HAMILTONIAN AND NON - HAMILTONIAN MODELS FOR WATER WAVES

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ABSTRACT

A general theory for determining Hamiltonian model equations from noncanonical perturbation expansions of Hamiltonian systems is applied to the Boussinesq expansion for long, small amplitude waves in shallow water, leading to the Korteweg-de-Vries equation. New Hamiltonian model equations, including a natural "Hamiltonian version" of the KdV equation, are proposed. The method also provides a direct explanation of the complete integrability (soliton property) of the KdV equation. Depth dependence in both the Hamiltonian models and the second order standard perturbation models is discussed as a possible mechanism for wave breaking.

1. INTRODUCTION

In recent years there has been increasing interest in the application of the methods of Hamiltonian mechanics to the dynamical equations of nondissipative continuum mechanics. One of the primary impetuses behind this development has been the discovery of a number of nonlinear evolution equations, known as "soliton" equations, including the celebrated Korteweg-de Vries (KdV) equation, which can be regarded as completely integrable, infinite dimensional Hamiltonian systems. These equations arise with surprising frequency as model equations for a wide variety of complicated, nonlinear physical phenomena including fluids, plasmas, optics and so on - see [7]. As has become increasingly apparent - see [13] and the references therein - the full physical systems themselves also admit Hamiltonian formulations. What is less well understood, however, is how the Hamiltonian structures for the physical systems and their model equations are related. As will be shown here, at least for the KdV model for water wave motion, this relationship is far from obvious, and can actually be used to explain the complete integrability of the model equation.

One of the most useful aspects of the Hamiltonian approach is the Noether correspondence between one - parameter symmetry groups and conservation laws. In earlier work with Benjamin on the free boundary problem for surface water waves, [4], [15], these symmetry group techniques were combined with Zakharov's Hamiltonian formulation of the problem, [20], to prove that in two dimensions there are precisely eight nontrivial conservation laws (seven if one includes surface tension). The present work

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arose in an investigation, still in progress, into how these conservation laws behave under the Boussinesq perturbation expansion leading to the KdV equation; in particular do they correspond to any of the infinity of conservation laws of this latter model?

In Boussinesq's method, one first introduces small parameters corresponding to the underlying assumptions of long, small amplitude waves in shallow water. Truncating the resulting perturbation expansion leads to the Boussinesq model system, describing bi-directional wave motion. The KdV equation comes from restricting to a "submanifold" of approximately unidirectional waves. It came as a shock to discover that the Boussinesq system, which forms the essential half-way point in the derivation, fails to be Hamiltonian; in particular there is no conservation of energy. Subsequent investigation revealed that if one expands the energy functional which serves as the Hamiltonian for the water wave problem and truncates to the right order, the resulting functional does not agree with either of the Hamiltonians available for the KdV equation. These all indicate a fundamental incongruity in the Hamiltonian structures in the physical system and its model equations. Alternative models, such as the BBM equation, [3], have the same problems. (It should be remarked that Segur, [17], employs a different derivation involving two time scales, and does derive a linear combination of the two KdV Hamiltonians from the water wave energy. It remains to be seen how the two approaches can be reconciled.)

In order to appreciate what is happening, consider the conceptually simpler case of a finite dimensional system

$$\dot{x} = J(x,\epsilon)\ \nabla H(x,\epsilon) = F(x,\epsilon)\ , \tag{1.1}$$

in which both the Hamiltonian function $H(x,\epsilon)$ and the skew-symmetric matrix $J(x,\epsilon)$ determining the underlying Hamiltonian structure may depend on the small parameter ϵ. We are specifically not writing (1.1) in the canonical (Darboux) variables (p,q), because a) this simplification is not available in the infinite dimensional case needed to treat evolution equations, and b) it tends to obscure the basic issues. Let

$$x = y + \epsilon\,\varphi(y) + \epsilon^2\,\psi(y) + \dots \tag{1.2}$$

be a given perturbation expansion. In standard perturbation theory, [9], one simply substitutes (1.2) into (1.1), expands in powers of ϵ to some requisite order, and truncates. After some elementary manipulations (see section 3) one finds the first order perturbation

$$\dot{y} = F_0\,(y) + \epsilon F_1(y)\ , \tag{1.3}$$

in which F_0 and F_1 are readily expressed in terms of F and φ. If we similarly expand the Hamiltonian

$$H(x,\epsilon) = H_0(y) + \epsilon\,H_1(y) + \epsilon^2\,H_2(y) + \dots\ ,$$

we find that unless the perturbation is canonical, which is the only type of perturbation allowed in classical or celestial mechanics, [18], the first order truncation $H_0 + \epsilon H_1$ is not a constant of the motion. In the present theory, the form of the perturbation expansion is more or less prescribed, so we cannot restrict our attention to only canonical perturbations, but we still wish to find perturbation equations of Hamiltonian form. The theory will thus have applications to the construction of model equations in a wide range of physical systems. To accomplish this goal, we must also expand the Hamiltonian operator

$$J(x,\epsilon) \mapsto J_0(y) + \epsilon J_1(y) + \epsilon^2 J_2(y) + \dots .$$

Truncating, we get the first order <u>cosymplectic</u> <u>perturbation</u> <u>equations</u>

$$\dot{y} = (J_0 + \epsilon J_1)\nabla(H_0 + \epsilon H_1) = J_0 \nabla H_0 + \epsilon(J_0 \nabla H_1 + J_1 \nabla H_0) + \epsilon^2 J_1 \nabla H_1 \quad . \qquad (1.4)$$

(Strictly speaking, for a general perturbation the operator $J_0 + \epsilon J_1$ may not satisfy all the requisite properties to be Hamiltonian. However, (1.4) always retains the key property of conserving the Hamiltonian $H_0 + \epsilon H_1$. In our water wave example, the perturbed operator is Hamiltonian, so we can ignore this technical complication here. See section 3 and the companion paper, [16], for a detailed discussion of this point.) The Hamiltonian perturbation equations (1.4) agree with the standard equations (1.3) up to terms in ϵ , i.e. $F_0 = J_0 \nabla H_0$, $F_1 = J_0 \nabla H_1 + J_1 \nabla H_0$, but have an additional ϵ^2 term so as to still be Hamiltonian. Note that these ϵ^2 terms are <u>not</u> the same as the second order terms in the standard expansion; these would include $J_0 \nabla H_2 + J_2 \nabla H_0$, which would again destroy the Hamiltonian nature of the system.

In the Boussinesq expansion, if we let (1.1) represent the original water wave problem, then the Boussinesq system will be represented by the non-Hamiltonian equation (1.3). There is thus a corresponding Hamiltonian model, like (1.4) incorporating quadratic terms in the relevant small parameters. For comparative purposes, we will also derive the second order terms in the standard expansion. Similarly, the KdV equation actually corresponds to the non-Hamiltonian perturbation equation (1.3). There is a corresponding "Hamiltonian version" of the KdV equation which incorporates higher order terms - see (4.26). In all of these new models, there is a dependence of the equation on the depth at which one looks at it - this leads to speculations on the nature of wave-breaking.

