAN, EDS., *Geometric Invariance in Computer Vision*, MIT Press, Cambridge, MA, 1992.
- [46] W. J. NIESSEN, B. TER HAAR ROMENY, L. M. J. FLORACK, AND A. H. SALDEN, *Nonlinear Diffusion of Scalar Images Using Well-Posed Differential Operators*, Technical report, Utrecht University, the Netherlands, October 1993.
- [47] J. OLIENSIS, *Local reproducible smoothing without shrinkage*, IEEE Trans. Pattern Anal. Machine Intell., 15 (1993), pp. 307–312.
- [48] V. I. OLIKER, *Evolution of nonparametric surfaces with speed depending on curvature I. The Gauss curvature case*, Indiana Univ. Math. J., 40 (1991), pp. 237–258.
- [49] V. I. OLIKER, *Self-similar solutions and asymptotic behavior of flows of nonparametric surfaces driven by Gauss or mean curvature*, Proc. Sympos. Pure Math., 54 (1993), pp. 389–402.
- [50] V. I. OLIKER AND N. N. URALTSEVA, *Evolution of nonparametric surfaces with speed depending on curvature II. The mean curvature case*, Comm. Pure Appl. Math., 46 (1993), pp. 97–135.
- [51] V. I. OLIKER AND N. N. URALTSEVA, *Evolution of nonparametric surfaces with speed depending on curvature, III. Some remarks on mean curvature and anisotropic flows*, in Degenerate Diffusions, IMA Vol. Math. Appl. 47, Springer-Verlag, New York, 1993, pp. 141–156.
- [52] V. I. OLIKER, personal communication, 1994.
- [53] P. J. OLVER, *Applications of Lie Groups to Differential Equations*, 2nd ed., Springer-Verlag, New York, 1993.

- [54] P. J. OLVER, *Equivalence, Invariants, and Symmetry*, Cambridge University Press, Cambridge, UK, 1995.
- [55] P. J. OLVER, G. SAPIRO, AND A. TANNENBAUM, *Classification and uniqueness of invariant geometric flows*, C. R. Acad. des Sci. Paris Sér. I Math., 319 (1994), pp. 339–344.
- [56] P. J. OLVER, G. SAPIRO, AND A. TANNENBAUM, *Differential invariants signatures and flows in computer vision: A symmetry group approach*, in *Geometry Driven Diffusion in Computer Vision*, B. ter Haar Romeny, ed., Kluwer, Dordrecht, the Netherlands, 1994, pp. 205–306.
- [57] S. J. OSHER AND J. A. SETHIAN, *Fronts propagation with curvature dependent speed: Algorithms based on Hamilton-Jacobi formulations*, J. Comput. Phys., 79 (1988), pp. 12–49.
- [58] B. TER HAAR ROMENY, ED., *Geometry Driven Diffusion in Computer Vision*, Kluwer, Dordrecht, the Netherlands, 1994.
- [59] G. SAPIRO AND A. TANNENBAUM, *On affine plane curve evolution*, J. Funct. Anal., 119 (1994), pp. 79–120.
- [60] G. SAPIRO AND A. TANNENBAUM, *Affine Shortening of Non-Convex Plane Curves*, EE Publication 845, Dept. of Electrical Engineering, Technion, I. I. T., Haifa 32000, Israel, July 1992.
- [61] G. SAPIRO AND A. TANNENBAUM, *Affine invariant scale-space*, Internat. J. Comput. Vision, 11 (1993), pp. 25–44.
- [62] G. SAPIRO AND A. TANNENBAUM, *Image smoothing based on an affine invariant flow*, in *Proceedings of Conference on Information Sciences and Systems*, Johns Hopkins University, Baltimore, MD, March 1993, pp. 196–201.
- [63] G. SAPIRO AND A. TANNENBAUM, *On invariant curve evolution and image analysis*, Indiana Univ. Math. J., 42 (1993), pp. 985–1009.
- [64] G. SAPIRO AND A. TANNENBAUM, *Area and length preserving geometric invariant scale-spaces*, IEEE Trans. Pattern Anal. Machine Intell., 17 (1995), pp. 67–72.
- [65] G. SAPIRO, A. TANNENBAUM, Y. L. YOU, AND M. KAVEH, *Experiments on geometric image enhancement*, in *First IEEE-International Conf. on Image Proc.*, Austin, TX, November 1994, pp. 472–476.
- [66] J. A. SETHIAN, *A review of recent numerical algorithms for hypersurfaces moving with curvature dependent speed*, J. Differential Geom., 31 (1989), pp. 1310–161.
- [67] H. M. SONER AND P. E. SOUGANIDIS, *Singularities and uniqueness of cylindrically symmetric surfaces moving by mean curvature*, Comm. Partial Differential Equations, 18 (1993), pp. 859–894.
- [68] M. SPIVAK, *A Comprehensive Introduction to Differential Geometry*, Publish or Perish Inc., Berkeley, CA, 1979.
- [69] J. I. E. URBAS, *On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures*, Math. Z., 205 (1990), pp. 355–372.
- [70] J. I. E. URBAS, *An expansion of convex hypersurfaces*, J. Differential Geom., 33 (1991), pp. 91–125.
- [71] J. I. E. URBAS, *Correction to An expansion of convex hypersurfaces*, J. Differential Geom., 35 (1992), pp. 763–765.
- [72] A. P. WITKIN, *Scale-space filtering*, in *International Joint Conference on Artificial Intelligence*, Karlsruhe, FRG, 1983, pp. 1019–1021.
- [73] A. L. YUILLE, *The creation of structure in dynamic shape*, in *Proceedings of the International Conference on Computer Vision*, Tampa, FL, 1988, pp. 685–689.
- [74] A. L. YUILLE AND T. A. POGGIO, *Scaling theorems for zero crossings*, IEEE Trans. Pattern Anal. Machine Intell., 8 (1986), pp. 15–25.
- [75] O. FAUGERAS, *On the evolution of simple curves if the real projective plane*, Comptes rendus de l'Acad. des Sciences de Paris, 317 (1993), pp. 565–570.