What are some of the advantages of this Hamiltonian approach to perturbation theory? The most important is that the Hamiltonian perturbation (1.4) conserves energy, whereas the standard perturbation (1.3) will not in general. (This holds whether or not $J_0 + \epsilon J_1$ is a true Hamiltonian operator.) In two dimensions, if the orbits of the unperturbed system (1.1) are closed curves surrounding a fixed point, then the Hamiltonian perturbation will have the same orbit structure, whereas the solutions to (1.3) can slowly spiral into or away from the fixed point. In higher dimensions, KAM theory shows that "most" solutions of a small Hamiltonian perturbation of a completely integrable system remain quasi-periodic, whereas the standard perturbation can again exhibit spiralling behavior. At the other extreme, only Hamiltonian perturbations of an ergodic system stand any chance of being ergodic in the right way as the solutions of (1.3) will mix up energy levels. Of course, both perturbation expansions are valid to the same order, and hence give equally valid approximations to the short time behavior of the system. Based on the above observations, the Hamiltonian perturbation appears to do a better job modelling long-time and qualitative behavior of the system. However, no rigorous theorems are available, with the infinite dimensional version being especially unclear.

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2. HAMILTONIAN MECHANICS

We begin by briefly reviewing the elements of finite dimensional Hamiltonian mechanics in general coordinate systems. The theory requires a minimal amount of differential geometry, and we refer the reader to Arnold's excellent book, [1], for a complete exposition. The subsequent extension to the infinte dimensional version needed to treat evolution equations is most easily done using the formal calculus of variations developed in [8], [14], which we outline in section B .

A. Finite Dimensional Theory

Given an n-dimensional manifold M , the "phase space", a Hamiltonain structure will be determined by a symplectic two-form Ω on M , the determining conditions being that Ω be nondegenerate and closed: $d\Omega = 0$. In local coordinates $x = (x_1, \ldots, x_n)$,

$$\Omega = \frac{1}{2}\, dx^T \wedge K(x)\, dx = \frac{1}{2} \sum_{i,j} K_{ij}(x)\, dx_i \wedge dx_j \quad ,$$

where $K(x)$ is a skew-symmetric matrix: $K^T = -K$. Nondegeneracy means that $K(x)$ is invertible for each x (which requires M to be even-dimensional), while closure requires K to satisfy the system of linear partial differential equations

$$\partial_i K_{jk} + \partial_k K_{ij} + \partial_j K_{ki} = 0 \, , \qquad i,\, j,\, k = 1 \, , \ldots , n \quad , \tag{2.1}$$

in which $\partial_i = \partial/\partial x_i$, etc. For a given Hamiltonian function $H : M \to \mathbb{R}$, Hamilton's equations take the form

$$\dot{x} = J(x)\ \nabla H(x)$$

in which the Hamiltonian operator $J(x)$ is the inverse to the matrix appearing in the symplectic two-form: $J(x) = K(x)^{-1}$. Similarly the Poisson bracket

$$\{F,G\} = \nabla F^T J \nabla G = \sum_{i,j} J_{ij}\, \partial_i F\, \partial_j G \tag{2.2}$$

uses the inverse matrix to that appearing in Ω . This Poisson bracket satisfies the usual properties of bilinearity, skew-symmetry and the Jacobi identity that are essential to the development of Hamiltonian mechanics.

Of course, in the finite-dimensional set-up, Darboux' theorem implies the existence of canonical local coordinates $(p,q) = (p_1, \ldots, p_m, q_1, \ldots, q_m)$, $n = 2m$, on M (the conjugate positions and momenta of classical mechanics) in terms of which, the symplectic two form has the simple form

$$\Omega = \sum_{i=1}^{m} d p_i \wedge d q_i \ .$$

Equivalently, K is the standard symplectic matrix

$$K_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} .$$

Note that now $J_0 = K_0^{-1} = -K_0$, so Hamilton's equations take the familiar form

$$\dot{p}_i = \partial H/\partial q_i \quad , \quad \dot{q}_i = -\partial H/\partial p_i \ , \quad i = 1, \ldots, m \ .$$

This introduction of canonical coordinates, especially with the blurring of the distinction between the Hamiltonian operator and its inverse, gives a welcome simplification in the computational aspects of the theory. However, an important lesson to be learned from the infinite dimensional, evolutionary version of Hamiltonian mechanics, in which no good version of Darboux' theorem is currently available, is that it is unwise to rely too strongly on canonical coordinates as the apparent simplifications tend to obscure some of the main issues.

The appearance of the inverse to the Hamiltonian operator $K(x)$ in the symplectic two-form Ω causes some unnecessary complications, especially in the evolutionary version of the theory in which J is a differential operator. These can be circumvented by turning to the dual Poisson structure on M determined by the cosymplectic two-vector

$$\Theta = \frac{1}{2} \partial_x^T \wedge J(x)\, \partial_x = \frac{1}{2} \sum_{i,j} J_{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \quad .$$

(In more classical language, Θ is an alternating contravariant two-tensor, i.e. a section of the bundle dual to the bundle of two-forms.) We no longer require that Θ be nondegenerate, as we no longer need to invert J , but we do need a condition analogous to the closure of the symplectic two-form. An easy computation shows that in local coordinates, in the case J is invertible, (2.1) is equivalent to the nonlinear system of differential equations

$$\sum_{\ell=1}^{n} \{ J_{i\ell} \partial_\ell J_{jk} + J_{k\ell} \partial_\ell J_{ij} + J_{j\ell} \partial_\ell J_{ki} \} = 0 \ , \qquad i,\ j,\ k = 1\ , \ldots,\ n \qquad (2.3)$$

These conditions, which we impose now in general, can be expressed in coordinate-free terms using the Schouten-Nijenhuis bracket:

$$[\Theta, \Theta] = 0 \ . \qquad (2.4)$$

We will not attempt to define this bracket here - see [11], [16] for details - but remark that for a pair of two-vectors Θ , $\tilde{\Theta}$, $[\Theta , \tilde{\Theta}]$ is bilinear and symmetric in its arguments. Any two-vector Θ satisfying (2.4) (or, equivalently, (2.3) in local coordinates) is called cosymplectic. Each such two-vector defines a Poisson bracket:

$\{F,G\} = \langle dF \wedge dG, \Theta \rangle$ (or (2.2) in local coordinates) with all the usual properties.

B. Evolution Equations

Let $x = (x_1, \ldots, x_p) \in X = \mathbb{R}^p$ be the independent spatial variables and $u = (u^1, \ldots, u^q) \in U = \mathbb{R}^q$ be the dependent variables in the equations under consideration. Let $u^{(n)}$ denote all the partial derivatives $u^i_J = \partial_J u^i$, $\partial_J = \partial_{j_1} \cdots \partial_{j_m}$, $\partial_j = \partial/\partial x_j$, $1 \le j_\nu \le p$, of order $m \le n$. Let $\mathcal{C}$ denote the algebra of smooth functions $P(x,u^{(n)})$, n arbitrary, depending on x,u and derivatives of u. Let $\mathcal{C}^m$ denote the space of m-tuples $Q = (Q_1, \ldots, Q_m)$ of functions in $\mathcal{C}$. A system of evolution equations takes the form

$$\frac{\partial u}{\partial t} = Q(x,u^{(n)}) , \tag{2.5}$$

where $Q \in \mathcal{C}^q$. For a given function $P \in \mathcal{C}$, the total derivative $D_i P$, $1 \le i \le p$, is obtained by differentiating P with respect to x_i, treating u as a function of x. For example, $D_x(uu_x) = u_x^2 + uu_{xx}$.

The role of the Hamiltonian function is played by a functional $\mathcal{H} = \int H(x,u^{(u)})dx$. Suppose the integration takes place over a domain $A \subset X$ with boundary ∂A. By the divergence theorem, provided u and its derivatives vanish on ∂A, adding a total divergence $\mathrm{Div}\, P = D_1 P_1 + \cdots + D_p P_p$, to the integrand H will not affect the value of the functional $\mathcal{H}[u]$. We thus define an equivalence relation on the space $\mathcal{C}$ of integrands such that $H \sim \tilde{H}$ whenever $H - \tilde{H} = \mathrm{Div}\, P$ for some $P \in \mathcal{C}^p$. Let $\mathfrak{F}$ denote the space of equivalence classes, which we identify with the space of functionals. The natural projection $\mathcal{C} \to \mathfrak{F}$ is denoted, suggestively, by an integral sign, so $\int H\,dx \in \mathfrak{F}$ denotes the equivalence class of $H \in \mathcal{C}$. In the space of functionals, we are allowed to integrate by parts: $\int P(D_i Q)dx = -\int Q(D_i P)dx$, $P, Q \in \mathcal{C}$, and ignore boundary contributions.

The same kind of constructions carry over to differential forms. A <u>differential one-form</u> is a finite sum of the form

$$\omega = \Sigma P^i_J \, du^i_J \quad , \quad P^i_J \in \mathcal{C} .$$

For example, if $P(x,u^{(n)}) \in \mathcal{C}$, then its exterior derivative is the one-form

$$dP = \Sigma \frac{\partial P}{\partial u^i_J} du^i_J = D_P \cdot du , \tag{2.6}$$

where $du = (du^1, \ldots, du^q)$, and D_P denotes the <u>Frechet derivative</u> of P with respect to u, which is a $1 \times q$ matrix of differential operators with entries $\Sigma(\partial P/\partial u^i_J) \cdot D^J$, $D^J = D_{j_1} \cdots D_{j_m}$. For example, if $P = uu_{xx}$, then

$$dP = u\,du_{xx} + u_{xx}du = (uD_x^2 + u_{xx})du ,$$

so $D_P = u D_x^2 + u_{xx}$. In this formulation, the total derivatives D_j act as Lie derivatives, so

$$D_j(PdQ) = (D_j P)\, dQ + P\, d(D_j Q) .$$

In particular, they commute with the exterior derivative.

Define an equivalence relation between one-forms by $\omega \sim \tilde{\omega}$ if and only if $\omega - \tilde{\omega} = \mathrm{Div}\,\mu$ for some p-tuple μ of one-forms. The equivalence classes are called <u>functional one-forms</u>, with projection again denoted by an integral sign $\int \omega\, dx$. The exterior derivative d, as it commutes with total derivatives, restricts to a map from functionals to functional one-forms; if $\mathcal{P} = \int P\, dx \in \mathfrak{F}$, then integrating (2.6) by parts, we find

$$d\mathcal{P} = \int (\delta\mathcal{P} \cdot du)\, dx = \int (E(P) \cdot du)\, dx ,$$

in which $\delta = \delta/\delta u$ is the variational derivative, and $E_i(P) = \Sigma(-D)^J(\partial P/\partial u^i_J)$ the corresponding i-th Euler operator. These constructions extend naturally to differential k-forms, and in fact the exterior derivative restricts to give an exact complex on the spaces of functional forms, [14].

A symplectic form is thus a closed functional two-form

$$\Omega = \frac{1}{2}\int (du^T \wedge K\, du)\, dx$$

in which K is a skew-adjoint $q \times q$ matrix of (differential) operators. (The <u>adjoint</u> K^* of an operator is defined so that $\int P \cdot (KQ)\, dx = \int Q \cdot (K^*P)\, dx$ for all P, $Q \in \mathcal{Q}^q$.) Whenever it will not cause confusion, we will for simplicity omit $\int dx$ in the formula for Ω. If K is independent of u, the closure condition is automatically satisfied. Hamilton's equations take the form

$$u_t = J\,\delta\mathcal{H} ,$$

in which $J = K^{-1}$ is the skew-adjoint Hamiltonian operator, $\mathcal{H} = \int H\, dx$ the Hamiltonian functional and δ, the variational derivative, replacing the gradient. Similarly, the Poisson bracket between functionals is

$$\{\mathcal{P}, \mathcal{Q}\} = \int \delta\mathcal{P} \cdot J(\delta\mathcal{Q})\, dx , \quad \mathcal{P}, \mathcal{Q} \in \mathfrak{F} . \tag{2.7}$$

Usually, the operator J is a bona fide matrix of differential operators, so its inverse is a more elusive object. To avoid introducing it, we must construct the dual cosymplectic two-vector. Note first that each functional one-form is uniquely equivalent to one of the form

$$\omega_P = \int (P \cdot du)\, dx , \quad P \in \mathcal{Q}^q . \tag{2.8}$$

The space dual to the space of functional one-forms is the space of <u>evolutionary vector fields</u>, i.e. formally infinite sums of the form

$$Q \cdot \partial_u \equiv \sum_{i,J} D^J Q_i \frac{\partial}{\partial u^i_J} , \quad Q = (Q_1, \ldots, Q_q) \in \mathcal{Q}^q .$$

These act on $\mathcal{Q}$, and commute with all total derivatives, hence give a well-defined action on $\mathfrak{F}$. The exponential of such a vector field is found by integrating the system of evolution equations (2.5) in some appropriate space of functions.

A two-vector is an alternating bi-linear map from the space of functional one-forms to the space of functionals. Each two-vector is uniquely determined by a skew-adjoint $q \times q$ matrix operator J, so that the two-vector

$$\Theta = \frac{1}{2} \partial_u^T \wedge J \partial_u$$

determines the map

$$\Theta(\omega_P, \omega_Q) = \int P J Q \, dx \quad , \quad P, Q \in \mathcal{A}^q , \tag{2.9}$$

cf. (2.8). (These two-vectors are not necessarily given as wedge products of vector fields.) The condition that the operator J be Hamiltonian, so the Poisson bracket (2.7) satisfy the Jacobi identity, is given by the vanishing of an appropriate Schouten - Nijenhuis bracket (2.4), which we do not attempt to define here - see [8], [16]. The bracket has the same bilinearity and symmetry properties as before, so the basic condition is nonlinear in J. In particular, skew adjoint operators J not depending on u are always Hamiltonian. However, if J does depend on u one needs to explicitly check the cosymplectic condition.

Example Consider the KdV equation in simplified form

$$u_t = u_{xxx} + u u_x .$$

This is Hamiltonian in two distinct ways:

$$u_t = J_0 \, \delta \mathcal{H}_1 = J_1 \, \delta \mathcal{H}_0 .$$

The Hamiltonian functionals are

$$\mathcal{H}_0 = \int \frac{1}{2} u^2 \, dx \ , \quad \mathcal{H}_1 = \int \left(\frac{1}{6} u^3 - \frac{1}{2} u_x^2 \right) dx ,$$

with corresponding operators

$$J_0 = D_x \quad , \qquad J_1 = D_x^3 + \frac{2}{3} u D_x + \frac{1}{3} u_x .$$

Here J_0 is clearly Hamiltonian since it does not depend on u. The proof that J_1 is also Hamiltonian can be found in [8], [12].

Finally, we need to discuss how these objects transform under a change of variables. Given a transformation $v = F(x, u^{(n)})$, $F \in \mathcal{A}^q$, note that by (2.6) $dv = D_F du$. Thus a functional one-form changes as

$$\omega_P = \int [P \ dv] dx = \int [P \, D_F \, du] dx = \int [D_F^* P \ . \ du] \ dx .$$

A similar computation works for functional two-forms, etc. For two-vectors, comparing the above with (2.9), we see that

$$\partial_u \wedge J \partial_u = \partial_v \wedge D_F J D_F^* \partial_v \tag{2.10}$$

provides the change of variables formula. In practice since D_F depends explicitly on u rather than v, (2.10) is not overly useful unless one can invert the relation $v = F(x, u^{(n)})$, either explicitly or as a perturbation series.

3. HAMILTONIAN PERTURBATION THEORY

We now show how standard perturbation theory can be appropriately modified to give Hamiltonian model equations for Hamiltonian systems under noncanonical perturbation expansions. We will not worry about the convergence of the expansions, or the validity of the resulting approximations except on a formal level. This, of course, just the first step in the derivation of model equations for the physical systems under consideration. One is then left with the far more difficult question of how close the solutions of the model equation are to the solutions of the original system. This latter question lies beyond the scope of this paper.

Consider a Hamiltonian system

$$\dot{x} = J(x,\varepsilon)\ \nabla H(x,\varepsilon) \equiv F(x,\varepsilon) \tag{3.1}$$

in which ε is a small parameter. For simplicity, we concentrate on the finite dimensional version, although the evolutionary theory proceeds exactly the same with $u(x,t)$ replacing x, the Hamiltonian functional replacing H, and the gradient being replaced by the variational derivative. Suppose we are given a perturbation expansion

$$x = y + \varepsilon\varphi(y) + \varepsilon^2\psi(y) + \dots . \tag{3.2}$$

To derive the standard perturbation equations, we substitute (3.2) into (3.1) and expand in powers of ε. To first order, we have

$$(1 + \varepsilon\nabla\varphi)\ \dot{y} = F_0(y) + \varepsilon\tilde{F}_1(y) , \tag{3.3}$$

in which, by the chain rule,

$$F_0(y) = F(y,0) = J_0(y)\ \nabla H_0(y) , \quad \tilde{F}_1(y) = F_\varepsilon(y,0) + \nabla F(y,0)\ \varphi(y) ,$$

with self-evident notation. Alternatively, one can invert $1 + \varepsilon\nabla\varphi$ in (3.3), re-expand and truncate, to obtain the "equivalent" system

$$\dot{y} = F_0(y) + \varepsilon\ F_1(y) , \tag{3.4}$$

where $F_1 = \tilde{F}_1 - \nabla\varphi \cdot F_0$.

Unless the expansion (3.2) happens to be canonical (i.e. preserve the Hamiltonian structure) neither of these perturbation equations will be Hamiltonian in general. If we expand the Hamiltonian itself,

$$H(x,\varepsilon) = H_0(y) + \varepsilon H_1(y) + \varepsilon^2 H_2(y) + \dots ,$$

we find that the first order truncation $H_0 + \varepsilon H_1$ is not in general conserved. In order to maintain the basic Hamiltonain character of the equation under perturbation, we must look at how the Hamiltonian operator behaves under perturbation. Substituting (3.2) into the cosymplectic two-vector $\Theta = \frac{1}{2}\partial_x \wedge J(x,\varepsilon)\partial_x$, we find

$$\Theta(x,\varepsilon) = \Theta_0(y) + \varepsilon\Theta_1(y) + \varepsilon^2\Theta_2(y) + \dots \tag{3.5}$$

or, in local coordinates,

$$\frac{1}{2}\partial_x \wedge J(x,\varepsilon)\ \partial_x = \frac{1}{2}\partial_y \wedge (J_0(y) + \varepsilon J_1(y) + \dots)\ \partial_y$$

using the basic change of variables formulae.

One annoying complication to the general theory is that because the cosymplectic condition (2.4) is nonlinear in Θ, one cannot arbitrarily truncate the expansion (3.5) and expect to maintain the vanishing of the bracket. Here we will ignore this somewhat technical complication, and assume that $\Theta_0 + \epsilon\Theta_1$ is cosymplectic. See [16] for a resolution of the problem in general.

Granted this, the first order cosymplectic perturbation to the Hamiltonian system (3.1) is given by combining the first order expansion of the Hamiltonian with the first order expansion of the Hamiltonian operator in the cosymplectic two-vector. This yields the cosymplectic perturbation equations

$$\dot{y} = (J_0 + \epsilon J_1)\nabla(H_0 + \epsilon H_1) = J_0 \nabla H_0 + \epsilon(J_1 \nabla H_0 + J_0 \nabla H_1) + \epsilon^2 J_1 \nabla H_1 \quad . \tag{3.6}$$

An easy calculation shows that this system agrees with the ordinary perturbation equations (3.4) to first order, but includes further terms in ϵ^2 in order to retain the Hamiltonian character of the system. (In the case $\Theta_0 + \epsilon\Theta_1$ is not a true cosymplectic two-vector, the perturbation equations (3.6) still conserve the Hamiltonian $H_0 + \epsilon H_1$, but we no longer have any nice Poisson bracket to work with.)

Alternatively, we can expand the symplectic two-form

$$\Omega(x,\epsilon) = \Omega_0(y) + \epsilon\Omega_1(y) + \dots \quad ,$$

or, in local coordinates

$$-\tfrac{1}{2}\, dx^T \wedge K(x,\epsilon)dx = -\tfrac{1}{2}\, dy^T \wedge (K_0 + \epsilon K_1 + \dots)\, dy \quad .$$

Combining this with the expansion of the Hamiltonian, we find the symplectic perturbation equations

$$(K_0 + \epsilon K_1)\, \dot{y} = \nabla H_0 + \epsilon \nabla H_1 \quad . \tag{3.7}$$

These again agree with the ordinary perturbation equations (3.4) to first order. (This may not be completely obvious; however note that to leading order $\dot{y} = J_0 \nabla H_0$, so wherever we see a term like $\epsilon\dot{y}$ in the system we can replace it by $\epsilon J_0 \nabla H_0$ without affecting the formal validity of the expansion.) Since the closure condition $d\Omega = 0$ is linear in the coefficients of Ω, in this case it is permissible to truncate the series for Ω and still retain the property of being symplectic. Thus (3.7) is in all cases Hamiltonian. While it is permissible to invert the operator $K_0 + \epsilon K_1$ in (3.7), one cannot re-expand and truncate and expect the resulting system to be Hamiltonian. The symplectic perturbation (3.7), while somewhat easier to deal with theoretically, in general leads to more unpleasant systems as the operator $K_0 + \epsilon K_1$, which can depend on y itself, is applied to the temporal derivative $\dot{y}$.

4. WATER WAVES

By the water wave problem we mean the irrotational motion of an incompressible, inviscid fluid under the influence of gravity. The model equations to be discussed here are for long, small amplitude two-dimensional waves over a shallow horizontal bottom. After rescaling, this free boundary problem takes the form, [2], [19],

$$\beta\varphi_{xx}+\varphi_{yy} = 0\,, \qquad 0<y<1+\alpha\eta\,, \tag{4.1}$$

$$\varphi_y = 0\,, \qquad y=0\,, \tag{4.2}$$

$$|\nabla\varphi|\to 0\,, \qquad |x|\to\infty\,, \tag{4.3}$$

$$\left.\begin{aligned} &\varphi_t+\tfrac{1}{2}\alpha\varphi_x^2+\tfrac{1}{2}\alpha\beta^{-1}\varphi_y^2+\eta-\tau\beta\eta_{xx}(1+\alpha^2\beta\eta_x^2)^{-\frac{3}{2}} = 0\,, &(4.4)\\ &\eta_t = \beta^{-1}\varphi_y-\alpha\eta_x\varphi_x\,, &(4.5)\end{aligned}\right\}\quad y=1+\alpha\eta\,.$$

Here x is the horizontal and y the vertical coordinate, so the bottom is at $y=0$; $\varphi(x,y,t)$ is the velocity potential and $1+\alpha\eta(x,t)$ the free surface elevation. The small parameters α and β represent respectively the ratio of wave amplitude to undisturbed fluid depth, and the square of the ratio of fluid depth to wave length; they are assumed to be of the same order of smallness. Finally τ represents a dimensionless surface tension coefficient, with $\tau = 0$ corresponding to the case of no surface tension.

A. The Standard Perturbation Expansion

In Boussinesq's scheme cf. [19], the first step is to construct a formal series solution to the elliptic boundary value problem (4.1 - 3). In terms of the velocity potential at height $0 \le \theta \le 1$, $\psi(x,t) \equiv \varphi(x,\theta,t)$, the solution is easily found to be

$$\varphi = \psi+\tfrac{1}{2}\beta(\theta^2-y^2)\psi_{xx}+\tfrac{1}{24}\beta^2(5\theta^4-6\theta^2y^2+y^4)\psi_{xxxx}+ \\ +\tfrac{1}{120}\beta^3(61\theta^6-75\theta^4y^2+15\theta^2y^4-y^6)\;\psi_{xxxxxx}+\dots\,. \tag{4.6}$$

Substituting (4.6) into (4.4 - 5), expanding in powers of α, β, truncating to second order and differentiating the first equation yields the model system

$$0 = u_t+\eta_x+\alpha uu_x+\beta[\tfrac{1}{2}(\theta^2-1)u_{xxt}-\tau\eta_{xxx}]+\alpha\beta[\tfrac{1}{2}(\theta^2-1)uu_{xxx}+\tfrac{1}{2}(\theta^2+1)u_xu_{xx}- \\ -(\eta u_{xt})_x]+\tfrac{1}{24}\beta^2(5\theta^4-6\theta^2+1)u_{xxxxt}\,, \tag{4.7}$$

$$0=\eta_t+u_x+\alpha(\eta u)_x+\tfrac{1}{6}\beta(3\theta^2-1)u_{xxx}+\tfrac{1}{2}\alpha\beta(\theta^2-1)(\eta u_{xx})_x+\tfrac{1}{120}\beta^2(25\theta^4-10\theta^2+1)u_{xxxxx}, \tag{4.8}$$

in which $u = \psi_x = \varphi_x(x,\theta,t)$ is the horizontal velocity at depth θ. (In the derivation of this Boussinesq system, usually only done to first order, we have ignored problems concerning precise domains of definition of the functions involved, cf. [10].)

We can play around with the system (4.7-8) by expanding terms according to the equations themselves and retruncating. For instance, to eliminate the t-derivative terms $u_{xt}, u_{xxt}, u_{xxxxt}$ in (4.7) we can differentiate it with respect to x, solve for u_{xt}, etc., and resubstitute. This leads to

$$0=u_t+\eta_x+\alpha uu_x+\beta[\tfrac{1}{2}(1-\theta^2)-\tau]\eta_{xxx}+\alpha\beta[(\eta\eta_{xx})_x+(2-\theta^2)u_xu_{xx}]+ \\ +\beta^2[\tfrac{1}{24}(\theta^4-6\theta^2+5)+\tfrac{1}{2}\tau(\theta^2-1)]\eta_{xxxxx}\,. \tag{4.9}$$

The system (4.8 - 9), which is valid to the same order as (4.7 - 8), is an evolutionary version of the basic Boussinesq system. See Bona and Smith, [5], for a further discussion of the possibilities.

The Boussinesq system is valid for waves moving in both directions. To specialize to waves moving to the right, we have to leading order $\eta = u$, so we seek a "submanifold" of approximately unidirectional solutions, determined by an expansion of the form $\eta = u + \alpha A + \beta B + \dots$. The coefficients are functions of u and its x-derivatives, and are determined so that (4.8 - 9) will agree to second order. A straight forward calculation shows that

$$\eta = u + \frac{1}{4}\alpha u^2 + \beta\left(\frac{1}{2}\theta^2 - \frac{1}{3} + \frac{1}{2}\tau\right) u_{xx} + \alpha\beta\left[\left(\frac{1}{4}\theta^2 - \frac{5}{6} + \frac{1}{4}\tau\right)uu_{xx} - \left(\frac{17}{48} + \frac{3}{16}\tau\right) u_x^2\right] + \beta^2\left[\frac{1}{360}(75\theta^4 - 60\theta^2 + 4) + \left(\frac{1}{4}\theta^2 - \frac{1}{6}\right)\tau + \frac{3}{8}\tau^2\right] u_{xxxx} \qquad (4.10)$$

is the required expansion. Then, to second order, (4.8 - 9) are the same equation, namely

$$0 = u_t + u_x + \frac{3}{2}\alpha uu_x + \left(\frac{1}{6} - \frac{1}{2}\tau\right)\beta u_{xxx} + \left(\frac{5}{12} - \frac{1}{4}\tau\right)\alpha\beta uu_{xxx} + \left(\frac{53}{24} - \frac{3}{2}\theta^2 - \frac{13}{8}\tau\right)\alpha\beta u_x u_{xx} + \left(\frac{19}{360} - \frac{1}{12}\tau - \frac{1}{8}\tau^2\right)\beta^2 u_{xxxxx} . \qquad (4.11)$$

This is the second order perturbation expansion for unidirectional waves; if one retains only first order terms we are left with the Korteweg-de Vries equation, and the above constitutes the traditional derivation of KdV as a model for water waves. Note especially that the KdV model is independent of which depth θ the horizontal velocity u is measured. Thus to first order, solitary waves at all depths move in tandem at the same speed. (Note for large surface tension $\tau > \frac{1}{3}$, these appear as waves of depression, [2].) In the second order model (4.11), depth variations only appear multiplying the obscure term $u_x u_{xx}$. It would be extremely interesting to study the effects of varying θ on the solutions of (4.11). Presumably, if the relationship between wave amplitude and wave velocity for the solitary wave solutions were to depend on θ , this would indicate a tendency to develop some form of internal shearing between solitary waves at different depths, which could lead to a better understanding of the mechanisms behind wave breaking. Unfortunately, the solitary wave solutions of (4.11) cannot be found by direct quadrature, so we must rely on numerical investigations - these will be reported on in a future paper.

An alternative, perhaps more common procedure is to take the surface elevation η as the principal variable. Inverting (4.10) and substituting into (4.8 - 9), we find the unidirectional model

$$0 = \eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \left(\frac{1}{6} - \frac{1}{2}\tau\right)\beta\eta_{xxx} - \frac{3}{8}\alpha^2\eta^2\eta_x + \left(\frac{5}{12} - \frac{1}{2}\tau\right)\alpha\beta\eta\eta_{xxx} + \left(\frac{23}{24} + \frac{5}{8}\tau\right)\alpha\beta\eta_x\eta_{xx} + \left(\frac{19}{360} - \frac{1}{12}\tau - \frac{1}{8}\tau^2\right)\beta^2\eta_{xxxxx} . \qquad (4.12)$$

Note that to first order, both the η and u equations coincide:

$$0 = \eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \left(\frac{1}{6} - \frac{1}{2}\tau\right)\beta\eta_{xxx} . \qquad (4.13)$$

Again, as with the Boussinesq system, we can resubstitute to find alternative models valid to the same order. For instance, since to leading order $\eta_t = -\eta_x$, in the KdV equation (4.13) we can replace η_{xxx} by $-\eta_{xxt}$ to find the BBM equation,

$$0 = \eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x - \left(\frac{1}{6} - \frac{1}{2}\tau\right)\beta\eta_{xxt} \tag{4.14}$$

as an equally valid first order approximation. As discussed in [3], it offers several advantages over the KdV model, including a more realistic dispersion relation.

Since the perturbations discussed so far are for the most part not canonical, the Boussinesq systems (4.7 - 8) or (4.8 - 9) are not Hamiltonian, except in the special case $\theta = 1$ noted by Broer, [6] - see the next section. The KdV equation (4.13) is Hamiltonian of course, but neither of the second order approximations (4.11) or (4.12) are Hamiltonian in any obvious manner <u>except</u> for (4.11) at the curious depth

$$\theta^{*} = \sqrt{\frac{11}{12} - \frac{1}{2}\tau} \quad . \tag{4.15}$$

In this case, it takes the form

$$u_t + D_x(\delta H/\delta u) = 0 ,$$

where the Hamiltonian is

$$\mathcal{H} = \int_{-\infty}^{\infty} \left\{\frac{1}{2}u^2 + \frac{1}{4}\alpha u^3 + \left(\frac{1}{4}\tau - \frac{1}{12}\right)\beta u_x^2 + \left(\frac{1}{8}\tau - \frac{5}{24}\right)\alpha\beta uu_x^2 + \left(\frac{19}{720} - \frac{1}{24}\tau - \frac{1}{16}\tau^2\right)\beta^2 u_{xx}^2\right\}dx \quad .$$

This depth will reappear later.

B. <u>Hamiltonian Perturbations - Bidirectional Models</u>

We now implement the results of section 3 to discuss the Hamiltonian perturbation theory for the water wave problem. In Zakharov's formulation, the surface elevation $\eta(x,t)$ and the surface values of the velocity potential $\varphi_S(x,t) \equiv \varphi(x, 1+\alpha\eta(x,t), t)$ are the canonical variables, and (4.1 - 5) are equivalent to the Hamiltonian system

$$\frac{\partial\varphi_S}{\partial t} = -\frac{\delta\mathcal{H}}{\delta\eta} \quad , \quad \frac{\partial\eta}{\partial t} = \frac{\delta\mathcal{H}}{\delta\varphi_S} \quad , \tag{4.16}$$

in which the Hamiltonian functional is the total energy

$$\mathcal{H} = \int_S \left\{\frac{1}{2}\varphi(\beta^{-1}\varphi_y - \alpha\eta_x\varphi_x) + \frac{1}{2}\eta^2 + \alpha^{-2}\tau[(1+\alpha^2\beta\eta_x^2)^{\frac{1}{2}} - 1]\right\}dx \quad . \tag{4.17}$$

In (4.17), the S subscript on the integral means all terms are evaluated on the free surface, i.e. at $y = 1+\alpha\eta(x,t)$, and then integrated from $-\infty$ to ∞. The values of φ within the fluid, and thus the values of the derivatives of φ on S, are determined from the surface values φ_S by solving the auxilliary boundary value problem (4.1 - 3) - see [4] for elaboration.

In discussing the Hamiltonian perturbations for the water-wave problem, for simplicity we restrict to first order expansions. First, substituting the basic expansion (4.6) into the Hamiltonian (4.17), we get the first order expansion

$$\mathcal{H}^{(1)} = \int_{-\infty}^{\infty} \left\{\frac{1}{2}u^2 + \frac{1}{2}\eta^2 + \frac{1}{2}\alpha\eta u^2 + \left(\frac{1}{3} - \frac{1}{2}\theta^2\right)\beta u_x^2 + \frac{1}{2}\beta\tau\eta_x^2\right\}dx \quad . \tag{4.18}$$

For the symplectic version of the Boussinesq system the two form appropriate to (4.16),

$$\Omega = d\eta \wedge d\varphi_S$$

is expanded in powers of α, β, leading to first order truncation

$$\Omega^{(1)} = \tfrac{1}{2}\, d\eta \wedge (d\psi + \tfrac{1}{2}\beta(\theta^2 - 1)\, d\psi_{xx}) = d\eta \wedge (D_x^{-1} + \tfrac{1}{2}\beta(\theta^2 - 1) D_x)\, du \ . \tag{4.19}$$

(We are omitting the integral signs in the two-forms for simplicity.) This yields the "symplectic Boussinesq" system

$$\begin{aligned} 0 &= \eta_t + u_x + \alpha(u\eta)_x + \tfrac{1}{2}\beta(\theta^2 - 1)\eta_{xxt} + \beta(\theta^2 - \tfrac{2}{3})\, u_{xxx} \ , \\ 0 &= u_t + \eta_x + \alpha u u_x + \tfrac{1}{2}\beta(\theta^2 - 1)\, u_{xxt} - \beta\tau\eta_{xxx} \ . \end{aligned} \tag{4.20}$$

(Actually, (4.20) is the x-derivative of the basic symplectic equations (3.7) associated with (4.18 - 19).) Note that (4.20) agrees with the standard Boussinesq system (4.7 - 8) to first order after manipulations similar to those discussed in [5].

As for the cosymplectic form, the two-vector

$$\Theta = \partial_\eta \wedge \partial_{\varphi_S}$$

has first order expansion

$$\Theta^{(1)} = \partial_\eta \wedge \{D_x + \tfrac{1}{2}\beta(1 - \theta^2) D_x^3\}\, \partial_u \ ,$$

cf. (2.10). This is cosymplectic since the underlying differential operator is constant coefficient, and leads to the "cosymplectic Boussinesq" system.

$$0 = \eta_t + u_x + \alpha(\eta u)_x + \beta(\tfrac{1}{2}\theta^2 - \tfrac{1}{6})u_{xxx} + \tfrac{1}{2}\alpha\beta(1 - \theta^2)(\eta u)_{xxx} - \tfrac{1}{3}\beta(3\theta^4 - 5\theta^2 + 2)u_{xxxxx} \tag{4.22}$$

$$0 = u_t + \eta_x + \alpha u u_x + \beta[\tfrac{1}{2}(1 - \theta^2) - \tau]\eta_{xxx} + \tfrac{1}{4}\alpha\beta(1 - \theta^2)[u^2]_{xxx} - \tfrac{1}{2}\beta^2(1 - \theta^2)\tau\eta_{xxxxx} \ .$$

Note that although the first order terms in (4.22) and (4.8 - 9) agree, the quadratic terms in α, β are very different. One special case of note is when $\theta = 1$, which is (to first order) equivalent to doing the expansion in terms of the canonical variables η, φ_S; the (co-)symplectic form does not change and (4.20) and (4.22) reduce to the Boussinesq equations

$$\begin{aligned} 0 &= \eta_t + u_x + \alpha(\eta u)_x + \tfrac{1}{3}\beta u_{xxx} , \\ 0 &= u_t + \eta_x + \alpha u u_x - \beta\tau\eta_{xxx} , \end{aligned} \tag{4.23}$$

whose Hamiltonian form was first noticed by Broer, [6]. The more general Hamiltonian models (4.20,22) are new.

C. Hamiltonian Perturbations - Unidirectional Models

The procedure for determining unidirectional models remains the same - we seek an expansion of η in terms of u such that the two equations in the Boussinesq system agree, in this case to first order. Moreover, since the Hamiltonian Boussinesq systems already agree with the standard Boussinesq systems to first order, the required expansion is the same as (4.10), or, rather, its first order truncation

$$\eta = u + \frac{1}{4}\alpha u^2 + \left(\frac{1}{2}\theta^2 - \frac{1}{3} + \frac{1}{2}\tau\right)\beta u_{xx} \quad . \tag{4.24}$$

(One slight annoyance here is that there does not appear to be any way of directly finding (4.24) from the Hamiltonian functional itself short of explicitly writing out the system.)

Substituting (4.24) into the Hamiltonian (4.18), to first order

$$\mathcal{H}^{(1)} = \int_{-\infty}^{\infty} \left(u^2 + \frac{3}{4}\alpha u^3 + \left(\frac{2}{3} - \theta^2\right)\beta u_x^2\right) dx \tag{4.25}$$

is the unidirectional Hamiltonian functional. (In (4.25) the term uu_{xx} was integrated by parts using (4.3).)

Consider first the cosymplectic perturbation. The Frechet derivative, (2.6), of (4.24) is the operator

$$D_F = 1 + \frac{1}{2}\alpha u + \left(\frac{1}{2}\theta^2 - \frac{1}{3} + \frac{1}{2}\tau\right)\beta D_x^2 \quad .$$

The inverse can be written in a series in α , β , with first order truncation

$$D_F^{-1} = 1 - \frac{1}{2}\alpha u - \left(\frac{1}{2}\theta^2 - \frac{1}{3} + \frac{1}{2}\tau\right)\beta D_x^2 \quad .$$

Comparing with (2.10), we see that (4.21) becomes

$$\Theta^{(1)} = \frac{\partial}{\partial u} \wedge \left[D_x - \frac{1}{4}\alpha(u D_x + D_x u) + \left(\frac{5}{6} - \theta^2 - \frac{1}{2}\tau\right)\beta D_x^3\right] \frac{\partial}{\partial u} \quad .$$

This is cosymplectic for the same reason the J_1 for the KdV equation is. Combining this with (4.25), we obtain the following "Hamiltonian form" of the KdV equation

$$u_t + \left[D_x - \frac{1}{4}\alpha(u D_x + D_x u) + \left(\frac{5}{6} - \theta^2 - \frac{1}{2}\tau\right)\beta D_x^2\right]\cdot\left[u + \frac{9}{8}\alpha u^2 + \left(\theta^2 - \frac{2}{3}\right)\beta u_{xx}\right] = 0 \ ,$$

or, explicitly,

$$\begin{aligned} u_t + u_x + \frac{3}{2}\alpha uu_x + \left(\frac{1}{6} - \frac{1}{2}\tau\right)\beta u_{xxx} - \frac{45}{32}\alpha^2 u^2 u_x + \left(\frac{53}{24} - \frac{11}{4}\theta^2 - \frac{9}{8}\tau\right)\alpha\beta uu_{xxx} + \\ + \left(\frac{139}{24} - 7\theta^2 - \frac{27}{8}\tau\right)\alpha\beta u_x u_{xx} + \left(\frac{5}{6} - \theta^2 - \frac{1}{2}\tau\right)\left(\theta^2 - \frac{2}{3}\right)\beta^2 u_{xxxxx} = 0 \quad . \end{aligned} \tag{4.26}$$

The first order terms in (4.26) agree with the KdV model, but there are additional, depth dependent second order terms required to maintain the Hamiltonian form of the equation. Note that these differ from the second order terms in the standard perturbation (4.11). The derivation of the Hamiltonian model in which η is the primary variable is similar. We have two-vector

$$\tilde{\Theta}^{(1)} = \frac{\partial}{\partial \eta} \wedge \left[D_x + \frac{1}{4}\alpha(\eta D_x + D_x \eta) + \left(\frac{1}{6} - \frac{1}{2}\tau\right)\beta D_x^3\right] \frac{\partial}{\partial \eta} \quad ,$$

and Hamiltonian functional

$$\tilde{\mathcal{H}}^{(1)} = \int_{-\infty}^{\infty} \left(\eta^2 + \frac{1}{4}\alpha \eta^3\right) dx \ . \tag{4.27}$$

These give the Hamiltonian model

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{xxx} + \frac{1}{16}\alpha\beta(\eta^2)_{xxx} + \frac{15}{32}\alpha^2\eta^2\eta_x = 0 \ . \tag{4.28}$$

The first order expansion (4.27) of the water wave energy functional does not agree with either of the KdV Hamiltonians! (In the derivation of (4.26) or (4.28), an extra factor of $\frac{1}{4}$ multiplies all terms except the t-derivative. This can be rigorously justified by duality since we are restricting to a submanifold of the full (u,η)-space.)

Alternatively, we can consider the symplectic form of the perturbation equations. An easy computation gives two-form

$$\vec{\Omega}^{(1)} = du \wedge [D_x^{-1} + \tfrac{1}{4}\alpha(u D_x^{-1} + D_x^{-1} u) + (\theta^2 - \tfrac{5}{6} + \tfrac{1}{2}\tau)\beta D_x] du \quad .$$

Combining this with the Hamiltonian (4.25), we obtain a Hamiltonian version of the BBM equation

$$[D_x^{-1} + \tfrac{1}{4}\alpha(u D_x^{-1} + D_x^{-1} u) + (\theta^2 - \tfrac{5}{6} + \tfrac{1}{2}\tau)\beta D_x] u_t + u + \tfrac{9}{8}\alpha u^2 + (\theta^2 - \tfrac{2}{3})\, u_{xxx} = 0 \quad .$$

This can be converted into a bona fide differential equation by differentiating, and recalling that $u = \psi_x$:

$$\psi_{xt} + \tfrac{1}{2}\alpha\, \psi_x \psi_{xt} + \tfrac{1}{4}\alpha\, \psi_{xx}\psi_t + (\theta^2 - \tfrac{5}{6} + \tfrac{1}{2}\tau)\beta\, \psi_{xxxt} + \psi_{xx} + \tfrac{9}{4}\alpha\, \psi_x \psi_{xx} + (\theta^2 - \tfrac{2}{3})\beta\, \psi_{xxxx} = 0 \quad .$$

This example well illustrates the earlier remark that while the symplectic perturbation is easier to handle theoretically, the resulting equations are much more unpleasant.

There is a long list of unanswered questions concerning these new model equations. What do their solitary wave solutions look like, and how do they interact? Undoubtedly, they are not solitons. How do the general solutions compare with those of the KdV or BBM equations? Does the appearance of a depth dependence in the higher order terms have any significance? And, finally, do they provide better models for the long-time or qualitative behavior of water waves? All these await future research.

5. COMPLETE INTEGRABILITY

We now turn to the question of why the KdV equation, despite its appearance as the non-Hamiltonian perturbation equation, happens to be a Hamiltonian system. Return to the general set-up, as summarized in (1.3,4), recalling that $F_1 = J_0 \nabla H_1 + J_1 \nabla H_0$. One possibility for (1.3) to be Hamiltonian is if the two constituents of F_1 are multiples of each other:

$$J_0 \nabla H_1 = \sigma\, J_1 \nabla H_0 \quad . \tag{5.1}$$

In this special case, we can invoke a theorem of Magri on the complete integrability of bi-Hamiltonian systems, [8], [12].

Theorem 5.1 Suppose the system $\dot{x} = K_1(x)$ can be written in Hamiltonian form in two distinct ways: $K_1 = J_0 \nabla H_1 = J_1 \nabla H_0$. Suppose further that $J_0 + \mu J_1$ is Hamiltonian for all constant μ. Then the recursion relation $K_n = J_0 \nabla H_n = J_1 \nabla H_{n-1}$ defines an infinite sequence of commuting flows $\dot{x} = K_n(x)$, with mutually conserved Hamiltonians H_n, in involution with respect to either the J_0- or J_1- Poisson bracket. (It should also be assumed that J_0 can always be inverted in the recursion relation, but this usually holds.)

In this special case, both the standard perturbation equation (1.3) and its cosymplectic counterpart (1.4) are linear combinations of the flows K_0, K_1, K_2, and hence, provided enough of the commuting Hamiltonians H_n are independent, are both completely integrable Hamiltonian systems.

For the water wave problem, in the Korteweg - de Vries model the first order terms are in the correct ratio only at the "magic" depth θ^* given by (4.17). At this depth, the Hamiltonian equation (4.26) is a linear combination of a fifth, third and first order KdV equation in the usual hierarchy. Just why this should happen to be the exact same depth at which the standard second order perturbation equation (4.11), (which cannot be completely integrable as no u^2u_x term appears) is Hamiltonian is a complete mystery. For more general depths θ, the condition (5.1) must be "fudged" in order to conclude complete integrability.

Nevertheless, the basic result leads to an interesting speculation. In a large number of physical examples, the zeroth order perturbation equations are linear, while the first order equations turn out to be completely integrable soliton equations such as KdV, sine - Gordon, non-linear Schrödinger, etc. In the cases when these do arise, is it because condition (5.1) or some generalization thereof is in force? If true, this would provide a good explanation for the appearance of soliton equations as models in such a large number of physical systems, as well as providing a convenient check for soliton-behavior in less familiar examples. A good check for this conjecture would be in Zakharov's derivation, [20], of the nonlinear Schrödinger equation as the modulational equation for periodic water waves.

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